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A TURNPIKE PROPERTY OF APPROXIMATE SOLUTIONS OF AN OPTIMAL CONTROL PROBLEM ARISING IN ECONOMIC DYNAMICS

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ABSTRACT. We study the structure of approximate solutions for a class of continuous-time optimal control problems. These optimal control problems arise in economic dynamics and describe a model proposed by Robinson, Solow and Srinivasan. We are interested in turnpike properties of the approximate solutions which are independent of the length of the interval, for all sufficiently large intervals.

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1. INTRODUCTION

The study of variational and optimal control problems defined on infinite (large) intervals has recently been a rapidly growing area of research. See, for example, [4–10, 13, 14, 18, 21, 26–30, 32, 39] and the references mentioned therein. These problems arise in engineering [1, 19, 41], in models of economic growth [2, 12, 15–17, 22, 33, 36, 38–40], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [3, 37] and in the theory of thermodynamical equilibrium for materials [11, 20, 23–25]. In this paper we study an optimal control problem arising in economic dynamics. This problem corresponds to the model of economic growth introduced in [31, 34, 35]. Discrete-time versions of this model were recently studied in [15–17, 38].

We are interested in a turnpike property of the approximate solutions of these problems which is independent of the length of the interval $[T_1, T_2]$ for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the optimal control problems are determined mainly by the cost function, and are essentially independent of T_2, T_1 and endpoint values. Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 [33] where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of

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economic dynamics [22, 39]. It has recently been shown that the turnpike property is a general phenomenon which holds for large classes of variational and optimal control problems [39].

We begin with some preliminary notation. Let R (R_+) be the set of real (non-negative) numbers and let R^n be a finite-dimensional Euclidean space with non-negative orthant $R^n_+ = \{x \in R^n : \ x_i \geq 0, \ i = 1, \dots, n\}$. For any $x, y \in R^n$, let the inner product $xy = \sum_{i=1}^n x_i y_i$.

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in R^n$. We say that $x \geq y$ if $x_i \geq y_i$ for all $i = 1, \dots, n$. We say that $x > y$ if $x \geq y$ and $x \neq y$, and say that $x \gg y$ if $x_i > y_i$ for all $i = 1, \dots, n$.

Let $e(i)$, $i = 1, \dots, n$, be the ith unit vector in R^n, and $\mathbf{e}$ be an element of R^n_+ all of whose coordinates are unity. For any $x \in R^n$, let $\|x\|$ denote the Euclidean norm of x.

Denote by $\mathrm{mes}(E)$ the Lebesgue measure of a Lebesgue measurable set $E \subset R$.

Let $a = (a_1, \dots, a_n) \gg 0$, $b = (b_1, \dots, b_n) \gg 0$, $b_1 \geq b_2 \cdots \geq b_n$, $d \in (0, 1)$, $c_i = b_i/(1 + da_i)$, $i = 1, \dots, n$.

The optimal control problem studied in the paper corresponds to the model of an economy capable of producing a finite number n of alternative types of machines. For every $i = 1, \dots, n$, one unit of machine of type i requires $a_i > 0$ units of labor to construct it, and together with one unit of labor, each unit of it can produce $b_i > 0$ units of a single consumption good. Thus, the production possibilities of the economy are represented by an (labor) input-coefficients vector, $a = (a_1, \dots, a_n) \gg 0$ and an output-coefficients vector, $b = (b_1, \dots, b_n) \gg 0$. For each nonnegative number t let $x(t) = (x_1(t), \dots, x_n(t)) \geq 0$ denote the amounts of the n types of machines that are available in time t, and let $y(t) = (y_1(t), \dots, y_n(t))$ be the amounts of the n types of machines used for production of the consumption good, $by(t)$, at time t. We also assume that the total labor force of the economy is unity. We assume that all machines depreciate at a rate $d \in (0, 1)$. Thus the effective labor cost of producing a unit of output on a machine of type i is given by $(1 + da_i)/b_i$: the direct labor cost of producing unit output, and the indirect cost of replacing the depreciation of the machine in this production. We work with the reciprocal of the effective labor cost, the effective output that takes the depreciation into account, and denote it by c_i for the machine of type i. Following [15–17, 38] throughout this paper, we assume that there is a unique machine type σ at which effective labor cost $(1 + da_i)/b_i$ is minimal, or at which the effective output per man $b_i/(1 + da_i)$ is maximal. Thus we assume:

There exists $\sigma \in \{1, \dots, n\}$ such that for all

$$i \in \{1, \dots, n\} \setminus \{\sigma\}, \ c_\sigma > c_i. \tag{1.1}$$

(For more details and explanations see [15–17, 38].)

We now give a formal description of our technological structure.

Define

$$\Omega = \{(x,z) \in R^n_+ \times R^n : \ z + dx \geq 0 \text{ and } a(z+dx) \leq 1\}. \tag{1.2}$$

For each $(x,z) \in \Omega$ define

$$\Lambda(x,z) = \{y \in R^n_+ : \ y \leq x \text{ and } \mathbf{e}y \leq 1 - a(z+dx)\}. \tag{1.3}$$

Let I be either $[0,\infty)$ or $[T_1,T_2]$ with $T_2 > T_1 \geq 0$. A pair of functions $(x(\cdot), y(\cdot))$ is called a program if $x : I \to R^n$ is an absolutely continuous (a.c.) function on any bounded subinterval of I, $y : I \to R^n$ is a Lebesgue measurable function and if

$$(x(t), x'(t)) \in \Omega \text{ for almost every } t \in I, \tag{1.4}$$

$$y(t) \in \Lambda(x(t), x'(t)) \text{ for almost every } t \in I. \tag{1.5}$$

In the sequel if $I = [T_1,T_2]$, then the program $(x(\cdot), y(\cdot))$ is denoted by

$$(x(t), y(t))_{t=T_1}^{T_2}$$

and if $I = [0,\infty)$, then the program $(x(\cdot), y(\cdot))$ is denoted by $(x(t), y(t))_{t=0}^{\infty}$.

Let $w : [0,\infty) \to R$ be a continuous strictly increasing concave and differentiable function which represents the preferences of the planner.

For any $(x,z) \in \Omega$ define

$$u(x,z) = \max\{w(by) : \ y \in \Lambda(x,z)\}.$$

A golden-rule stock is $\widehat{x} \in R^n_+$ such that $(\widehat{x}, 0)$ is a solution to the problem:

maximize $u(x,z)$ subject to

(i) $z \geq 0$; (ii) $(x,z) \in \Omega$.

In [15] it was established the following result.

