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IMPROVEMENTS OF COMPOSITION RULE FOR THE CANAVATI FRACTIONAL DERIVATIVES AND APPLICATIONS TO OPIAL-TYPE INEQUALITIES

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ABSTRACT. This paper gives improvements of a composition rule for the Canavati fractional derivatives and presents improvements and weighted versions of Opial-type inequalities involving the Canavati fractional derivatives.

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1. INTRODUCTION AND PRELIMINARIES

This work, motivated by the monograph of Anastassiou [2], presents improvements of a composition rule for the Canavati fractional derivatives which relax restrictions on the orders of fractional derivatives in the composition rule. See Section 2. Also, we will give versions of Opial-type inequalities known for Riemann-Liouville fractional derivatives, and we will present improvements of some Opial-type inequalities for the Canavati fractional derivatives which have the general form

$$\int_a^b w_1(t)|(D^{\mu_1} f)(t)||(D^{\mu_2} f)(t)|\,dt \le K \left(\int_a^b w_2(t)|(D^{\nu} f)(t)|^p\,dt\right)^{\frac{2}{p}},$$

where w_1 and w_2 are weight functions, and $D^\gamma f$ denotes the Canavati fractional derivative of f of order γ.

First, following [5], we survey some facts about fractional derivatives. Let $g \in C([0,1])$. Let $\nu > 0$, $n = [\nu]$, $[\cdot]$ the integral part, and $\overline{\nu} = \nu - n$, $0 \le \overline{\nu} < 1$. Define the *Riemann-Liouville fractional integral of g* of order ν by

$$(J_\nu g)(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1}\, g(t)\,dt, \quad x \in [0,1],$$

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where Γ is the gamma function $\Gamma(\nu) = \int_0^\infty e^{-t}\, t^{\nu-1}\, dt$. Define the subspace $C^\nu([0,1])$ of $C^n([0,1])$ as

$$C^\nu([0,1]) = \{g \in C^n([0,1]) : J_{1-\overline{\nu}}\, g^{(n)} \in C^1([0,1])\}\,.$$

For $g \in C^\nu([0,1])$ the *Canavati ν-fractional derivative* of g is defined by

$$D^\nu g = D J_{1-\overline{\nu}} g^{(n)}\,,$$

where $D = d/dx$. Since we will compare our results with ones from [2] we will now give a more general definition of the Riemann-Liouville fractional integral. Let $[a,b] \subseteq \mathbb{R}$ and $x_0, x \in [a,b]$ such that $x \geq x_0$ where x_0 is fixed. For $f \in C([a,b])$ the *generalized Riemann-Liouville fractional integral of f* of order ν is given by

$$(J_\nu^{x_0} f)(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1}\, f(t)\, dt, \quad x \in [x_0, b]\,.$$

Analogously, define the subspace $C_{x_0}^\nu([a,b])$ of $C^n([a,b])$ as

$$C_{x_0}^\nu([a,b]) = \{f \in C^n([a,b]) : J_{1-\overline{\nu}}^{x_0}\, f^{(n)} \in C^1([x_0,b])\}.$$

For $f \in C_{x_0}^\nu([a,b])$ the *generalized Canvati ν-fractional derivative of f* over $[x_0,b]$ is given by

$$D_{x_0}^\nu f = D J_{1-\overline{\nu}}^{x_0} f^{(n)}\,.$$

Notice that

$$(J_{1-\overline{\nu}}^{x_0} f^{(n)})(x) = \frac{1}{\Gamma(1-\overline{\nu})} \int_{x_0}^x (x-t)^{-\overline{\nu}} f^{(n)}(t)\, dt$$

exists for $f \in C_{x_0}^\nu([a,b])$.

Lemma 1.1 ([5], [2])**.** *(i) $D_{x_0}^n f = f^{(n)}$ for $n \in \mathbb{N}$.*
(ii) Let $f \in C_{x_0}^\nu([a,b])$, $\nu > 0$ and $f^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$, $n = [\nu]$. Then

$$f(x) = (J_\nu^{x_0} D_{x_0}^\nu f)(x).$$

That is,

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} (D_{x_0}^\nu f)(t)\, dt, \tag{1.1}$$

for all $x \in [a,b]$ with $x \geq x_0$.

Lemma 1.2 ([5], [9])**.** *Let $f \in C([a,b])$, $\mu, \nu > 0$. Then $J_\mu^{x_0}(J_\nu^{x_0} f) = J_{\mu+\nu}^{x_0}(f)$.*

The proof of the composition rule contained in the following lemma can be found in [2].

Lemma 1.3. *Let $\gamma \geq 0$, $\nu \geq 1$ be such that $\nu - \gamma \geq 1$. Let $f \in C_{x_0}^\nu([a,b])$ be such that $f^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$. Then*

$$(D_{x_0}^\gamma f)(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_{x_0}^x (x-t)^{\nu-\gamma-1} (D_{x_0}^\nu f)(t)\, dt, \tag{1.2}$$

hence $(D_{x_0}^\gamma f)(x) = (J_{\nu-\gamma}^{x_0} D_{x_0}^\nu f)(x)$ and is continuous in x on $[x_0,b]$.

