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EXISTENCE OF THREE POSITIVE SOLUTIONS FOR M-POINT TIME SCALE BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

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ABSTRACT. In this paper, by using the Leggett-Williams fixed point theorem and Five Functionals fixed point theorem, we establish the existence of three positive solutions for m-point time scale boundary value problems on infinite intervals. As an application, we also give some examples to demonstrate our results.

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1. INTRODUCTION

We consider the following time scale m-point boundary value problem (BVP)

$$(1.1)\qquad \begin{cases} (\varphi(x^{\triangle}(t)))^{\nabla} + \phi(t)f(t,x(t),x^{\triangle}(t)) = 0,\ t \in (0,\infty)_{\mathbb{T}} \\ x(0) = \sum_{i=1}^{m-2}\alpha_i x^{\triangle}(\eta_i),\ \lim_{t\in\mathbb{T},t\to\infty} x^{\triangle}(t) = 0, \end{cases}$$

where $\mathbb{T}$ is time scale, $f \in \mathcal{C}([0,\infty)_{\mathbb{T}} \times [0,\infty) \times [0,\infty), [0,\infty))$, $\alpha_i \geq 0$ $(1 \leq i \leq m-2)$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < \infty, \varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism with $\varphi(0) = 0$. A projection $\varphi : \mathbb{R} \to \mathbb{R}$ is called an increasing homeomorphism and positive homomorphism if the following conditions are satisfied:

(i) If $x \leq y$, then $\varphi(x) \leq \varphi(y)$, for all $x, y \in \mathbb{R}$;
(ii) φ is continuous bijection and its inverse mapping is also continuous;
(iii) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in \mathbb{R}_+$, where $\mathbb{R}_+ = [0,\infty)$.

We will assume that the following conditions are satisfied:

(H1) $\phi \in \mathcal{C}([0,\infty)_{\mathbb{T}}, [0,\infty))$, $\int_0^{\infty} \phi(s)\nabla s < \infty$, $\int_0^{\infty} \varphi^{-1}(\int_{\tau}^{\infty} \phi(s)\nabla s)\triangle\tau < \infty$;
(H2) $f(t,(1+t)u,v) \leq \omega(\max\{|u|,|v|\})$ with $\omega \in \mathcal{C}([0,\infty),[0,\infty))$ nondecreasing;
(H3) $0 < \gamma(\theta_1,\theta_2) = \min_{(t,u,v)\in[\frac{1}{k},k]_{\mathbb{T}}\times[\frac{\theta_1}{k},\theta_2]\times[0,\theta_2]} f(t,(1+t)u,v)$, for $0 < \theta_1 < \theta_2$; and
(H4) $f(t,(1+t)u,v) \in \mathcal{C}([0,\infty)_{\mathbb{T}} \times [0,\infty) \times [0,\infty),[0,\infty))$, $f(t,0,0) \neq 0$ on any subinterval of $(0,\infty)$.

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Throughout the paper, let $\mathbb{T}$ (time scale) be a nonempty closed subset of $\mathbb{R}$. We assume that $\mathbb{T}$ has the topology which inherits from the standard topology on $\mathbb{R}$. For notation, we shall use the convention that, for each interval J of $\mathbb{R}$, J will denote the time scale interval, that is, $J := J \cap \mathbb{T}$.

The study of dynamic equations on time scales goes back to its founder Stefan Hilger [7] Theoretically, this new theory cannot only unify continuous and discrete equations, but have also exhibited much more complicated dynamics on time scales. Moreover, the study of dynamic equations on time scales has led to several important applications, e.g., insect population models, neural networks, heat transfer and epidemic models (see [3, 4] and references therein). Some preliminary definitions and theorems on time scales also can be found in books [3, 4] which are excellent references for calculus of time scales.

Much of the theory of time scale dynamic equations on finite intervals have been presented in [1, 2, 6, 8, 9, 11, 12] and references therein. However there is significantly less literature available on the basic theory of time scale dynamic equations on infinite intervals (see [5, 10, 13, 14] and references therein). To the authors' knowledge, no one has studied the existence of positive solutions for m-point time scale boundary value problems for an increasing homeomorphism and positive homomorphism on half-line. The present work is motivated by recent papers [5, 13].

Guo, Yu and Wang [5] considered the following m-point boundary value problem on infinite intervals

$$(1.2)\qquad \begin{cases} (\varphi_p(x'(t)))' + \phi(t)f(t, x(t), x'(t)) = 0, \ 0 < t < +\infty \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x'(\eta_i), \ \lim_{t\to+\infty} x'(t) = 0, \end{cases}$$

where $\varphi_p(s) = |s|^{p-2}s, \ p > 1, \ \phi : \mathbb{R}_+ \to \mathbb{R}_+, \ f(t, u, v) : \mathbb{R}_+^3 \to \mathbb{R}_+$ is a continuous function, $\mathbb{R}_+ = [0, +\infty), \ \alpha_i \geq 0$ and $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < +\infty$ are given. By using Avery- Peterson fixed point theorem, they obtained the existence of at least three positive solutions under some sufficient conditions.