Theorem 1.1. *There exists a unique golden-rule stock* $\widehat{x} = (1/(1+da_\sigma))e(\sigma)$.

It is not difficult to see that $\widehat{x}$ is a solution to the problem

$$w(by) \to \max, \ y \in \Lambda(\widehat{x}, 0).$$

Set

$$\widehat{y} = \widehat{x}.$$

For $i = 1, \dots, n$ set

$$\widehat{q}_i = a_i b_i (1 + da_i)^{-1}, \ \widehat{p}_i = w'(b\widehat{x})\widehat{q}_i. \tag{1.6}$$

Let

$$\widehat{p} = (\widehat{p}_1, \dots, \widehat{p}_n),$$

$$\xi_\sigma = 1 - d - 1/a_\sigma. \tag{1.7}$$

In Lemma 1 of [15] it was established the following important auxiliary result.

Lemma 1.2. *$w(b\widehat{x}) \ge w(by) + \widehat{p}z$ for any $(x, z) \in \Omega$ and for any $y \in \Lambda(x, z)$.*

In this paper we use the following auxiliary result obtained in [40].

Lemma 1.3. *Let $m_0 > 0$. Then there exists $m_1 > 0$ such that for each $T > 0$ and each program $(x(t), y(t))_{t=0}^{T}$ satisfying $x(0) \le m_0\mathbf{e}$ the inequality $x(t) \le m_1\mathbf{e}$ holds for all $t \in [0, T]$.*

We use the following notion of good programs (functions) introduced in [12] (see also [18, 23–25, 39]).

A program $(x(t), y(t))_{t=0}^{\infty}$ is called good if there exist $M \in R$ such that

$$\int_0^T (w(by(t)) - w(b\widehat{y}))dt \ge M \text{ for all } T \ge 0.$$

A program is called bad if

$$\lim_{T\to\infty} \int_0^T (w(by(t)) - w(b\widehat{y}))dt = -\infty.$$

The following two results were established in [40].

Theorem 1.4. *Any program $(x(t), y(t))_{t=0}^{\infty}$ that is not good is bad.*

Theorem 1.5. *For any initial stock $x_0 \in R^n_+$ there is a good program $(x(t), y(t))_{t=0}^{\infty}$ satisfying $x(0) = x_0$.*

In the sequel we use a notion of an overtaking optimal program (function) introduced in [2, 12, 36] (see also [18, 39]).

A program $(\tilde{x}(t), \tilde{y}(t))_{t=0}^{\infty}$ is overtaking optimal if for each program $(x(t), y(t))_{t=0}^{\infty}$ satisfying $x(0) = \tilde{x}(0)$ the following inequality holds:

$$\limsup_{T\to\infty} [\int_0^T w(by(t))dt - \int_0^T w(b\tilde{y}(t))dt] \le 0.$$

The following two theorems are the main results of [40].

Theorem 1.6. *Assume that a program $(x(t), y(t))_{t=0}^{\infty}$ is good. Then*

$$(i) \lim_{t\to\infty} x(t) = \widehat{x}.$$

(ii) Let $\epsilon \in (0, 1)$ and $L > 1$. Then there is $T_0 > 0$ such that for each $T \ge T_0$

$$mes(\{t \in [T, T + L] : \|y(t) - \widehat{x}\| > \epsilon\}) \le \epsilon.$$

Theorem 1.7. *For any initial stock $x_0 \in R^n_+$ there exists an overtaking optimal program $(x(t), y(t))_{t=0}^{\infty}$ satisfying $x(0) = x_0$.*

Let $z \in R^n_+$ and $T > 0$. Set

(1.8)
$$U(z,T) = \sup\Big\{\int_0^T w(by(t))dt : \ (x(t),y(t))_{t=0}^T \text{ is a program such that } x(0) = z\Big\}.$$

By Lemma 1.3, Theorem 1.5, (1.3), (1.5) and (1.8), $U(z,T)$ is a finite number.

Let $x_0, x_1 \in R^n_+$ and let $0 \le T_1 < T_2$. Define

$$U(x_0,x_1,T_1,T_2) = \sup\Big\{\int_{T_1}^{T_2} w(by(t))dt : \ (x(t),y(t))_{t=T_1}^{T_2}$$

(1.9)
$$\text{is a program such that } x(T_1) = x_0,\ x(T_2) \ge x_1\Big\}.$$

Here we assume that supremum over empty set is $-\infty$. By Lemma 1.3, (1.3), (1.5) and (1.9), $U(x_0,x_1,T_1,T_2) < \infty$. It is also clear that for any $z \in R^n_+$ and any $T > 0$, $U(z,T) = U(z,0,0,T)$.

We will establish the following two results which describe the structure of approximate optimal solutions of optimal control problems on sufficiently large intervals.

Theorem 1.8. *Let M, ϵ be positive numbers and let $\Gamma \in (0,1)$. Then there exist $T_* > 0$ and a positive number γ such that for each $T > 2T_*$, each $z_0, z_1 \in R^n_+$ satisfying $z_0 \le M\mathbf{e}$ and $az_1 \le \Gamma d^{-1}$ and each program $(x(t),y(t))_{t=0}^T$ which satisfies*

$$x(0) = z_0,\ x(T) \ge z_1, \qquad \int_0^T w(by(t))dt \ge U(z_0,z_1,0,T) - \gamma$$

there are numbers τ_1, τ_2 such that $\tau_1 \in [0,T_*]$, $\tau_2 \in [T - T_*, T]$,

$$\|x(t) - \widehat{x}\| \le \epsilon \text{ for all } t \in [\tau_1,\tau_2]$$

and that for each number S satisfying $\tau_1 \le S \le \tau_2 - L$,

$$mes(\{t \in [S,S+L] : \|y(t) - \widehat{x}\| > \epsilon\}) \le \epsilon.$$

Moreover, if $\|x(0) - \widehat{x}\| \le \gamma$, then $\tau_1 = 0$ and if $\|x(T) - \widehat{x}\| \le \gamma$, then $\tau_2 = T$. .