Our first goal is to improve this composition rule for the Canavati fractional derivatives. Using the Laplace transform we prove that one doesn't need all vanishing derivatives of the function f at point x_0 and that condition $\nu - \gamma \geq 1$ can be relaxed. This will be used in all presented Opial-type inequalities involving the Canavati fractional derivatives.

Next, we give two estimations in an Opial-type inequality motivated by Pang and Agarwal's extension [8, Theorem 1.1] of an inequality due to Fink [6] given for classical derivatives (here Theorem 3.1 and Theorem 3.2). Comparison of the obtained constants is also given.

Also, applying differently Hölder's inequality we obtain an improvement of the next weighted Opial-type inequality for the Canavati fractional derivatives (see [2]).

Theorem 1.4 ([2, Theorem 3.5]). *Let $\mu_1, \mu_2 \geq 0$, $\nu \geq 1$ be such that $\nu - \mu_1$, $\nu - \mu_2 \geq 1$ and $f \in C^{\nu}_{x_0}([a,b])$ with $f^{(i)}(x_0) = 0$, $i = 0, 1, \dots, n-1$, $n := [\nu]$. Here $x_0, x \in [a,b]$, with $x \geq x_0$. Let w be a nonnegative continuous function on $[a,b]$. Denote*

$$Q := \left(\int_{x_0}^{x} w(\tau)^2 \, d\tau \right)^{\frac{1}{2}}.$$

Then

$$\int_{x_0}^{x} w(\tau) |(D^{\mu_1}_{x_0} f)(\tau)| |(D^{\mu_2}_{x_0} f)(\tau)| \, d\tau \leq K_1 \left(\int_{x_0}^{x} ((D^{\nu}_{x_0} f)(t))^2 \, dt \right),$$

where K_1 is given by

$$K_1 = \frac{Q \, (x - x_0)^{2\nu - \mu_1 - \mu_2 - \frac{1}{2}}}{\sqrt[3]{6} \, \Gamma(\nu - \mu_1) \, \Gamma(\nu - \mu_2) \, (\nu - \mu_1 - \frac{5}{6})^{\frac{1}{6}} \, (\nu - \mu_2 - \frac{5}{6})^{\frac{1}{6}} \, (4\nu - 2\mu_1 - 2\mu_2 - \frac{7}{3})^{\frac{1}{2}}}.$$

Finally, let us emphasize in the next remark that generalized Riemann-Liouville fractional integral and generalized Canavati fractional derivative can, by simple substitutions, be reduced to the non-generalized case.

Remark 1.5. Note that generalized Riemann-Liouville fractional integral and generalized Canavati fractional derivative depend only on interval $[x_0, b]$. It is easy to see that using transformation $L : [0,1] \to [x_0, b]$, $L(t) = (b - x_0)t + x_0$ (or inverse transformation), the generalized Rieman-Liouville fractional integral and the generalized Canavati fractional derivative can be expressed using non-generalized cases. For example

$$\left(J^{x_0}_{\nu} f \right)(x) = (b - x_0)^{\nu} \left(J_{\nu} \tilde{f} \right) \left(\frac{x - x_0}{b - x_0} \right),$$

where $\tilde{f}(t) = (b - x_0)t + x_0$. Therefore, all our results stated for non-generalized fractional derivative have an interpretation for the generalized fractional derivative and vice versa.

2. MAIN RESULT

We relax some conditions in composition rule for the Canavati fractional derivative given in Lemma 1.3.

Theorem 2.1. *Let $\nu > \gamma > 0$, $n = [\nu]$, $m = [\gamma]$. Let $f \in C^{\nu}([0,1])$ be such that $f^{(i)}(0) = 0$ for $i = m, m+1, \ldots, n-1$. Then*

(*i*)

$$f \in C^{\gamma}([0,1]) \tag{2.1}$$

(*ii*)

$$(D^{\gamma} f)(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_0^x (x-t)^{\nu-\gamma-1} (D^{\nu} f)(t)\, dt \tag{2.2}$$

for every $x \in [0,1]$.

Proof. Set $\overline{\nu} = \nu - n$, $\overline{\gamma} = \gamma - m$. To prove (*i*), suppose first $m = n$ (which gives $\overline{\nu} > \overline{\gamma}$). We have

$$J_{1-\overline{\gamma}} f^{(n)} = J_{\overline{\nu}-\overline{\gamma}} J_{1-\overline{\nu}} f^{(n)} = J_{\overline{\nu}-\overline{\gamma}+1} \left(D J_{1-\overline{\nu}} f^{(n)} \right), \tag{2.3}$$

where the last equality in (2.3) follows using integration by parts and $J_{1-\overline{\nu}} f^{(n)}(0) = 0$. Since $D J_{1-\overline{\nu}} f^{(n)} \in C^0([0,1])$ and $\overline{\nu} - \overline{\gamma} + 1 > 1$ it follows $J_{1-\overline{\gamma}} f^{(n)} \in C^1([0,1])$ by [5, Proposition 1]. If $m < n$, then using $f^{(m)}(0) = \cdots = f^{(n-1)}(0) = 0$ and integration by parts it easily follows that $f^{(m)} = J_{n-m} f^{(n)}$. We have

$$J_{1-\overline{\gamma}} f^{(m)} = J_{1-\overline{\gamma}} J_{n-m} f^{(n)} = J_{1-\overline{\gamma}+n-m} f^{(n)}.$$

The result again follows from [5, Proposition 1] since $1 - \overline{\gamma} + n - m > 1$ and $f^{(n)} \in C^0([0,1])$.