Zhao and Ge [13] considered the following second-order m-point boundary value problem on time scales

$$(1.3)\qquad \begin{cases} (\phi_p(u^\triangle(t)))^\nabla + q(t)f(u(t), u^\triangle(t)) = 0, \ t \in (0, \infty)_\mathbb{T} \\ u(0) = \beta u^\triangle(\eta), \ \lim_{t\in\mathbb{T}, t\to\infty} u^\triangle(t) = 0, \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s, \ p > 1, \ (\phi_p)^{-1} = \phi_q, \ \frac{1}{p} + \frac{1}{q} = 1, \eta \in \mathbb{T}, \ \beta \in \mathbb{R}, \ \beta > 0$, and $f \in \mathcal{C}([0, \infty) \times [0, \infty), [0, \infty))$. By using Leggett-Williams fixed point theorem [8], they obtained the existence of at least three positive solutions under some sufficient conditions.

Motivated by the above results, in this paper, we obtained the existence of at least three positive solutions for the BVP (1.1) by using Leggett-Williams fixed point theorem [8] and Five Functionals fixed point theorem [2]. As an application, examples are worked out finally. The remainder of this paper is organized as follows. Section 2 is devoted to some preliminary discussions. We give and prove our main results in Section 3 and Section 4.

2. PRELIMINARIES

In this section we present some definitions and lemmas, which will be needed in the proof of the main results.

Definition 2.1. Let $\mathbb{B}$ be a real Banach space. A nonempty closed set $P \subset \mathbb{B}$ is a cone provided that

(1) $u \in P$ and $\lambda \geq 0$ implies $\lambda u \in P$;
(2) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset \mathbb{B}$ induces an ordering in $\mathbb{B}$ given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space $\mathbb{B}$, provided that $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v),$$

for all $u, v \in P,\ 0 \leq t \leq 1$.

Similarly, we say the map γ a nonnegative continuous convex functional on a cone P of a real Banach space $\mathbb{B}$ provided that

$$\gamma(tu + (1-t)v) \leq t\gamma(u) + (1-t)\gamma(v),$$

for all $u, v \in P,\ 0 \leq t \leq 1$.

Let $\mathbb{B}$ be the Banach space defined by

$$\mathbb{B} = \{x \in \mathcal{C}^{\triangle}[0, \infty) : \sup_{t \in [0,\infty)_{\mathbb{T}}} \frac{|x(t)|}{1+t} < \infty,\ \lim_{t \in \mathbb{T}, t \to \infty} x^{\triangle}(t) = 0\}, \tag{2.1}$$

with the norm $\|x\| = \max\{\|x\|_1, \|x^{\triangle}\|_\infty\}$, where

$$\|x\|_1 = \sup_{t \in [0,\infty)_{\mathbb{T}}} \frac{|x(t)|}{1+t}, \quad \|x^{\triangle}\|_\infty = \sup_{t \in [0,\infty)_{\mathbb{T}}} |x^{\triangle}(t)|$$

and let

$$P = \{x \in \mathbb{B} :\ x \textit{ is nonnegative, concave, nondecreasing on } [0, \infty)_{\mathbb{T}}\}. \tag{2.2}$$

Lemma 2.3. *Let $y \in \mathcal{C}([0, \infty)_{\mathbb{T}}, [0, \infty))$ and $\int_0^\infty y(t)\nabla t < \infty$, then BVP*

$$\begin{cases} \varphi(x^{\triangle}(t)))^{\nabla} + y(t) = 0,\ t \in (0, \infty)_{\mathbb{T}} \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x^{\triangle}(\eta_i),\ \lim_{t \in \mathbb{T}, t \to \infty} x^{\triangle}(t) = 0, \end{cases} \tag{2.3}$$

has a unique solution

$$x(t)=\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\eta_i}^{\infty}y(s)\nabla s)+\int_0^t\varphi^{-1}(\int_s^{\infty}y(\tau)\nabla\tau)\triangle s.$$

Solving the BVP (1.1) is equivalent to finding fixed points of the operator $T:P\to\mathbb{B}$ defined by for $t\in[0,\infty)_{\mathbb{T}}$

$$(Tx)(t)=\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\eta_i}^{\infty}\phi(\tau)f(\tau,x(\tau),x^{\triangle}(\tau))\nabla\tau)$$
$$+\int_0^t\varphi^{-1}(\int_s^{\infty}\phi(\tau)f(\tau,x(\tau),x^{\triangle}(\tau))\nabla\tau)\triangle s. \tag{2.4}$$

Lemma 2.4. *For $x\in P$, $\|x\|_1\le M\|x^{\triangle}\|_\infty$, where $M=\max\{\sum_{i=1}^{m-2}\alpha_i,\ 1\}$.*

Proof. Since $x\in P$, it holds

$$\frac{x(t)}{1+t}=\frac{1}{1+t}(\int_0^t x^{\triangle}(s)\triangle s+\sum_{i=1}^{m-2}\alpha_i x^{\triangle}(\eta_i))\le\frac{t+\sum_{i=1}^{m-2}\alpha_i}{1+t}\|x^{\triangle}\|_\infty\le M\|x^{\triangle}\|_\infty.$$

The result is proved. □

Lemma 2.5. *If* (H1), (H2) *and* (H4) *hold, then $T:P\to P$ is completely continuous.*

Proof. We divide the proof into four steps.

Step 1: We show that $TP\subset P$.

For $x\in P$, we have

$(Tx)^{\triangle}(\infty)=0,$

$(\varphi(Tx^{\triangle}(t)))^{\nabla}=-\phi(t)f(t,x(t),x^{\triangle}(t))\le 0,$

$(Tx)^{\triangle}(t)=\varphi^{-1}(\int_t^{\infty}\phi(\tau)f(\tau,x(\tau),x^{\triangle}(\tau))\nabla\tau)\ge 0$

$(Tx)(0)=\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\eta_i}^{\infty}\phi(\tau)f(\tau,x(\tau),x^{\triangle}(\tau))\nabla\tau)=\sum_{i=1}^{m-2}\alpha_i(Tx)^{\triangle}(\eta_i)\ge 0$

Hence Tx is nonnegative, concave, and nondecreasing on $[0,\infty)_{\mathbb{T}}$, i.e., $TP\subset P$.