Theorem 1.9. *Let M_0, M_1, ϵ be positive numbers, $L > 1$ and let $\Gamma \in (0,1)$. Then there exist $T_* > L$, a natural number Q and $l > 0$ such that for each $T > T_*$, each $z_0, z_1 \in R^n_+$ satisfying $z_0 \le M\mathbf{e}$ and $az_1 \le \Gamma d^{-1}$ and each program $(x(t),y(t))_{t=0}^T$ which satisfies*

$$x(0) = z_0,\ x(T) \ge z_1, \qquad \int_0^T w(by(t))dt \ge U(z_0,z_1,0,T) - M_1$$

there exists a finite sequence of closed intervals $[S_i, S_i']$, $i = 1,\dots,q$ such that $q \le Q$, $S_i' - S_i \le l$, $i = 1,\dots,q$, $S_i' \le S_{i+1}$ for each integer i satisfying $1 \le i \le q-1$,

$$\|x(t) - \widehat{x}\| \le \epsilon,\ t \in [0,T] \setminus \cup_{i=1}^q [S_i,S_i']$$

and if $S \in [0, T-L]$ *satisfies*

$$[S, S+L] \subset [S_i', S_{i+1}] \text{ with } 1 \leq i < q,$$

then

$$mes(\{t \in [S, S+L] : \ \|y(t) - \widehat{x}\| > \epsilon\}) \leq \epsilon.$$

2. PRELIMINARIES

For any $(x, z) \in \Omega$ and any $y \in \Lambda(x, z)$ set

$$\delta(x, y, z) = w(b\widehat{y}) - w(by) - \widehat{p}z. \tag{2.1}$$

By (2.1) and Lemma 1.2,

$$\delta(x, y, z) \geq 0 \text{ for each } (x, z) \in \Omega \text{ and each } y \in \Lambda(x, z). \tag{2.2}$$

It is not difficult to prove the following result.

Lemma 2.1 (40, Lemma 2.1)**.** *Let* $T > 0$ *and* $(x(t), y(t))_{t=0}^T$ *be a program. Then*

$$\int_0^T (w(by(t)) - w(b\widehat{y}))dt = -\int_0^T \delta(x(t), y(t), x'(t))dt - \widehat{p}(x(T) - x(0)).$$

Lemmas 2.1 and 1.3 imply the following result.

Lemma 2.2 (40, Proposition 2.1)**.** *A program* $(x(t), y(t))_{t=0}^\infty$ *is good if and only if*

$$\int_0^\infty \delta(x(t), y(t), x'(t))dt := \lim_{T \to \infty} \int_0^T \delta(x(t), y(t), x'(t))dt$$

is finite.

Lemma 2.3 (40, Proposition 3.1)**.** *Let* $T > 0$, $m_0 > 0$ *and let*

$$\{(x^{(i)}(t), y^{(i)}(t))_{t=0}^T\}_{i=1}^\infty$$

be a sequence of programs satisfying

$$x^{(i)}(0) \leq m_0 \mathbf{e} \text{ for all integers } i \geq 0.$$

Then there exist a program $(x(t), y(t))_{t=0}^T$ *and a strictly increasing sequence of natural numbers* $\{i_k\}_{k=1}^\infty$ *such that*

$$x^{(i_k)}(t) \to x(t) \text{ as } k \to \infty \text{ uniformly on } [0, T],$$

$$(x^{(i_k)})' \to x' \text{ as } k \to \infty \text{ weakly in } L^2([0, T]; R^n),$$

$$y^{(i_k)} \to y \text{ as } k \to \infty \text{ weakly in } L^2([0, T]; R^n).$$

Lemma 2.4 (40, Proposition 3.2)**.** *Let $T > 0$, $m_0 > 0$,*

$$\{(x^{(i)}(t), y^{(i)}(t))_{t=0}^{T}\}_{i=1}^{\infty}$$

be a sequence of programs satisfying

$$x^{(i)}(0) \le m_0\mathbf{e} \text{ for all integers } i \ge 0$$

and let $(x(t), y(t))_{t=0}^{T}$ be a program such that

$$x^{(i)}(t) \to x(t) \text{ as } i \to \infty \text{ uniformly on } [0, T],$$

$$(x^{(i)})' \to x' \text{ as } i \to \infty \text{ weakly in } L^2([0, T]; R^n),$$

$$y^{(i)} \to y \text{ as } i \to \infty \text{ weakly in } L^2([0, T]; R^n).$$

Then

$$\int_0^T w(by(t))dt \ge \limsup_{i\to\infty} \int_0^T w(by^{(i)}(t))dt.$$

Let $x_0 \in R^n_+$. Define

$$\Delta(x_0) = \inf\{\int_0^\infty \delta(x(t), y(t), x'(t))dt : \ (x(t), y(t))_{t=0}^{\infty}$$

$$\text{is program and } x(0) = x_0\}. \tag{2.3}$$

By Theorem 1.5 and Lemma 2.1, $\Delta(x_0)$ is well-defined and finite.

Lemma 2.5 (40, Proposition 3.3)**.** *Let $x_0 \in R^n_+$. Then there exists a program $(x(t), y(t))_{t=0}^{\infty}$ such that $x(0) = x_0$ and*

$$\int_0^\infty \delta(x(t), y(t), x'(t))dt = \Delta(x_0).$$

Lemma 2.6 (15, Lemma 2)**.** *The von Neumann facet*

$$\{(x, z) \in \Omega : \ \text{there is } y \in \Lambda(x, z) \text{ such that } \delta(x, y, z) = 0\}$$

is a subset of

$$\{(x, z) \in \Omega : \ x_i = z_i = 0 \text{ for all } i \in \{1, \dots, n\} \setminus \{\sigma\},$$

$$z_\sigma = (1/a_\sigma) + (\xi_\sigma - 1)x_\sigma\}$$

with the equality if the function w is linear. If the function w is strictly concave, then the facet is the singleton $\{(\widehat{x}, 0)\}$.

Lemma 2.7 (40, Lemma 4.2)**.** *Let $(x(t), y(t))_{t=0}^{\infty}$ be a good program and $T_i > 2i$ for all natural numbers i. Let*

$$(x^{(i)}(t), y^{(i)}(t)) = (x(t + T_i), y(t + T_i)), \ t \in [-i, i]$$

for each natural number i.

Then there exist a strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$, a locally a. c. function $\bar{x} : R \to R^n$ and a measurable Lebesgue function $\bar{y} : R \to R^n$ such that for each natural number j,

$$x^{(i_k)}(t) \to \bar{x}(t) \text{ as } i \to \infty \text{ uniformly on } [-j, j],$$

$$(x^{(i_k)})' \to \bar{x}' \text{ as } k \to \infty \text{ weakly in } L^2([-j, j]; R^n).$$

Moreover,

$$0 \le \bar{y}(t) \le \bar{x}(t) \text{ for a.e. } t \in R,$$

$$\bar{x}'(t) + d\bar{x}(t) \ge 0 \text{ for a.e. } t \in R,$$

$$e\bar{y}(t) + a(\bar{x}'(t) + d\bar{x}(t)) \le 1 \text{ for a.e. } t \in R$$

and

$$\delta(\bar{x}(t), \bar{y}(t), \bar{x}'(t)) = 0 \text{ for a.e. } t \in R.$$

Lemma 2.8 (40, Lemma 4.3)**.** *Let $x : R \to R^n$ be a locally a.c. function and $y : R \to R^n$ be a Lebesgue measurable function such that*

$$(x(t), x'(t)) \in \Omega \text{ for a. e. } t \in R,$$

$$y(t) \in \Lambda(x(t), x'(t)) \text{ for a. e. } t \in R,$$

$$\sup\{\|x(t)\| : t \in R\} < \infty,$$

$$\delta(x(t), y(t), x'(t)) = 0 \text{ for a. e. } t \in R.$$

Then $x(t) = \widehat{x}$ for all $t \in R$ and $y(t) = \widehat{x}$ for almost every $t \in R$.