We will prove (2.2) using the Laplace transform. Set $g = f^{(m)}$. Now (2.2) can be written as

$$\left(D J_{1-\overline{\gamma}} g \right)(x) = \left(J_{\nu-\gamma} D J_{1-\overline{\nu}} g^{(n-m)} \right)(x) = g^{(n-m-1)}(0) = 0. \tag{2.4}$$

where $x \in [0,1]$ and $g(0) = g'(0) = \cdots = g^{(n-m-1)}(0) = 0$. Define auxiliary function $h : [0,\infty) \to \mathbb{R}$ with

$$h(x) = \begin{cases} g(x), & x \in [0,1] \\ \sum_{k=0}^{n-m} \frac{g^{(k)}(1)}{k!} (x-1)^k, & x \geq 1 \end{cases}. \tag{2.5}$$

Obviously $h \in C^{n-m}([0,\infty))$, $h(0) = h'(0) = \cdots = h^{(n-m-1)}(0) = 0$. Also h has polynomial growth at ∞, so the Laplace transform of h exists. The identity (2.4) will follow if we prove that

$$\frac{1}{\Gamma(1-\overline{\gamma})} \frac{d}{dx} \int_0^x (x-t)^{-\overline{\gamma}} h(t)\, dt$$

$$= \frac{1}{\Gamma(\nu-\gamma)\Gamma(1-\overline{\nu})}\int_0^x (x-t)^{\nu-\gamma-1}\frac{d}{dt}\int_0^t (t-y)^{-\overline{\nu}} h^{(n-m)}(y)\,dy\,dt \tag{2.6}$$

holds for every $x \geq 0$. Using standard properties of the Laplace transform we have

$$\begin{aligned}
&\mathcal{L}\left(\frac{1}{\Gamma(1-\overline{\gamma})}\frac{d}{dx}\int_0^x (x-t)^{-\overline{\gamma}} h(t)\,dt\right)(s) \\
&= \frac{1}{\Gamma(1-\overline{\gamma})}\, s\,\mathcal{L}\left(\int_0^x (x-t)^{-\overline{\gamma}} h(t)\,dt\right)(s) \\
&= \frac{s}{\Gamma(1-\overline{\gamma})}\mathcal{L}\left(x^{-\overline{\gamma}}\right)(s)\mathcal{L}(h)(s) \;=\; s^{\overline{\gamma}}\,\mathcal{L}(h)(s).
\end{aligned} \tag{2.7}$$

On the other hand we have

$$\begin{aligned}
&\mathcal{L}\left(\frac{1}{\Gamma(1-\overline{\nu})\Gamma(\nu-\gamma)}\int_0^x (x-t)^{\nu-\gamma-1}\frac{d}{dt}\int_0^t (t-y)^{-\overline{\nu}} h^{(n-m)}(y)\,dy\,dt\right)(s) \\
&= \frac{1}{\Gamma(1-\overline{\nu})\Gamma(\nu-\gamma)}\,\mathcal{L}\left(x^{\nu-\gamma-1}\right)(s)\mathcal{L}\left(\frac{d}{dt}\int_0^t (t-y)^{-\overline{\nu}} h^{(n-m)}(y)\,dy\,dt\right)(s) \\
&= \frac{s^{\gamma-\nu}}{\Gamma(1-\overline{\nu})}\, s\,\mathcal{L}\left(x^{-\overline{\nu}}\right)(s)\mathcal{L}\left(h^{(n-m)}\right)(s) \\
&= s^{\gamma-\nu}\frac{s}{s^{1-\overline{\nu}}}\, s^{n-m}\,\mathcal{L}(h)(s) \;=\; s^{\overline{\gamma}}\mathcal{L}(h)(s).
\end{aligned} \tag{2.8}$$

Using (2.7) and (2.8) it follows that both sides of (2.6) have the same Laplace transform and since both sides are continuous function, we conclude that equality holds in (2.6) for every $x \geq 0$. This completes the proof of the theorem. □

3. OPIAL-TYPE INEQUALITIES

The first theorem is an Opial-type inequality due to Fink who proved it for ordinary derivatives [6]. Although our proof is similar to the one given in [8, Theorem 1.1], we sketch a proof for the reader's convenience. In [2, Section 5] the result is given for Riemann–Liouville's fractional derivatives without a discussion of the best possible cases. Using different technique we prove another estimation of the same type of inequality and compare obtained estimations.