Step 2 : We show that $T:P\to P$ is continuous.

Let $x_n\to x$ as $n\to+\infty$ in P, then there exists r_0 such that $\sup_{n\in\mathbb{N}\setminus\{0\}}\|x_n\|<r_0$. By (H2), we get that $f(t,(1+t)u,v)\le\omega(r_0)$, and we have

$$\int_0^{\infty}\phi(\tau)|f(\tau,x_n,x_n^{\triangle})-f(\tau,x,x^{\triangle})|\nabla\tau\le 2\omega(r_0)\int_0^{\infty}\phi(\tau)\nabla\tau.$$

Therefore by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}|\varphi((Tx_n)^{\triangle}(t))-\varphi((Tx)^{\triangle}(t))| &= |\int_t^{\infty}\phi(\tau)f(\tau,x_n,x_n^{\triangle})-f(\tau,x,x^{\triangle})\nabla\tau|\\ &\le \int_0^{\infty}\phi(\tau)|f(\tau,x_n,x_n^{\triangle})-f(\tau,x,x^{\triangle})|\nabla\tau\end{aligned}$$

$$\to \ 0, \ \ n \to \infty.$$

Furthermore $\|Tx_n - Tx\| \le M\|(Tx_n)^{\triangle} - (Tx)^{\triangle}\|_{\infty} \to 0$, as $n \to \infty$. So T is continuous.

Step 3 : We show that $T : P \to P$ is relatively compact.

Let Ω be any bounded subset of P, then there exists $K > 0$ such that $\|x\| \le K$. By (H1), for $\forall x \in \Omega$, we have

$$\|(Tx)^{\triangle}\|_{\infty} = \varphi^{-1}(\int_0^{\infty} \phi(\tau) f(\tau, x(\tau), x^{\triangle}(\tau))\nabla\tau) \le \omega(K)\varphi^{-1}(\int_0^{\infty} \phi(\tau)\nabla\tau) < \infty.$$

Therefore, $\|T\Omega\| \le M\|(T\Omega)^{\triangle}\|_{\infty} < \infty$. So $T\Omega$ is uniformly bounded.

Now we show that $(T\Omega)$ is equicontinuous on $[0,\infty)_{\mathbb{T}}$. For any $R > 0, t_1, t_2 \in [0, R]_{\mathbb{T}}$, and for all $x \in \Omega$, without loss of generality we may assume that $t_2 > t_1$.

$$\begin{aligned}
&|\frac{(Tx)(t_1)}{1+t_1} - \frac{(Tx)(t_2)}{1+t_2}| \\
&\le \sum_{i=1}^{m-2} \alpha_i \varphi^{-1}(\int_0^{\infty} \phi(\tau) f(\tau, x(\tau), x^{\triangle}(\tau))\nabla\tau)|\frac{1}{1+t_1} - \frac{1}{1+t_2}| \\
&\quad \int_0^{t_1} \varphi^{-1}(\int_s^{\infty} \phi(\tau) f(\tau, x(\tau), x^{\triangle}(\tau))\nabla\tau)\triangle s|\frac{1}{1+t_1} - \frac{1}{1+t_2}| \\
&\quad + \frac{1}{1+t_2}\int_{t_1}^{t_2} \varphi^{-1}(\int_s^{\infty} \phi(\tau) f(\tau, x(\tau), x^{\triangle}(\tau))\nabla\tau)\triangle s \\
&\le \sum_{i=1}^{m-2} \alpha_i \varphi^{-1}(\omega(K))\varphi^{-1}(\int_0^{\infty} \phi(\tau)\nabla\tau)|\frac{1}{1+t_1} - \frac{1}{1+t_2}| \\
&\quad + \varphi^{-1}(\omega(K))\int_0^{t_1} \varphi^{-1}(\int_s^{\infty} \phi(\tau)\nabla\tau)\triangle s|\frac{1}{1+t_1} - \frac{1}{1+t_2}| \\
&\quad + + \frac{1}{1+t_2}\varphi^{-1}(\omega(K))\int_{t_1}^{t_2} \varphi^{-1}(\int_s^{\infty} \phi(\tau)\nabla\tau)\triangle s \\
&\to 0, \quad \text{uniformly as } t_1 \to t_2,
\end{aligned}$$

$$\begin{aligned}
|\varphi((Tx)^{\triangle}(t_1)) - \varphi((Tx)^{\triangle}(t_2))| &= |\int_{t_1}^{t_2} \phi(\tau) f(\tau, x(\tau), x^{\triangle}(\tau))\nabla\tau| \\
&\le \omega(K)|\int_{t_1}^{t_2} \phi(\tau)\nabla\tau| \\
&\to 0, \quad \text{uniformly as } t_1 \to t_2.
\end{aligned}$$

So $T\Omega$ is equicontinuous on any compact interval of $[0,\infty)_{\mathbb{T}}$.