3. **AUXILIARY RESULTS**

Lemma 3.1. *Let $\Gamma \in (0, 1)$. Then there exists a number $k(\Gamma) > 0$ such that for each $z_0 \in R^n_+$ and each $z_1 \in R^n_+$ satisfying $az_1 \le \Gamma d^{-1}$ there is a program $(x(t), y(t))_{t=0}^{\infty}$ such that $x(0) = z_0$ and $x(t) \ge z_1$ for all $t \ge k(\Gamma)$.*

Proof. There exists $k(\Gamma) > 0$ such that

$$1 - e^{-dk(\Gamma)} > \Gamma. \tag{3.1}$$

Assume that $z_0 \in R^n_+$ and $z_1 \in R^n_+$ satisfies

$$az_1 \le \Gamma d^{-1}. \tag{3.2}$$

Put

$$z_2 = \Gamma^{-1} z_1, \tag{3.3}$$

$$x(t) = e^{-dt}(z_0 - z_2) + z_2,\ y(t) = 0,\ t \in [0, \infty), \tag{3.4}$$

By (3.4) for all $t \ge 0$,

$$x(t) \ge 0, \tag{3.5}$$

(3.6) $$x'(t) + dx(t) = -de^{-dt}(z_0 - z_2) + de^{-dt}(z_0 - z_2) + dz_2 = dz_2.$$

In view of (3.6), (3.2) and (3.3) for all $t \geq 0$,

$$x'(t) + dx(t) \geq 0,$$

$$a(x'(t) + dx(t)) = adz_2 = ad(\Gamma^{-1}z_1) \leq 1.$$

Together with (1.2), (1.3), (1.4), (1.5), (3.4) and (3.5) these inequalities imply that $(x(t), y(t))_{t=0}^{\infty}$ is a program. By (3.4),

(3.7) $$x(0) = z_0.$$

It follows from (3.4), (3.1) and (3.2) that for all $t \geq k(\Gamma)$

$$x(t) \geq (1 - e^{-dt})z_2 \geq (1 - e^{-dk(\Gamma)})z_2 \geq \Gamma^{-1}(1 - e^{-dk(\gamma)})z_1 \geq z_1.$$

Lemma 3.1 is proved. □

In the sequel with each $\Gamma \in (0, 1)$ we associate a number $k(\Gamma) > 0$ for which the assertion of Lemma 3.1 holds.

Lemma 3.2. *There is $m > 0$ such that for each $z \in R_+^n$ and each number $T > 0$*

(3.8) $$U(z, T) \geq Tw(b\widehat{x}) - m.$$

Proof. By Theorem 1.1

(3.9) $$a\widehat{x} = a_\sigma \widehat{x}_\sigma = a_\sigma(1 + da_\sigma)^{-1} < d^{-1}.$$

By (3.9) there is $\Gamma \in (0, 1)$ such that

(3.10) $$a\widehat{x} \leq \Gamma d^{-1}.$$

Choose

$$m > k(\Gamma)[|w(0)| + |w(k(\Gamma))| + |w(b\widehat{x})|].$$

Let $z \in R_+^n$. By (3.10), the choice of $k(\Gamma)$ and Lemma 3.1 there exists a program $(x(t), y(t))_{t=0}^{k(\Gamma)}$ such that

(3.11) $$x(0) = z, \; x(k(\Gamma)) \geq \widehat{x}.$$

Let $T > 0$. We show that (3.8) holds. There are cases

(3.12) $$T \leq k(\Gamma)$$

and

(3.13) $$T > k(\Gamma).$$

Assume that (3.12) holds. Then by (3.11), (3.12) and the choice of m

$$\begin{aligned}U(z,T) &\geq \int_0^T w(by(t))dt \geq Tw(0) \geq T(-|w(0)|) \geq -k(\Gamma)|w(0)| \\ &= Tw(b\widehat{x}) + [-k(\Gamma)|w(0)| - Tw(b\widehat{x})] \geq Tw(b\widehat{x}) - k(\Gamma)|w(0)| - k(\Gamma)|w(b\widehat{x})| \\ &\geq Tw(b\widehat{x}) - m\end{aligned}$$

and (3.8) holds.

Assume that (3.13) holds. For all $t > k(\Gamma)$ set

$$x(t) = \widehat{x} + e^{-d(t-k(\Gamma))}(x(k(\Gamma)) - \widehat{x}),\ y(t) = \widehat{x}. \tag{3.14}$$

By (3.14), (3.11) for all $t \in (k(\Gamma), \infty)$,

$$0 \leq y(t) \leq x(t). \tag{3.15}$$

It follows from (3.14), Theorem 1.1 and (1.2) that for all $t \in (k(\Gamma), \infty)$

$$x'(t) + dx(t) = d\widehat{x} = d(1 + da_\sigma)^{-1}e_\sigma, \tag{3.16}$$

$$a(x'(t) + dx(t)) \leq 1,\ (x(t), x'(t)) \in \Omega. \tag{3.17}$$

By (3.16), (3.14) and Theorem 1.1 for all $t \in (k(\Gamma), \infty)$

$$a(x'(t) + dx(t)) + \mathbf{e}y(t) = ad\widehat{x} + (1 + da_\sigma)^{-1}$$

$$= a_\sigma d(1 + da_\sigma)^{-1} + (1 + da_\sigma)^{-1} = 1$$

and together with (1,3), (3.17) and (3.15) this implies that

$$y(t) \in \Lambda(x(t), x'(t)).$$

Thus we have shown that $(x(t), y(t))_{t=0}^\infty$ is a program. By (3.11), (3.13), (3.14) and the choice of m,

$$U(z,T) \geq \int_0^T w(by(t))dt = \int_0^{k(\Gamma)} w(by(t))dt + (T - k(\Gamma))w(b\widehat{x})$$

$$\geq Tw(b\widehat{x}) + k(\Gamma)(w(0) - w(b\widehat{x})) \geq Tw(b\widehat{x}) - m.$$