Theorem 3.1. *Let $1/p + 1/q = 1$ with p, $q > 1$. Let $\nu > \mu_2 \geq \mu_1 + 1 \geq 1$, $n = [\nu]$ and $m = [\mu_1]$. Let $f \in C^\nu([0,1])$ be such that $f^{(i)}(0) = 0$ for $i = m, m+1, \ldots, n-1$ and let $x \in [0,1]$. Then*

$$\int_0^x |(D^{\mu_1}f)(\tau)||(D^{\mu_2}f)(\tau)|\,d\tau \leq A_1\, x^{2\nu-\mu_1-\mu_2-1+\frac{2}{q}} \left(\int_0^x |(D^\nu f)(\tau)|^p\,d\tau\right)^{\frac{2}{p}}, \tag{3.1}$$

where A_1 is given by

$$A_1 = \frac{1}{2^{\frac{1}{p}}\,\Gamma(\nu-\mu_1)\,\Gamma(\nu-\mu_2+1)\,[q(\nu-\mu_2)+1]^{\frac{1}{q}}\,[q(2\nu-\mu_1-\mu_2-1)+2]^{\frac{1}{q}}}.$$

Inequality (3.1) *is sharp in the case* $\mu_2 = \mu_1 + 1$ *and the equality is attained for*

$$f(s) = \frac{1}{\Gamma(\nu)} \int_0^s (s-t)^{\nu-1} (x-t)^{\frac{q}{p}(\nu-\mu_2)} dt. \tag{3.2}$$

Proof. Set $\alpha_j = \nu - \mu_j - 1$, $j = 1, 2$. Notice that $\alpha_1 - \alpha_2 - 1 \geq 0$. Let $0 \leq t \leq s \leq x$. Then (see [6])

$$\int_0^x \left[(\tau - t)_+^{\alpha_1} (\tau - s)_+^{\alpha_2} + (\tau - s)_+^{\alpha_1} (\tau - t)_+^{\alpha_2} \right] d\tau \leq \frac{(x-t)^{\alpha_1} (x-s)^{\alpha_2+1}}{\alpha_2 + 1} . \tag{3.3}$$

In the following calculation we abbreviate

$$c_1 := \left[\Gamma(\nu - \mu_2) \Gamma(\nu - \mu_1) \right]^{-1} , \quad c_2 := \left[\Gamma(\nu - \mu_2 + 1) \Gamma(\nu - \mu_1) \right]^{-1} ,$$
$$c_3 := q(\nu - \mu_2) + 1 , \quad \epsilon := 2\nu - \mu_1 - \mu_2 - 1 + \frac{1}{q} .$$

For $\tau \in [0, x]$, by (2.2), we have

$$(D^{\mu_j} f)(\tau) = \frac{1}{\Gamma(\nu - \mu_j)} \int_0^x (\tau - t)_+^{\alpha_i} (D^\nu f)(t) \, dt .$$

Using this representation, the auxiliary inequality (3.3), and Hölder's inequality, we obtain

(3.4)

$$\begin{aligned}
&\int_0^x |(D^{\mu_1} f)(\tau)| |(D^{\mu_2} f)(\tau)| \, d\tau \\
&\leq c_1 \int_0^x \left(\int_0^x |(D^\nu f)(t)| (\tau - t)_+^{\alpha_1} \, dt \right) \left(\int_0^x |(D^\nu f)(s)| (\tau - s)_+^{\alpha_2} \, ds \right) d\tau \\
&= c_1 \int_0^x |(D^\nu f)(t)| \Bigg\{ \int_t^x |(D^\nu f)(s)| \\
&\cdot \left(\int_0^x [(\tau - t)_+^{\alpha_1} (\tau - s)_+^{\alpha_2} + (\tau - s)_+^{\alpha_1} (\tau - t)_+^{\alpha_2}] \, d\tau \right) ds \Bigg\} \, dt
\end{aligned}$$

(3.5)

$$\begin{aligned}
&\leq c_2 \int_0^x |(D^\nu f)(t)| \left(\int_t^x |(D^\nu f)(s)| (x-t)^{\alpha_1} (x-s)^{\alpha_2+1} \, ds \right) dt \\
&\leq c_2 \int_0^x |(D^\nu f)(t)| (x-t)^{\alpha_1} \left(\int_t^x |(D^\nu f)(s)|^p \, ds \right)^{\frac{1}{p}} \left(\int_t^x (x-s)^{q(\alpha_2+1)} \, ds \right)^{\frac{1}{q}} dt
\end{aligned}$$