Step 4 : We show that $T : P \to P$ is equiconvergent at ∞. For any $x \in \Omega$,

$$\begin{aligned}
\lim_{t\to\infty} |\frac{(Tx)(t)}{1+t}| &= \lim_{t\to\infty} \frac{1}{1+t}\int_0^{t} \varphi^{-1}(\int_s^{\infty} \phi(\tau) f(\tau, x(\tau), x^{\triangle}(\tau))\nabla\tau)\triangle s \\
&\le M\varphi^{-1}(\omega(K)) \lim_{t\to\infty}(\int_t^{\infty} \phi(\tau)\nabla\tau) = 0,
\end{aligned}$$

$$\lim_{t\to\infty}|(Tx)^{\triangle}(t)| = \lim_{t\to\infty}\varphi^{-1}(\int_t^\infty \phi(\tau)f(\tau,x(\tau),x^{\triangle}(\tau))\nabla\tau) \leq \varphi^{-1}(\omega(K))\lim_{t\to\infty}(\int_t^\infty \phi(\tau)\nabla\tau) = 0.$$

So $T\Omega$ is equiconvergent at infinity. As a consequence of Step 1-4, we get $T : P \to P$ is completely continuous. The proof is complete. □

3. MAIN RESULTS

In this section, we prove the existence of at least three positive solutions to the BVP (1.1) by applying the Leggett-Williams fixed point theorem [8]. Applications of the Leggett-Williams fixed point theorem can be found in recent papers (see Refs.[12]).

Let $\mathbb{B} = (\mathbb{B}, \|.\|)$, $P \subset \mathbb{B}$ be defined as (2.1) and (2.2), respectively. Let α be a nonnegative continuous concave functional on P and $a, b, c > 0$ constants. Define

$$P_c = \{x \in P : \|x\| < c\},$$

$$P(\alpha, a, b) = \{x \in P : a \leq \alpha(x), \|x\| \leq b\}.$$

To prove our results, we need the following fixed point theorem.

Theorem 3.1 (Leggett-Williams [8]). *Let $\mathbb{B} = (\mathbb{B}, \|.\|)$ be a Banach space, $P \subset \mathbb{B}$ a cone of $\mathbb{B}$ and $c > 0$ a constant. Suppose that there exists a nonnegative continuous concave functional α on P with $\alpha(x) \leq \|x\|$, for $x \in \overline{P}_c$ and let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous map. Assume that there exist a, b, d with $0 < a < b < d \leq c$, such that:*

(S1) $\{x \in P(\alpha, b, d) : \alpha(x) > b\} \neq \emptyset$ *and* $\alpha(Tx) > b$, *for all* $x \in P(\alpha, b, d)$*;*
(S2) $\|Tx\| < a$, *for all* $x \in \overline{P}_a$*;*
(S3) $\alpha(Tx) > b$, *for all* $x \in P(\alpha, b, c)$, *with* $\|Tx\| > d$.

Then T has at least three fixed points $x_1, x_2, x_3 \in P$, such that

$$\|x_1\| < a, \quad \alpha(x_2) > b, \quad \|x_3\| > a \text{ and } \alpha(x_3) < b.$$

We define the nonnegative continuous concave functional $\alpha : P \to [0, \infty)$ by

$$\alpha(u) = \frac{k}{k+1}\min_{t\in[\frac{1}{k},\infty)_{\mathbb{T}}}|u(t)|, \quad \forall u \in P,$$

where $\frac{1}{k} \in \mathbb{T}$ be fixed, such that $0 < k < \infty$. It is easy to see that

$$\alpha(u) = \frac{k}{k+1}u(\frac{1}{k}) \leq \|u\|.$$

For convenience, we define

$$\theta = \varphi^{-1}(\int_0^\infty \phi(\tau)\nabla\tau),$$

$\lambda = \frac{1}{k+1}\varphi^{-1}(\gamma(b,c)\int_{\frac{1}{k}}^{k}\phi(\tau)\nabla\tau)$,

$\mu = \sup_{t\in[0,\infty)_{\mathbb{T}}}\frac{1}{1+t}(\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\eta_i}^{\infty}\phi(\tau)\nabla\tau) + \int_0^t\varphi^{-1}(\int_s^{\infty}\phi(\tau)\nabla\tau)\triangle s)$.

Theorem 3.2. *Assume that* (H1)-(H3) *hold. Let* $0 < a < b \le \frac{\lambda}{\mu\omega(c)}d < d \le c$, f *satisfies the following conditions:*

(B1) $f(t,(1+t)u,v) < \varphi(\frac{a}{M\mu})$ *for all* $(t,u,v)\in[0,\infty)_{\mathbb{T}}\times[0,a]\times[0,a]$;
(B2) $f(t,(1+t)u,v) \le \varphi(\frac{c}{M\mu})$ *for all* $(t,u,v)\in[0,\infty)_{\mathbb{T}}\times[0,c]\times[0,c]$;
(B3) $f(t,(1+t)u,v) > \varphi(\frac{b}{\lambda})$ *for all* $(t,u,v)\in[\frac{1}{k},k]_{\mathbb{T}}\times[\frac{b}{k},d]\times[0,d]$;
(B4) $\theta\le\mu$.