Thus (3.8) holds. Lemma 3.2 is proved. □

Lemma 3.3. *Let $\Gamma \in (0, 1)$. Then there exists $m > 0$ such that for each $z_0 \in R^n_+$, each $z_1 \in R^n_+$ satisfying $az_1 \leq \Gamma d^{-1}$ and each number $T > k(\Gamma)$,*

$$U(z_0, z_1, 0, T) \geq Tw(b\widehat{x}) - m.$$

Proof. By Lemma 3.2 there is $m_0 > 0$ such that

$$U(z,T) \geq Tw(b\widehat{x}) - m_0 \text{ for each } z \in R^n_+ \text{ and each number } T > 0. \tag{3.18}$$

Put

$$m = m_0 + 1 + k(\Gamma)(w(b\widehat{x}) - w(0)). \tag{3.19}$$

Assume that

$$z_0,\ z_1 \in R^n_+,\ az_1 \leq \Gamma d^{-1},\ T > k(\Gamma). \tag{3.20}$$

By the choice of m_0 (see (3.18)) and (3.20) there is a program $(x(t), y(t))_{t=0}^{T-k(\Gamma)}$ such that

$$x(0) = z_0,\ \int_0^{T-k(\Gamma)} w(by(t))dt \geq U(z_0, T-k(\Gamma)) - 1 \geq (T-k(\Gamma))w(b\widehat{x}) - m_0 - 1. \tag{3.21}$$

By the choice of $k(\Gamma)$, Lemma 3.1 and (3.2) there is a program $(x(t), y(t))_{t=T-k(\Gamma)}^{T}$ such that

$$x(T) \geq z_1. \tag{3.22}$$

Clearly, $(x(t), y(t))_{t=0}^{T}$ is a program. In view of (3.21), (3.22) and (3.19),

$$\begin{aligned} U(z_0, z_1, 0, T) &\geq \int_0^T w(by(t))dt = \int_0^{T-k(\Gamma)} w(by(t))dt + \int_{T-k(\Gamma)}^T w(by(t))dt \\ &\geq (T-k(\Gamma))w(b\widehat{x}) - m_0 - 1 + k(\Gamma)w(0) \\ &= Tw(b\widehat{x}) - k(\Gamma)(w(b\widehat{x}) - w(0)) - m_0 - 1 = Tw(b\widehat{x}) - m. \end{aligned}$$

Lemma 3.3 is proved. □

Lemma 3.4. *Let $m_0 > 0$. Then there exists $m_2 > 0$ such that for each number $T > 0$ and each program $(x(t), y(t))_{t=0}^{T})$ which satisfies $x(0) \leq m_0\mathbf{e}$ the following inequality holds:*

$$\int_0^T [w(by(t)) - w(b\widehat{x})]dt \leq m_2.$$

Proof. By Lemma 1.3 there exists $m_1 > 0$ such that for each number $T > 0$ and each program $(x(t), y(t))_{t=0}^{T}$ which satisfies $x(0) \leq m_0\mathbf{e}$ we have

$$x(t) \leq m_1\mathbf{e} \text{ for all } t \in [0,T]. \tag{3.23}$$

Choose a number

$$m_2 > 2\|\widehat{p}\|m_1 n. \tag{3.24}$$

Assume that $T > 0$ and that a program $(x(t), y(t))_{t=0}^{T}$ satisfies $x(0) \leq m_0\mathbf{e}$. Then (3.23) holds. By (2.2), (3.23), (3.24) and Lemma 2.1

$$\int_0^T (w(by(t)) - w(b\widehat{x}))dt \leq -\widehat{p}(x(T) - x(0)) \leq 2\|\widehat{p}\|nm_1 \leq m_2.$$

Lemma 3.4 is proved. □

It is not difficult to see that the following auxiliary result holds.

Lemma 3.5. *Assume that nonnegative numbers* T_1, T_2 *satisfy* $T_1 < T_2$,

$$(x(t), y(t))_{t=T_1}^{T_2}$$

is a program and that $u \in R_+^n$. *Then* $(x(t) + e^{-d(t-T_1)}u, y(t))_{t=T_1}^{T_2})$ *is also a program.*

In order to prove Lemma 3.5 it is sufficient to note that for a. e. $t \in [T_1, T_2]$,

$$(x(t) + e^{-d(t-T_1)}u)' + d(x(t) + e^{-d(t-T_1)}u) = x'(t) + dx(t).$$

Lemma 3.5 implies the following result.

Lemma 3.6. *Let* $0 \le T_1 < T_2$, $M > 0$, $x_0, x_1 \in R_+^n$ *and let*

$$(x(t), y(t))_{t=T_1}^{T_2}$$

be a program such that

$$x(T_1) = x_0,\ x(T_2) \ge x_1,\ \int_{T_1}^{T_2} w(by(t))dt \ge U(x_0, x_1, T_1, T_2) - M.$$

Then for each pair of numbers S_1, S_2 *satisfying*

$$T_1 \le S_1 < S_2 \le T_2$$

the following inequality holds:

$$\int_{S_1}^{S_2} w(by(t))dt \ge U(x(S_1), x(S_2), S_1, S_2) - M.$$

Lemma 3.7. *Let* $\epsilon > 0$. *Then there exists* $\delta > 0$ *such that for each* $z, z' \in R_+^n$ *satisfying*

$$\|z - \widehat{x}\|,\ \|z' - \widehat{x}\| \le \delta \tag{3.25}$$

and each $T \in [2^{-1}, 2]$ *there is a program* $(x(t), y(t))_{t=0}^{T}$ *such that*

$$x(0) = z,\ x(T) \ge z',\ \|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \le \epsilon,\ t \in [0, T],\ \|x'(t)\| \le \epsilon,\ t \in [0, T].$$

Proof. We may assume without loss of generality that

$$\epsilon < (1 + da_\sigma)^{-1}. \tag{3.26}$$

Choose a positive number δ such that

$$\delta \sum_{i=1}^{n} a_i < (\epsilon/16),\ 16\delta n < \epsilon. \tag{3.27}$$