(3.6)

$$\begin{aligned}
&= c_2 \, c_3^{-1/q} \int_0^x |(D^\nu f)(t)| (x-t)^\epsilon \left(\int_t^x |(D^\nu f)(s)|^p \, ds \right)^{\frac{1}{p}} dt \\
&\leq c_2 \, c_3^{-1/q} \left(\int_0^x |(D^\nu f)(t)|^p \left(\int_t^x |(D^\nu f)(s)|^p \, ds \right) dt \right)^{\frac{1}{p}} \left(\int_0^x (x-t)^{\epsilon q} \, dt \right)^{\frac{1}{q}} \\
&= c_2 \, c_3^{-1/q} (\epsilon q + 1)^{-1/q} \, x^{(\epsilon q+1)/q} \left\{ \frac{1}{2} \left(\int_0^x |(D^\nu f)(t)|^p \, dt \right)^2 \right\}^{\frac{1}{p}} .
\end{aligned}$$

It is obvious that in the case $\alpha_2 = \alpha_1 + 1$ we have equality in (3.3). Using equality condition for Hölder's inequality we have also equality in (3.5) for the function f given in (3.2) since obviously $(D^\nu f(s))^p = (x-s)^{q(\alpha_2+1)}$. Straightforward calculation shows that for this function equality holds also in (3.6). Equality in (3.4) in this case is obvious. □

Using a different technique we obtain yet another constant for the previous inequality:

Theorem 3.2. *Let $1/p + 1/q = 1$ with $p,\ q > 1$. Let $\mu_j > 0$, $\nu > \mu_j + 1 - \frac{1}{q}$, $n = [\nu]$ and $m = \min\{[\mu_1], [\mu_2]\}$. Let $f \in C^\nu([0,1])$ be such that $f^{(i)}(0) = 0$ for $i = m, m+1, \dots, n-1$ and let $x \in [0,1]$. Then*

$$\int_0^x |(D^{\mu_1} f)(\tau)||(D^{\mu_2} f)(\tau)|\, d\tau \le A_2\, x^{2\nu-\mu_1-\mu_2-1+\frac{2}{q}} \left(\int_0^x |(D^\nu f)(\tau)|^p\, d\tau \right)^{\frac{2}{p}},$$

where A_2 is given by

$$A_2 = \frac{q}{[q(2\nu-\mu_1-\mu_2-1)+2] \prod_{j=1}^2 \Gamma(\nu-\mu_j)\, [q(\nu-\mu_j-1)+1]^{\frac{1}{q}}}.$$

Proof. Set $\alpha_j = \nu - \mu_j - 1$, $j = 1,2$. For $\tau \in [0,x]$, by (2.2), we have

$$(D^{\mu_j} f)(\tau) = \frac{1}{\Gamma(\alpha_j+1)} \int_0^\tau (\tau-t)^{\alpha_j} (D^\nu f)(t)\, dt.$$

By Hölder's inequality we have

$$\begin{aligned}
&\int_0^x \prod_{j=1}^2 |(D^{\mu_j} f)(\tau)|\, d\tau \\
&\le \frac{1}{\prod_{j=1}^2 \Gamma(\alpha_j+1)} \int_0^x \prod_{j=1}^2 \left(\int_0^\tau (\tau-t)^{q\alpha_j}\, dt \right)^{\frac{1}{q}} \left(\int_0^\tau |(D^\nu f)(t)|^p\, dt \right)^{\frac{1}{p}} d\tau \\
&\le \frac{1}{\prod_{j=1}^2 \Gamma(\alpha_j+1)\,(q\alpha_j+1)^{\frac{1}{q}}} \left(\int_0^x |(D^\nu f)(t)|^p\, dt \right)^{\frac{2}{p}} \int_0^x \tau^{\alpha_1+\alpha_2+\frac{2}{q}}\, d\tau \\
&= A_2\, x^{2\nu-\mu_1-\mu_2-1+\frac{2}{q}} \left(\int_0^x |(D^\nu f)(t)|^p\, dt \right)^{\frac{2}{p}}.
\end{aligned}$$

□

Remark 3.3. The constants A_1 and A_2 from the two previous theorems are in general not comparable, but there are cases when we can do that. Notice

$$\frac{A_1}{A_2} = \frac{(\nu-\mu_1-1+\frac{1}{q})^{\frac{1}{q}} (\nu-\mu_2-1+\frac{1}{q})^{\frac{1}{q}} (2\nu-\mu_1-\mu_2-1+\frac{2}{q})^{\frac{1}{p}}}{2^{\frac{1}{p}} (\nu-\mu_2)(\nu-\mu_2+\frac{1}{q})^{\frac{1}{q}}}.$$

We want to find cases when $A_2 < A_1$. Set $\nu - \mu_2 = d_2$, $\mu_2 - \mu_1 = d_1 \geq 1$. Then $A_2 < A_1$ is equivalent to

$$\frac{1}{(d_1 + d_2 - 1 + \frac{1}{q})^{\frac{1}{q}} \, (d_1 + 2d_2 - 1 + \frac{2}{q})^{1-\frac{1}{q}}} < \frac{(d_2 - 1 + \frac{1}{q})^{\frac{1}{q}}}{2^{1-\frac{1}{q}} \, d_2 \, (d_2 + \frac{1}{q})^{\frac{1}{q}}}. \tag{3.7}$$

If d_1 is big enough, then the left side of (3.7) tends to zero, while the right side is independent of d_1. Therefore, in this case $A_2 < A_1$.