Then boundary value problem (1.1) *has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\|u_1\| < a,\quad \alpha(u_2) > b,\quad \|u_3\| > a \text{ and } \alpha(u_3) < b.$$

Proof. We first show that $T:\overline{P}_c\to\overline{P}_c$. If (B2) holds, then $f(t,(1+t)u,v)\le\varphi(\frac{c}{M\mu})$ for all $(t,u,v)\in[0,\infty)_{\mathbb{T}}\times[0,c]\times[0,c]$. In fact, by Lemma 2.5, we have $T\overline{P}_c\subset P$. Furthermore, $\forall u\in\overline{P}_c li$, we have $0\le\|u\|\le c$. By Lemma 2.4 and (B4), we have

$$\begin{aligned}\|Tu\| &= \max\{\|Tu\|_1, \|Tu^{\triangle}\|_{\infty}\}\\ &\le M\|Tu^{\triangle}\|_{\infty} = M\varphi^{-1}(\int_0^{\infty}\phi(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\\ &\le M\frac{c}{M\mu}\varphi^{-1}(\int_0^{\infty}\phi(\tau)\nabla\tau) = \frac{c}{\mu}\theta\le\frac{c}{\mu}\mu = c.\end{aligned}$$

Hence $T\overline{P}_c\subset\overline{P}_c$. In the same way, we can show that if (B1) holds, then $T\overline{P}_a\subset P_a$. Hence $\|Tu\| < a$, $\forall u\in\overline{P}_a$ and so (S2) of Theorem 3.1 holds.

Next, we show that (S1) of Theorem 3.1 holds. To check that the condition (S1) of Theorem 3.1, we choose

$$u(t) = \frac{b+d}{2}(1+t),\ t\in[0,\infty)_{\mathbb{T}}.$$

It is easy to see that $u(t)\in P(\alpha,b,d)$ with $\alpha(u) > b$. So $\{u\in P(\alpha,b,d):\alpha(u)>b\}\ne\emptyset$. Moreover, $\forall u\in P(\alpha,b,d)$, we have $\frac{b}{k}\le\frac{u(t)}{1+t}\le d$, $t\in[\frac{1}{k},k]_{\mathbb{T}}$, $\|u\|\le d$. By (B3), we have

$$\begin{aligned}\alpha(Tu) &= \frac{k}{k+1}\min_{t\in[\frac{1}{k},\infty)}(Tu)(t) = \frac{k}{k+1}(Tu)(\frac{1}{k})\\ &= \frac{k}{k+1}(\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\eta_i}^{\infty}\phi(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\\ &+ \int_0^{\frac{1}{k}}\varphi^{-1}(\int_s^{\infty}\phi(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\triangle s))\end{aligned}$$

$$\geq \frac{k}{k+1}(\frac{1}{k}\varphi^{-1}(\int_{\frac{1}{k}}^{k}\phi(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau))$$
$$> \frac{b}{\lambda}\frac{1}{k+1}\varphi^{-1}(\int_{\frac{1}{k}}^{k}\phi(\tau)\nabla\tau) = b.$$

Finally we show that (S3) of Theorem 3.1 holds. For $u \in P(\alpha,b,c)$ and $\|Tu\| > d$, we have

$$\frac{b}{k} \leq \frac{u(t)}{1+t} \leq c,\ t \in [\frac{1}{k},k]_{\mathbb{T}},\ \|u\| \leq c.$$

By (H3) and (B4), we have

$$\begin{aligned}
\alpha(Tu) &= \frac{k}{k+1}(Tu)(\frac{1}{k})\\
&= \frac{k}{k+1}(\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\eta_i}^{\infty}\phi(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\\
&\quad + \int_0^{\frac{1}{k}}\varphi^{-1}(\int_s^{\infty}\phi(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\triangle s)\\
&\geq \frac{k}{k+1}\int_0^{\frac{1}{k}}\varphi^{-1}(\int_s^{\infty}\phi(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\triangle s\\
&\geq \frac{1}{k+1}\varphi^{-1}(\int_{\frac{1}{k}}^{k}\phi(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\\
&\geq \frac{1}{k+1}\varphi^{-1}(\gamma(b,c)\int_{\frac{1}{k}}^{k}\phi(\tau)\nabla\tau)\\
&= \frac{\frac{1}{k+1}\varphi^{-1}(\gamma(b,c)\int_{\frac{1}{k}}^{k}\phi(\tau)\nabla\tau)}{\omega(c)(\sup_{t\in[0,\infty)_{\mathbb{T}}}\frac{1}{1+t}(\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\eta_i}^{\infty}\phi(\tau)\nabla\tau)+\int_0^t\varphi^{-1}(\int_s^{\infty}\phi(\tau)\nabla\tau)\triangle s))}\\
&\quad \times \omega(c)(\sup_{t\in[0,\infty)_{\mathbb{T}}}\frac{1}{1+t}(\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\eta_i}^{\infty}\phi(\tau)\nabla\tau)+\int_0^t\varphi^{-1}(\int_s^{\infty}\phi(\tau)\nabla\tau)\triangle s))\\
&\geq \frac{\lambda}{\mu\omega(c)}\sup_{t\in[0,\infty)_{\mathbb{T}}}\frac{1}{1+t}(\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\eta_i}^{\infty}\phi(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\\
&\quad + \int_0^t\varphi^{-1}(\int_s^{\infty}\phi(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\triangle s)\\
&= \frac{\lambda}{\mu\omega(c)}\|Tu\| > \frac{\lambda}{\mu\omega(c)}d \geq b.
\end{aligned}$$

To sum up, all the hypotheses of Theorem 3.1 are satisfied. The proof is complete. □