Assume that $T \in [2^{-1}, 2]$ and that $z, z' \in R_+^n$ satisfy (3.25). For all $t \in [0, T]$ put

$$y(t) = ((1 + da_\sigma)^{-1} - \epsilon)e(\sigma). \tag{3.28}$$

Clearly,

$$\|y(t) - \widehat{x}\| \leq \epsilon, \; t \in [0, T]. \tag{3.29}$$

Put

$$\xi = 4\delta e$$

and define

$$x(t) = z + t\xi, \; t \in [0, T]. \tag{3.30}$$

By (3.28). (3.26), (3.25), (3.27), (3.29), (3.30) and the choice of ξ,

$$0 \leq y(t) \leq z \leq x(t) \text{ for all } t \in [0, T]. \tag{3.31}$$

Equation (3.30) and the choice of ξ imply that for all $t \in [0, T]$

$$x'(t) + dx(t) = \xi + d(z + t\xi) \geq 0. \tag{3.32}$$

In view of (3.32) and (3.25) for all $t \in [0, T]$,

$$a(x'(t) + dx(t)) = adz + (1 + dt)a\xi \leq ad\widehat{x} + \delta d \sum_{i=1}^{n} a_i + (1 + 2d)a\xi.$$

Together with (3.28), Theorem 1.1, (3.29) and (3.27) this implies that for all $t \in [0, T]$

$$\mathbf{e}y(t) + a(x'(t) + dx(t)) \leq (1 + da_\sigma)^{-1} - \epsilon + ad(1 + da_\sigma)^{-1} + \delta \sum_{i=1}^{n} a_i + 3a\xi$$

$$= 1 - \epsilon + \delta \sum_{i=1}^{n} a_i + 3a\xi \leq 1.$$

By the relation above, (3.31), (3.32), (3.30) and (3.28), $(x(t), y(t))_{t=0}^{T}$ is a program. It follows from (3.30), (3.29), (3.25) and the choice of ξ that

$$x(0) = z, \; x(T) = z + T\xi \geq \widehat{x} - \delta\mathbf{e} + 2^{-1}\xi = \widehat{x} - \delta\mathbf{e} + 2^{-1}\xi \geq \widehat{x} + \delta\mathbf{e} \geq z.$$

By (3.30), (3.25) and (3.27) for all $t \in [0, T]$

$$\|x(t) - \widehat{x}\| \leq \|z - \widehat{x}\| + T\|\xi\| \leq \delta + 2\|\xi\| \leq \delta + 8\delta n \leq 16\delta n < \epsilon.$$

This completes the proof of Lemma 3.7. □

Lemma 3.8. *Let $m_0, m_1, \epsilon > 0$. Then there is a natural number τ such that for each program $(x(t), y(t))_{t=0}^{\tau}$ satisfying*

$$x(0) \leq m_0\mathbf{e}, \; \int_0^\tau w(by(t))dt \geq \tau w(b\widehat{x}) - m_1$$

there is $t \in [0, \tau]$ such that

$$\|x(t) - \widehat{x}\| \leq \epsilon.$$

Proof. Let us assume the contrary. Then for each natural number k there exists a program $(x^{(k)}(t), y^{(k)}(t))_{t=0}^{k}$ such that

$$x^{(k)}(0) \leq m_0\mathbf{e}, \ \int_0^k w(by^{(k)}(t))dt \geq kw(b\widehat{x}) - m_1,$$

$$\|x^{(k)}(t) - \widehat{x}\| > \epsilon \text{ for all } t \in [0, k]. \tag{3.33}$$

In view of (3.33) and Lemma 1.3 there is $m_2 > m_0$ such that for each natural number k

$$x^{(k)}(t) \leq m_2\mathbf{e}, \ t \in [0, k]. \tag{3.34}$$

By Lemma 3.4 there exits $m_3 > 0$ such that for each $T > 0$ and each program $(x(t), y(t))_{t=0}^{T}$ satisfying

$$x(0) \leq m_2\mathbf{e}$$

the following inequality holds:

$$\int_0^T (w(by(t)) - w(b\widehat{x}))dt \leq m_3. \tag{3.35}$$

Let k be a natural number and let a number s satisfy $0 < s < k$. It follows from (3.34) and the choice of m_3 (see (3.35)) that

$$\int_s^k [w(by^{(k)}(t)) - w(b\widehat{x})]dt \leq m_3.$$

Combined with (3.33) this implies that

$$\int_0^s [w(by^{(k)}(t)) - w(b\widehat{x})]dt = \int_0^k [w(by^{(k)}(t)) - w(b\widehat{x})]dt$$
$$- \int_s^k [w(by^{(k)}(t)) - w(b\widehat{x})]dt \geq -m_1 - m_3.$$

Thus for each natural number k and each $s \in (0, k)$

$$\int_0^s [w(by^{(k)}(t)) - w(b\widehat{x})]dt \geq -m_1 - m_3. \tag{3.36}$$

By extracting a subsequence and using (3.34), Lemma 2.3 and diagonalization process we obtain that there exist a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ and a program $(x^*(t), y^*(t))_{t=0}^{\infty}$ such that for any natural number q,

$$x^{(k_j)}(t) \to x^*(t) \ \text{ as } j \to \infty \text{ uniformly on } [0, q], \tag{3.37}$$

$$(x^{(k_j)})'(t) \to (x^*)'(t) \ \text{ as } j \to \infty \text{ weakly in } L^2([0, q]; R^n), \tag{3.38}$$

$$y^{(k_j)} \to y^* \text{ as } j \to \infty \text{ weakly in } L^2([0, q]; R^n). \tag{3.39}$$

In view of (3.34) and (3.37),

$$x^*(t) \leq m_2\mathbf{e} \text{ for all } t \geq 0. \tag{3.40}$$

By (3.34), (3.37), (3.38), (3.39) and Lemma 2.4 for all natural numbers q,

$$\int_0^q w(by^*(t))dt \geq \limsup_{j\to\infty} \int_0^q w(by^{k_j}(t))dt \geq qw(b\widehat{x})) - m_3 - m_1.$$

Together with Theorem 1.4 this implies that $(x^*(t), y^*(t))_{t=0}^{\infty}$ is a good program. By Theorem 1.6,

$$\lim_{t\to\infty} x^*(t) = \widehat{x}. \tag{3.41}$$

On the other hand it follows from (3.37) and (3.33) that

$$\|x^*(t) - \widehat{x}\| \geq \epsilon,\ t \in [0, \infty).$$

This contradicts (3.41). The contradiction we have reached proves Lemma 3.8. □

Lemma 3.9. *Let $\epsilon > 0$. Then there exists $\gamma > 0$ such that for each number $T > 2$ and each program $(x(t), y(t))_{t=0}^{T}$ which satisfies*

$$\|x(0) - \widehat{x}\| \leq \gamma,\ \|x(T) - \widehat{x}\| \leq \gamma, \tag{3.42}$$

$$\int_0^T w(by(t))dt \geq U(x(0), x(T), 0, T) - \gamma \tag{3.43}$$

the following inequality holds:

$$\int_0^T \delta(x(t), y(t), x'(t))dt \leq \epsilon. \tag{3.44}$$

Proof. Choose a positive number ϵ_0 such that

$$\epsilon_0 < (\epsilon/18)(\|\widehat{p}\| + 1)^{-1}, \tag{3.45}$$

$$\text{if } y \in R^n_+ \text{ and } \|y - \widehat{x}\| \leq \epsilon_0,\ \text{then } |w(b\widehat{x}) - w(by)| < \epsilon/16. \tag{3.46}$$