Let $d_1 = 1$, that is $\mu_2 = \mu_1 + 1$ (see the discussion of sharpness in Theorem 3.1). Then the reverse inequality in (3.7) is equivalent to

$$\frac{qd_2 + 1}{qd_2 - q + 1} > \left(1 + \frac{1}{qd_2}\right)^q,$$

which is equivalent to inequality

$$\left(\frac{qd_2 + 1 - q}{qd_2 + 1}\right)^{1/q} < \frac{qd_2}{1 + qd_2},$$

and this is a simple consequence of the Bernoulli inequality. This is in accordance with Theorem 3.1.

An illustrative case is $p = q = 2$. In this case (3.7) is equivalent to

$$12(d_1 - 1)\, d_2^2 + 2(2\, d_1^2 - 4\, d_1 + 1)\, d_2 - 2\, d_1^2 + d_1 > 0.$$

That is, $A_2 < A_1$ is equivalent to

$$d_2 > \tilde{d}_2 = \frac{-2d_1^2 + 4d_1 - 1 + \sqrt{4d_1^4 + 8d_1^3 - 16d_1^2 + 4d_1 + 1}}{12(d_1 - 1)}.$$

Notice that $\lim_{d_1 \to 1} \tilde{d}_2 = \infty$ and $\lim_{d_1 \to \infty} \tilde{d}_2 = \frac{1}{2}$. Roughly speaking, for $d_1 \approx 1$ or $d_2 \approx 0.5$ estimation in Theorem 3.1 is better than estimation in Theorem 3.2, otherwise the opposite conclusion holds. For example, for $d_1 = 2$ and $d_2 > \frac{-1+\sqrt{73}}{12} \approx 0.62867$ or for $d_2 = 1$ and $d_1 > \frac{-5+\sqrt{105}}{4} \approx 1.31174$ estimation in Theorem 3.2 is better than estimation in Theorem 3.1.

We also consider a weighted Opial-type inequality for the generalized fractional derivative given in [2] (here Theorem 1.4). Set

$$C = \frac{Q\,(x - x_0)^{2\nu - \mu_1 - \mu_2 - \frac{1}{2}}}{\Gamma(\nu - \mu_1)\,\Gamma(\nu - \mu_2)}$$

and recall that the constant in inequality from Theorem 1.4 is

$$K_1 = \frac{C}{\sqrt[3]{6}\,(\nu - \mu_1 - \frac{5}{6})^{\frac{1}{6}}\,(\nu - \mu_2 - \frac{5}{6})^{\frac{1}{6}}\,(4\nu - 2\mu_1 - 2\mu_2 - \frac{7}{3})^{\frac{1}{2}}}. \tag{3.8}$$

Several applications of Hölder's inequality on different factors with different indices will improve this constant.

Theorem 3.4. *Let $\mu_j > 0$, $\nu > \mu_j + \frac{5}{6}$, $n = [\nu]$ and $m = \min\{[\mu_1], [\mu_2]\}$. Let $f \in C^{\nu}_{x_0}([a,b])$ be such that $f^{(i)}(x_0) = 0$ for $i = m, m+1, \ldots, n-1$. Here $x_0, x \in [a,b] \subseteq \mathbb{R}$, $x \geq x_0$ where x_0 is fixed. If $w \in C([a,b])$ is a nonnegative function on $[a,b]$, then*

$$\int_{x_0}^{x} w(\tau)|(D^{\mu_1}_{x_0} f)(\tau)||(D^{\mu_2}_{x_0} f)(\tau)|\, d\tau \leq K_2 \int_{x_0}^{x} |(D^{\nu}_{x_0} f)(t)|^2\, dt,$$

where K_2 is given by

$$K_2 = \frac{C}{\sqrt[3]{6}\,(\nu - \mu_1 - \frac{5}{6})^{\frac{1}{6}}\,(\nu - \mu_2 - \frac{5}{6})^{\frac{1}{6}}\,(4\nu - 2\mu_1 - 2\mu_2 - 1)^{\frac{1}{2}}}. \tag{3.9}$$

Proof. Set $\alpha_j = \nu - \mu_j - 1$, $j = 1, 2$. For $\tau \in [x_0, x]$, by Remark 1.5 and (2.2), we have

$$(D^{\mu_j}_{x_0} f)(\tau) = \frac{1}{\Gamma(\alpha_j + 1)} \int_{x_0}^{\tau} (\tau - t)^{\alpha_j} (D^{\nu}_{x_0} f)(t)\, dt.$$

By Hölder's inequality we have

$$\begin{aligned}
&\int_{x_0}^{x} w(\tau) \prod_{j=1}^{2} |(D^{\mu_j}_{x_0} f)(\tau)|\, d\tau \\
&\leq \left(\int_{x_0}^{x} w(\tau)^2\, d\tau\right)^{\frac{1}{2}} \left(\int_{x_0}^{x} \prod_{j=1}^{2} |(D^{\mu_j}_{x_0} f)(\tau)|^2\, d\tau\right)^{\frac{1}{2}} \\
&\leq \frac{Q}{\prod_{j=1}^{2} \Gamma(\alpha_j + 1)} \left\{ \int_{x_0}^{x} \prod_{j=1}^{2} \left(\int_{x_0}^{\tau} (\tau - t)^{\alpha_j} |(D^{\nu}_{x_0} f)(t)|\, dt\right)^2 d\tau \right\}^{\frac{1}{2}}.
\end{aligned}$$