Example 3.3. Let $\mathbb{T} = \{\frac{n}{2} : n \in \mathbb{N}_0\} \cup [2,\infty)$, $\eta_1 = \frac{1}{2}$, $\eta_2 = \frac{3}{2}$, $\alpha_1 = \alpha_2 = 1$, $M = 2$, $\phi(t) = e^{-t}$ and $\varphi(x) = x^2$ in the boundary value problem (1.1). Now we consider the following problem

$$(3.1)\qquad \begin{cases} (\varphi(u^{\triangle}))^{\nabla}(t) + e^{-t} f(t, u(t), u^{\triangle}(t)) = 0, \ t \in (0, \infty)_{\mathbb{T}} \\ u(0) = u^{\triangle}(\frac{1}{2}) + u^{\triangle}(\frac{3}{2}), \ \ \lim_{t\in\mathbb{T}, t\to\infty} u^{\triangle}(t) = 0 \end{cases}$$

where

$$(3.2)\qquad f(t, (1+t)u, v) = \begin{cases} \frac{t}{1+t^2}(6 \times 10^2 u^{16} + \frac{v}{2\times 10^3}), \ \ u \le 1, 0 \le v, t \in \mathbb{T} \\ \frac{t}{1+t^2}(6 \times 10^2 + \frac{v}{2\times 10^3}), \ \ u > 1, 0 \le v, t \in \mathbb{T}. \end{cases}$$

Choose $a = \frac{1}{2},\ b = 3,\ k = 2,\ d = 1000,\ c = 2\times 10^3$. Then by simple calculations, we can obtain that

$$\gamma(b, c) = \gamma(3, 2\times 10^3) = 240,\ \theta = 0.895,\ \lambda = 3.11,\ \mu = 0.913,\ \omega(c) = \omega(2\times 10^3) = 300.5.$$

$$0 < \frac{1}{2} < 3 < 11.33 < 1000 < 2000.$$

(1) $f(t, (1+t)u, v) \le 0.0045 < \varphi(\frac{a}{M\mu})$, for $(t, u, v) \in [0, \infty)_{\mathbb{T}} \times [0, \frac{1}{2}] \times [0, \frac{1}{2}]$;
(2) $f(t, (1+t)u, v) \le 300.5 \le \varphi(\frac{c}{M\mu})$, for $(t, u, v) \in [0, \infty)_{\mathbb{T}} \times [0, 2\times 10^3] \times [0, 2\times 10^3]$;
(3) $f(t, (1+t)u, v) \ge 240 > \varphi(\frac{b}{\lambda})$, for $(t, u, v) \in [\frac{1}{2}, 2]_{\mathbb{T}} \times [\frac{3}{2}, 1000] \times [0, 1000]$;
(4) $\theta = 0.895 \le \mu = 0.913$.

Therefore the conditions of Theorem 3.2 are all satisfied. So BVP (3.1) has at least three positive solutions u_1, u_2 and u_3 such that

$$\|u_1\| < \frac{1}{2}, \ \ 3 < \alpha(u_2) = \frac{2}{3} u_2(\frac{1}{2}), \ \ \|u_3\| > \frac{1}{2}, \ \ \alpha(u_3) = \frac{2}{3} u_3(\frac{1}{2}) < 3.$$

4. MAIN RESULTS

We will use the Five Functionals fixed point theorem due to Avery [2] (which is the generalization of the Leggett-Williams fixed point theorem [13]) to prove the existence of at least three positive solutions to the BVP (1.1). An application of the Five Functionals fixed point theorem can be found in recent paper (see Refs. [6, 9, 11, 12]).

Let γ, β, θ be nonnegative continuous convex functionals on P and α, ψ be nonnegative continuous concave functionals on P. Then for nonnegative real numbers l, a, b, d and c we define the convex sets

$P(\gamma, c) = \{u \in P : \gamma(u) < c\}$,

$P(\gamma, \alpha, a, c) = \{u \in P : a \le \alpha(u),\ \gamma(u) \le c\}$,

$Q(\gamma, \beta, d, c) = \{u \in P : \beta(u) \le d,\ \gamma(u) \le c\}$,

$P(\gamma, \theta, \alpha, a, b, c) = \{u \in P : a \le \alpha(u),\ \theta(u) \le b,\ \gamma(u) \le c\}$,

$Q(\gamma, \beta, \psi, l, d, c) = \{u \in P : l \le \psi(u),\ \beta(u) \le d,\ \gamma(u) \le c\}$.

Theorem 4.1 (Five Functionals Fixed Point Theorem [2])**.** *Suppose there exist $c > 0$ and $r > 0$ such that*

$$\alpha(u) \le \beta(u) \text{ and } \|u\| \le r\gamma(u)$$

for all $u \in \overline{P(\gamma, c)}$. Suppose further that $T : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ be a completely continuous operator. If there exist nonnegative real numbers $a,\ b,\ d$ and l with $0 < d < a$ such that

(i) $\{u \in P(\gamma, \theta, \alpha, a, b, c) : \alpha(u) > a\} \neq \emptyset$ *and* $\alpha(Tu) > a$ *for* $u \in P(\gamma, \theta, \alpha, a, b, c)$;
(ii) $\{u \in Q(\gamma, \beta, \psi, l, d, c) : \beta(u) < d\} \neq \emptyset$ *and* $\beta(Tu) < d$ *for* $u \in Q(\gamma, \beta, \psi, l, d, c)$;
(iii) $\alpha(Tu) > a$ *for* $u \in P(\gamma, \alpha, a, c)$ *with* $\theta(Tu) > b$;
(iv) $\beta(Tu) < d$ *for* $u \in Q(\gamma, \beta, d, c)$ *with* $\psi(Tu) < l$.