By Lemma 3.7 there exists $\gamma \in (0, \epsilon_0)$ such that the following property holds:

(P1) for each $T \in [2^{-1}, 2]$ and each $z, z' \in R^n_+$ satisfying $\|z - \widehat{x}\|,\ \|z' - \widehat{x}\| \leq \gamma$ there exists a program $(u(t), v(t))_{t=0}^{T}$ such that

$$u(0) = z,\ u(T) \geq z',\ \|u(t) - \widehat{x}\|,\ \|v(t) - \widehat{x}\| \leq \epsilon_0,\ t \in [0, T],$$
$$\|u'(t)\| \leq \epsilon_0,\ t \in [0, T].$$

Assume that $T > 2$ and that a program $(x(t), y(t))_{t=0}^{T}$ satisfies (3.42) and (3.43). By (3.42) and property (P1) there exist programs

$$(u^{(1)}(t), v^{(1)}(t))_{t=0}^{1} \text{ and } (u^{(2)}(t), v^{(2)}(t))_{t=T-1}^{T}$$

such that

$$u^{(1)}(0) = x(0),\ u^{(1)}(1) \geq \widehat{x},\ \|u^{(1)}(t) - \widehat{x}\|,\ \|v^{(1)}(t) - \widehat{x}\| \leq \epsilon_0,\ t \in [0, 1], \tag{3.47}$$
$$\|(u^{(1)})'(t)\| \leq \epsilon_0,\ t \in [0, 1],$$
$$u^{(2)}(T-1) = \widehat{x},\ u^{(2)}(T) \geq x(T),\ \|u^{(2)}(t) - \widehat{x}\|,\ \|v^{(2)}(t) - \widehat{x}\| \leq \epsilon_0,$$

$$\|(u^{(2)})'(t)\| \le \epsilon_0,\ t \in [T-1, T]. \tag{3.48}$$

Define a program $(\bar{x}(t), \bar{y}(t))_{t=0}^{T}$ as follows. Put

$$\bar{x}(t) = u^{(1)}(t),\ \bar{y}(t) = v^{(1)}(t),\ t \in [0, 1], \tag{3.49}$$

$$\bar{x}(t) = \widehat{x} + e^{-d(t-1)}(u^{(1)}(1) - \widehat{x}),\ \bar{y}(t) = \widehat{x},\ t \in (1, T-1].$$

By Lemma 3.5, (3.49) and (3.47), $(\bar{x}(t), \bar{y}(t))_{t=0}^{T-1}$ is a program. In view of (3.49) and (3.47),

$$\bar{x}(T-1) \ge \widehat{x} = u^{(2)}(T-1). \tag{3.50}$$

For $t \in (T-1, T]$ put

$$\bar{x}(t) = u^{(2)}(t) + e^{-d(t-(T-1))}(\bar{x}(T-1) - u^{(2)}(T-1)),\ \bar{y}(t) = v^{(2)}(t). \tag{3.51}$$

Lemma 3.5, (3.50) and (3.51) imply that $(\bar{x}(t), \bar{y}(t))_{t=0}^{T}$ is a program. It follows from (3.49), (3.47), (3.50), (3.51) and (3.48) that

$$\bar{x}(0) = x(0),\ \bar{x}(T) \ge u^{(2)}(T) \ge x(T). \tag{3.52}$$

Equations (3.43) and (3.52) imply that

$$-\gamma \le \int_0^T w(by(t))dt - \int_0^T w(b\bar{y}(t))dt. \tag{3.53}$$

It follows from (3.51), (3.48), (3.49) and (3.47) that

$$\|\bar{x}(T) - \widehat{x}\| \le \|\bar{x}(T) - u^{(2)}(T)\| + \|u^{(2)}(T) - \widehat{x}\|$$

$$\le \|\bar{x}(T-1) - u^{(2)}(T-1)\| + \epsilon_0 = \|\bar{x}(T-1) - \widehat{x}\| + \epsilon_0 \le \|u^{(1)}(1) - \widehat{x}\| + \epsilon_0 \le 2\epsilon_0.$$

Lemma 2.1 and (3.53) imply that

$$-\gamma \le -\int_0^T (w(b\bar{y}(t)) - w(b\widehat{y}))dt - \int_0^T \delta(x(t), y(t), x'(t))dt - \widehat{p}(x(T) - x(0)).$$

Together with (3.49), (3.51), (3.47), (3.48), (3.45) and (3.46) this implies that

$$\begin{aligned}\int_0^T \delta(x(t), y(t), x'(t))dt &\le \gamma - \int_0^T (w(b\bar{y}(t)) - w(b\widehat{x}))dt + \|\widehat{p}\|\|x(T) - x(0)\| \\ &\le \gamma - \int_0^1 (w(b\bar{y}(t)) - w(b\widehat{x}))dt \\ &\quad - \int_{T-1}^T (w(b\bar{y}(t)) - w(b\widehat{x}))dt + \|\widehat{p}\| 2\epsilon_0 \\ &\le \gamma + \epsilon/16 + \epsilon/16 + \epsilon/8 < \epsilon.\end{aligned}$$

Lemma 3.9 is proved. □

Lemma 3.10. *Let $\epsilon > 0$ and $\tau_0 > 0$. Then there exist $\gamma > 0$ and $T_0 > \tau_0$ such that for each $T \geq T_0$ and each program $(x(t), y(t))_{t=0}^{T}$ which satisfies*

$$\|x(0) - \widehat{x}\|,\ \|x(T) - \widehat{x}\| \leq \gamma,\ \int_0^T \delta(x(t), y(t), x'(t))dt \leq \gamma$$

the following properties hold:

$$\|x(t) - \widehat{x}\| \leq \epsilon \text{ for all } t \in [0, T];$$

for each $S \in [0, T - \tau_0]$,

$$mes(\{t \in [S, S + \tau_0] :\ \|y(t) - \widehat{x}\| > \epsilon\}) \leq \epsilon.$$