Again, for $p = 3$ and $q = \frac{3}{2}$

$$\int_{x_0}^{\tau} (\tau - t)^{\alpha_j} |(D^{\nu}_{x_0} f)(t)|\, dt \leq \left(\int_{x_0}^{\tau} dt\right)^{\frac{1}{3}} \left(\int_{x_0}^{\tau} (\tau - t)^{\frac{3}{2}\alpha_j} |(D^{\nu}_{x_0} f)(t)|^{\frac{3}{2}}\, dt\right)^{\frac{2}{3}},$$

and for $p = 4$ and $q = \frac{4}{3}$ we have

$$\int_{x_0}^{\tau} (\tau - t)^{\frac{3}{2}\alpha_j} |(D^{\nu}_{x_0} f)(t)|^{\frac{3}{2}}\, dt \leq \left(\int_{x_0}^{\tau} (\tau - t)^{6\alpha_j}\, dt\right)^{\frac{1}{4}} \left(\int_{x_0}^{\tau} |(D^{\nu}_{x_0} f)(t)|^2\, dt\right)^{\frac{3}{4}}.$$

Therefore

$$\begin{aligned}
&\int_{x_0}^{x} w(\tau) \prod_{j=1}^{2} |(D^{\mu_j}_{x_0} f)(\tau)|\, d\tau \\
&\leq \frac{Q}{\prod_{j=1}^{2} \Gamma(\alpha_j + 1)} \Bigg\{ \int_{x_0}^{x} (\tau - x_0)^{\frac{4}{3}} \left(\int_{x_0}^{\tau} |(D^{\nu}_{x_0} f)(t)|^2\, dt\right)^2 \\
&\qquad \cdot \prod_{j=1}^{2} \left(\frac{(\tau - x_0)^{6\alpha_j + 1}}{6\alpha_j + 1}\right)^{\frac{1}{3}} d\tau \Bigg\}^{\frac{1}{2}} \\
&\leq \frac{Q}{\prod_{j=1}^{2} \Gamma(\alpha_j + 1)\, (6\alpha_j + 1)^{\frac{1}{6}}} \int_{x_0}^{x} |(D^{\nu}_{x_0} f)(t)|^2\, dt \left(\int_{x_0}^{x} (\tau - x_0)^{2\alpha_1 + 2\alpha_2 + 2}\, d\tau\right)^{\frac{1}{2}}
\end{aligned}$$

$$= \quad K_2 \int_{x_0}^{x} |(D_{x_0}^{\nu} f)(t)|^2 \, dt.$$

□

Theorem 3.5. *Let $\mu_j > 0$, $\nu > \mu_j + \frac{1}{2}$, $n = [\nu]$ and $m = \min\{[\mu_1], [\mu_2]\}$. Let $f \in C_{x_0}^{\nu}([a,b])$ be such that $f^{(i)}(x_0) = 0$ for $i = m, m+1, \ldots, n-1$. Here $x_0, x \in [a,b] \subseteq \mathbb{R}$, $x \geq x_0$ where x_0 is fixed. If $w \in C([a,b])$ is a nonnegative function on $[a,b]$, then*

$$\int_{x_0}^{x} w(\tau) |(D_{x_0}^{\mu_1} f)(\tau)| |(D_{x_0}^{\mu_2} f)(\tau)| \, d\tau \leq K_3 \int_{x_0}^{x} |(D_{x_0}^{\nu} f)(t)|^2 \, dt,$$

where K_3 is given by

$$K_3 = \frac{C}{2\,(\nu - \mu_1 - \frac{1}{2})^{\frac{1}{2}} (\nu - \mu_2 - \frac{1}{2})^{\frac{1}{2}} (4\nu - 2\mu_1 - 2\mu_2 - 1)^{\frac{1}{2}}}. \tag{3.10}$$

Proof. Write $\alpha_j = \nu - \mu_j - 1$, $j = 1, 2$. For $\tau \in [x_0, x]$, by Remark 1.5 and (2.2), we have

$$(D_{x_0}^{\mu_j} f)(\tau) = \frac{1}{\Gamma(\alpha_j + 1)} \int_{x_0}^{\tau} (\tau - t)^{\alpha_j} (D_{x_0}^{\nu} f)(t) \, dt.$$