Then T has at least three positive solutions u_1, u_2 and u_3 in $\overline{P(\gamma, c)}$ satisfying

$$\beta(u_1) < d, \quad \alpha(u_2) > a, \quad d < \beta(u_3) \text{ with } \alpha(u_3) < a.$$

Let $0 < k < \infty,\ \frac{1}{k} \in \mathbb{T}$ be fixed, $l = 0,\ r = 1$ and define the nonnegative, continuous, concave functionals $\alpha,\ \psi$ and the nonnegative, continuous, convex functionals $\gamma,\ \beta,\ \theta$ on P by

$$\alpha(u) = \frac{k}{k+1} \min_{t \in [\frac{1}{k}, \infty)_{\mathbb{T}}} u(t),\ \gamma(u) = \beta(u) = \theta(u) = \|u\|,\ \psi(u) \equiv 0. \tag{4.1}$$

In addition to this $\alpha(u) \le \beta(u)$ and $\|u\| \le r\ \gamma(u)$ for $u \in P$.

Set

$$h = \varphi^{-1}\Big(\int_{\frac{1}{k}}^{k} \phi(\tau)\nabla\tau\Big),\ N = [\varphi^{-1}\Big(\int_0^\infty \phi(\tau)\nabla\tau\Big)]^{-1}. \tag{4.2}$$

Theorem 4.2. *Let $0 < \sum_{i=1}^{m-2} \alpha_i = \xi < 1$. Assume that* (H1) *and* (H4) *hold. Suppose that there exist positive numbers $0 < d < a < c$ such that*

(D1) $f(t, (1+t)u, v) > \varphi(\frac{(k+1)a}{h})$ *for all* $(t, u, v) \in [\frac{1}{k}, k]_{\mathbb{T}} \times [\frac{a}{k}, c] \times [0, c]$;
(D2) $f(t, (1+t)u, v) < \varphi(\frac{dN}{M})$ *for all* $(t, u, v) \in [0, \infty)_{\mathbb{T}} \times [0, d] \times [0, d]$;
(D3) $f(t, (1+t)u, v) \le \varphi(\frac{cN}{M})$ *for all* $(t, u, v) \in [0, \infty)_{\mathbb{T}} \times [0, c] \times [0, c]$

Then boundary value problem (1.1) *has at least three positive solutions u_1, u_2 and u_3 such that*

$$\|u_1\| < d, \quad \alpha(u_2) > a, \quad d < \|u_3\| \text{ with } \alpha(u_3) < a.$$

Proof. The conditions of the Five Functionals fixed point theorem (Theorem 4.1) will be shown to be satisfied. Let $\mathbb{B}$, P and T be defined as in (2.1), (2.2) and (2.4), respectively. From Lemma 2.5, $T : P \to P$.

Let $u \in \overline{P(\gamma, c)}$, then $\gamma(u) \le c$, this implies $0 \le \frac{u(t)}{1+t} \le c,\ 0 \le u^{\triangle}(t) \le c$ for $t \in [0, \infty)$. We obtain by Lemma 2.4 and (D3),

$$\gamma(Tu) \ = \ \|Tu\| = \max\{\|Tu\|_1, \|Tu^{\triangle}\|_\infty\}$$

$$= \quad M\varphi^{-1}(\int_0^\infty \phi(\tau)f(\tau,u(\tau),u^\triangle(\tau))\nabla\tau) \leq M\frac{cN}{M}\varphi^{-1}(\int_0^\infty \phi(\tau)\nabla\tau) = c.$$

Therefore $T:\overline{P(\gamma,c)}\to\overline{P(\gamma,c)}$. In the following, we shall show that the conditions of Theorem 4.1 are satisfied with $b=c$.

We take $u(t)=\frac{c+a}{2}(1+t),\ t\in[0,\infty)_{\mathbb{T}}$. It is easy to see that $u(t)\in P,\ \|u\|=\frac{c+a}{2}<c,\ \alpha(u)=\frac{c+a}{2}>a$. That is,

$$\{u\in P(\gamma,\theta,\alpha,a,b,c):\alpha(u)>a\}=\{u\in P:\frac{k}{k+1}\min_{t\in[\frac{1}{k},\infty)_{\mathbb{T}}}u(t)>a,\ \|u\|\leq c\}\neq\emptyset.$$

For $u\in P(\gamma,\theta,\alpha,a,b,c)$ we have, by condition (D1),

$$\begin{aligned}
\alpha(Tu) \quad &= \quad \frac{k}{k+1}\min_{t\in[\frac{1}{k},\infty)_{\mathbb{T}}}(Tu)(t)=\frac{k}{k+1}(Tu)(\frac{1}{k})\\
&= \quad \frac{k}{k+1}(\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}\int_{\eta_i}^\infty \phi(\tau)f(\tau,u(\tau),u^\triangle(\tau))\nabla\tau)\\
&+ \quad \int_0^{\frac{1}{k}}\varphi^{-1}(\int_s^\infty \phi(\tau)f(\tau,u(\tau),u^\triangle(\tau))\nabla\tau)\triangle s)\\
&\geq \quad \frac{k}{k+1}(\int_0^{\frac{1}{k}}\varphi^{-1}(\int_{\frac{1}{k}}^k \phi(\tau)f(\tau,u(\tau),u^\triangle(\tau))\nabla\tau)\triangle s)\\
&= \quad \frac{1}{k+1}\varphi^{-1}(\int_{\frac{1}{k}}^k \phi(\tau)f(\tau,u(\tau),u^\triangle(\tau))\nabla\tau)\\
&> \quad \frac{1}{k+1}\frac{(k+1)a}{h}\varphi^{-1}(\int_{\frac{1}{k}}^k \phi(\tau)\nabla\tau)=a.
\end{aligned}$$

So,

$$\alpha(Tu)>a. \tag{4.3}$$

Hence, condition (i) of Theorem 4.1 holds.

We take $u(t)=\frac{d}{2}$. It is easy to see that $u(t)\in P,\ 0=\psi(u),\ \|u\|=\frac{d}{2}<d$. That is,

$$\{u\in Q(\gamma,\beta,\psi,l,d,c):\beta(u)<d\}=\{u\in P:\|u\|<d\}\neq\emptyset.$$

By condition (D2) and Lemma 2.4, we get for $u\in Q(\gamma,\beta,\psi,l,d,c)$,

$$\begin{aligned}
\beta(Tu) \quad &= \quad \|Tu\|=\max\{\|Tu\|_1,\|Tu^\triangle\|_\infty\}\leq M\|Tu^\triangle\|_\infty = MTu^\triangle(0)\\
&= \quad M\varphi^{-1}(\int_0^\infty \phi(\tau)f(\tau,u(\tau),u^\triangle(\tau))\nabla\tau) < M\frac{dN}{M}\varphi^{-1}(\int_0^\infty \phi(\tau)\nabla\tau)=d
\end{aligned}$$

Thus, condition (ii) of Theorem 4.1 is satisfied.

Since

$$P(\gamma,\alpha,a,c)=\{u\in P:\frac{k}{k+1}\min_{t\in[\frac{1}{k},\infty)_{\mathbb{T}}}u(t)\geq a,\ \|u\|\leq c\},$$

we get $\alpha(Tu) > a$ for $u \in P(\gamma, \alpha, a, c)$ according to (4.3). Therefore, condition (iii) of Theorem 4.1 is satisfied.

Finally, as far as (iv) is concerned, we omit the condition (iv) since $\psi(Tu) < l = 0$ is impossible. Since all conditions of Theorem 4.1 are verified, the BVP (1.1) has at least three positive solutions such that

$$\|u_1\| < d,\ \frac{k}{k+1} \min_{t\in[\frac{1}{k},\infty)_{\mathbb{T}}} u_2(t) > a,\ d < \|u_3\|,\ \frac{k}{k+1} \min_{t\in[\frac{1}{k},\infty)_{\mathbb{T}}} u_3(t) < a.$$

□

Example 4.3. Let $\mathbb{T} = [0,5] \cup \{6,7,8,9\} \cup [10,+\infty),\ r = 1,\ l = 0,\ \eta_1 = \frac{1}{3},\ \eta_2 = \frac{2}{3},\ \alpha_1 = \alpha_2 = \frac{1}{3},\ M = 1,\ \phi(t) = e^{-t},\ \varphi(x) = x^2$ in the boundary value problem (1.1). Now we consider the following problem

$$\begin{cases} (\varphi(u^{\triangle}))^{\nabla}(t) + e^{-t} f(t, u(t), u^{\triangle}(t)) = 0,\ t \in (0,\infty)_{\mathbb{T}} \\ u(0) = \frac{1}{3}u^{\triangle}(\frac{1}{3}) + \frac{1}{3}u^{\triangle}(\frac{2}{3}),\quad \lim_{t\in\mathbb{T},t\to\infty} u^{\triangle}(t) = 0 \end{cases} \tag{4.4}$$

where

$$f(t,(1+t)u,v) = \begin{cases} \frac{t}{1+t^2}(3\times 10^3 u^6 + \frac{v}{2\times 10^3}),\ \ u \le 1, 0 \le v,\ t \in \mathbb{T} \\ \frac{t}{1+t^2}(3\times 10^3 + \frac{v}{2\times 10^3}),\ \ u > 1, 0 \le v,\ t \in \mathbb{T}. \end{cases} \tag{4.5}$$

Choose $a = 5,\ d = \frac{1}{10},\ k = 3,\ c = 2\times 10^3$. Then by simple calculations, we can obtain that

$$h = 0.816,\ N = 1,$$
$$0 < \frac{1}{10} < 5 < 2\times 10^3.$$

(1) $f(t,(1+t)u,v) \ge 900 > \varphi(\frac{(k+1)a}{h}) = \varphi(\frac{20}{0.816}) = 600$, for $(t,u,v) \in [\frac{1}{3},3]_{\mathbb{T}} \times [\frac{5}{3}, 2\times 10^3] \times [0, 2\times 10^3]$;

(2) $f(t,(1+t)u,v) \le 0.0015 < \varphi(\frac{dN}{M}) = 0.01$, for $(t,u,v) \in [0,\infty)_{\mathbb{T}} \times [0,\frac{1}{10}] \times [0,\frac{1}{10}]$;

(3) $f(t,(1+t)u,v) \le 1500.5 \le \varphi(\frac{cN}{M}) = 4\times 10^6$, for $(t,u,v) \in [0,\infty)_{\mathbb{T}} \times [0, 2\times 10^3] \times [0, 2\times 10^3]$.

Hence, by Theorem 4.2, the BVP (4.3) and (4.4) has at least three positive solutions u_1, u_2 and u_3 such that

$$\|u_1\| < \frac{1}{10},\ \ \frac{3}{4} \min_{t\in[\frac{1}{3},\infty)_{\mathbb{T}}} u_2(t) > 5,\ \ \|u_3\| > \frac{1}{10},\ \ \frac{3}{4} \min_{t\in[\frac{1}{3},\infty)_{\mathbb{T}}} u_3(t) < 5.$$

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