By Hölder's inequalities we have

$$\begin{aligned}
&\int_{x_0}^{x} w(\tau) \prod_{j=1}^{2} |(D_{x_0}^{\mu_j} f)(\tau)| \, d\tau \\
\leq \quad & \int_{x_0}^{x} w(\tau) \prod_{j=1}^{2} \left(\frac{1}{\Gamma(\alpha_j + 1)} \int_{x_0}^{\tau} (\tau - t)^{\alpha_j} |(D_{x_0}^{\nu} f)(t)| \, dt \right) d\tau \\
\leq \quad & \frac{1}{\prod_{j=1}^{2} \Gamma(\alpha_j + 1)} \int_{x_0}^{x} w(\tau) \left(\int_{x_0}^{\tau} |(D_{x_0}^{\nu} f)(t)|^2 \, dt \right) \prod_{j=1}^{2} \left(\int_{x_0}^{\tau} (\tau - t)^{2\alpha_j} \, dt \right)^{\frac{1}{2}} d\tau \\
\leq \quad & \frac{1}{\prod_{j=1}^{2} \Gamma(\alpha_j + 1)\,(2\alpha_j + 1)^{\frac{1}{2}}} \left(\int_{x_0}^{x} |(D_{x_0}^{\nu} f)(t)|^2 \, dt \right) \int_{x_0}^{x} w(\tau)(\tau - x_0)^{\alpha_1 + \alpha_2 + 1} \, d\tau \\
\leq \quad & \frac{1}{\prod_{j=1}^{2} \Gamma(\alpha_j + 1)\,(2\alpha_j + 1)^{\frac{1}{2}}} \left(\int_{x_0}^{x} |(D_{x_0}^{\nu} f)(t)|^2 \, dt \right) \\
& \qquad \cdot \left(\int_{x_0}^{x} w(\tau)^2 \, d\tau \right)^{\frac{1}{2}} \left(\int_{x_0}^{x} (\tau - x_0)^{2\alpha_1 + 2\alpha_2 + 2} \, d\tau \right)^{\frac{1}{2}} \\
= \quad & K_3 \int_{x_0}^{x} |(D_{x_0}^{\nu} f)(t)|^2 \, dt.
\end{aligned}$$

□

Remark 3.6. Comparing three constants K_i, that is (3.8), (3.9) and (3.10), we conclude

$$K_3 < K_2 < K_1. \tag{3.11}$$

The second inequality is obvious. The first inequality is equivalent to

$$\frac{\sqrt[3]{6\nu-6\mu_1-5}}{2\nu-2\mu_1-1}\frac{\sqrt[3]{6\nu-6\mu_2-5}}{2\nu-2\mu_2-1}<1,$$

for $\nu-\mu_j>5/6$, $j=1,2$, which holds since $\frac{\sqrt[3]{3x-5}}{x-1}<1/2$ for $x>5/3$.

Finally, we give a very general Opial-type inequality involving the Canavati fractional derivatives which is analogous to [8, Theorem 1.3] for ordinary derivatives. The proof is omitted (see [4]).

Theorem 3.7. *Let $l\in\mathbb{N}$, $\mu_j>0$, $\nu>\mu_j$, $n=[\nu]$ and $m=\min\{[\mu_j]:j=1,\dots,l\}$. Let $f\in C^\nu([0,1])$ be such that $f^{(i)}(0)=0$ for $i=m,m+1,\dots,n-1$. Let w_1, w_2 be continuous positive weight functions on $[0,x]$ where $x\in[0,1]$. Let $r_j>0$, $r=\sum_{j=1}^l r_j$, $s_k>1$ and $\frac{1}{s_k}+\frac{1}{s_k'}=1$ for $k=1,2$. Let $p\in\mathbb{R}$ be such that $p>s_2$ and let $\sigma=1/s_2-1/p$. Suppose also $\nu>\mu_j+1-\sigma$, $j=1,\dots,l$. Denote*

$$Q=\left(\int_0^x w_1(\tau)^{s_1'}\,d\tau\right)^{\frac{1}{s_1'}},\quad P=\left(\int_0^x w_2(\tau)^{-\frac{s_2'}{p}}\,d\tau\right)^{\frac{r}{s_2'}}.$$

Then

$$\begin{aligned}&\int_0^x w_1(\tau)\prod_{j=1}^l|(D^{\mu_j}f)(\tau)|^{r_j}\,d\tau\\ &\le\ P\,A\left(\int_0^x w_1(\tau)\tau^\rho\,d\tau\right)\left(\int_0^x w_2(t)\,|(D^\nu f)(t)|^p\,dt\right)^{\frac{r}{p}}\\ &\le\ P\,Q\,A\,\frac{x^{\rho+\frac{1}{s_1}}}{(\rho s_1+1)^{\frac{1}{s_1}}}\left(\int_0^x w_2(t)\,|(D^\nu f)(t)|^p\,dt\right)^{\frac{r}{p}},\end{aligned}$$

where $\alpha_j=\nu-\mu_j-1$, $\rho=\Sigma_{j=1}^l\alpha_j r_j+\sigma r$ and A is given by

$$A=\frac{\sigma^{r\sigma}}{\prod_{j=1}^l\Gamma(\alpha_j+1)^{r_j}\,(\alpha_j+\sigma)^{r_j\sigma}}.$$

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