Proof. We may assume that $\epsilon < 1$. Choose

$$\epsilon_0 \in (0, 16^{-1}\epsilon). \tag{3.54}$$

By Lemma 3.7 and the continuity of the function $\delta(\cdot, \cdot, \cdot)$ there exists a sequence of positive numbers $\{\gamma_q\}_{q=1}^{\infty}$ such that

$$\gamma_q \leq 4^{-1}\gamma_{q-1} \text{ for all integers } q \geq 2, \tag{3.55}$$

$$\gamma_q \leq 4^{-q}\epsilon_0 \text{ for all integers } q \geq 1. \tag{3.56}$$

and that for any integer $q \geq 1$ the following property holds:

(P2) for each $z, z' \in R^n_+$ satisfying $\|z - \widehat{x}\|,\ \|z' - \widehat{x}\| \leq \gamma_q$ there exists a program $(x(t), y(t))_{t=0}^{1}$ such that

$$x(0) = z,\ x(1) \geq z',\ \|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \leq 4^{-q}\epsilon_0,\ t \in [0, 1],$$

$$\|x'(t)\| \leq 4^{-q}\epsilon_0,\ t \in [0, 1],$$

$$\int_0^1 \delta(x(t), y(t), x'(t))dt \leq 4^{-q}\epsilon_0.$$

Assume that the lemma does not hold. Then for each natural number q there exist

$$T_q \geq \tau_0 + q \tag{3.57}$$

and a program $(x^{(q)}(t), y^{(q)}(t))_{t=0}^{T_q}$ such that

$$\|x^{(q)}(0) - \widehat{x}\|,\ \|x^{(q)}(T_q) - \widehat{x}\| \leq \gamma_q, \tag{3.58}$$

$$\int_0^{T_q} \delta(x^{(q)}(t), y^{(q)}(t), (x^{(q)})'(t)dt \leq \gamma_q$$

and that at least one of the following properties hold:

$$\sup\{\|x^{(q)}(t) - \widehat{x}\| :\ t \in [0, T_q]\} > \epsilon; \tag{3.59}$$

(P3) there is $S \in [0, T_q - \tau_0]$ such that

$$\text{mes}(\{t \in [S, S + \tau_0] :\ \|y^{(q)}(t) - \widehat{x}\| > \epsilon\}) > \epsilon. \tag{3.60}$$

Extracting a subsequence and re-indexing we may assume without loss of generality that one of the following cases hold:

(3.59) holds for all natural numbers q; (P3) holds for all natural numbers q.

It follows from (3.58), (3.55) and (P2) that for any natural number q there exists a program $(u^{(q}(t), v^{(q)}(t))_{t=0}^{1}$ such that

$$u^{(q)}(0) = x^{(q)}(T_q),\ u^{(q)}(1) \geq x^{(q+1)}(0), \tag{3.61}$$

$$\|u^{(q)}(t) - \widehat{x}\|,\ \|v^{(q)}(t) - \widehat{x}\| \leq 4^{-q}\epsilon_0,\ t \in [0, 1], \tag{3.62}$$

$$\|(u^{(q)})'(t)\| \leq 4^{-q}\epsilon_0,\ t \in [0, 1],$$

$$\int_0^1 \delta(u^{(q)}(t), v^{(q)}(t), (u^{(q)})'(t))dt \leq 4^{-q}\epsilon_0. \tag{3.63}$$

We construct a program $(\bar{x}(t), \bar{y}(t)_{t=0}^{\infty}$ by induction. Set

$$\bar{x}(t) = x^{(1)}(t),\ \bar{y}(t) = y^{(1)}(t),\ t \in [0, T_1]. \tag{3.64}$$

Assume that q is a natural number and that we have already defined a program

$$(\bar{x}(t), \bar{y}(t))_{t=0}^{\sum_{i=1}^{q} T_i + q - 1}$$

such that

$$\bar{x}\left(\sum_{i=1}^{q} T_i + q - 1\right) \geq x^{(q)}(T_q). \tag{3.65}$$

(Clearly, for $q = 1$ our assumption holds.) For $t \in (\sum_{i=1}^{q} T_i + q - 1, \sum_{i=1}^{q} T_i + q]$ set

$$\begin{aligned} \bar{x}(t) &= u^{(q)}\left(t - \left(\sum_{i=1}^{q} T_i + q - 1\right)\right) \\ &\quad + e^{-d\left(t - \left(\sum_{i=1}^{q} T_i + q - 1\right)\right)} \left[\bar{x}\left(\sum_{i=1}^{q} T_i + q - 1\right) - x^{(q)}(T_q)\right], \\ \bar{y}(t) &= v^{(q)}\left(t - \left(\sum_{i=1}^{q} T_i + q - 1\right)\right). \end{aligned} \tag{3.66}$$

By (3.65), (3.66), Lemma 3.5 and (3.61) $(\bar{x}(t), \bar{y}(t))_{t=0}^{\sum_{i=1}^{q} T_i + q}$ is a program,

$$\begin{aligned} \bar{x}\left(\sum_{i=1}^{q} T_i + q\right) &= u^{(q)}(1) + e^{-d}\left[\bar{x}\left(\sum_{i=1}^{q} T_i + q - 1\right) - x^{(q)}(T_q)\right] \\ &\geq u^{(q)}(1) \geq x^{(q+1)}(0). \end{aligned} \tag{3.67}$$

For $t \in (\sum_{i=1}^{q} T_i + q, \sum_{i=1}^{q+1} T_i + q]$ set

$$\bar{x}(t) = x^{(q+1)}(t - (\sum_{i=1}^{q} T_i + q)) + e^{-d(t - (\sum_{i=1}^{q} T_i + q))}[\bar{x}(\sum_{i=1}^{q} T_i + q) - x^{(q+1)}(0)],$$

$$\bar{y}(t) = y^{(q+1)}\left(t - \left(\sum_{i=1}^{q} T_i + q\right)\right). \tag{3.68}$$

By (3.68), (3.67) and Lemma 3.5, $(\bar{x}(t), \bar{y}(t))_0^{\sum_{i=1}^{q+1} T_i + q}$ is a program and

$$\bar{x}\left(\sum_{i=0}^{q+1} T_i + q\right) \geq x^{(q+1)}(T_{q+1}). \tag{3.69}$$

Thus the program $(\bar{x}(t), \bar{y}(t))_{t=0}^{\infty}$ has been constructed by induction.

By (3.68) and (3.62) (with $q = 1$) for $t \in [T_1, T_1 + 1]$,

$$\begin{aligned} \|\bar{x}(t) - \widehat{x}\| &\leq \|\bar{x}(t) - u^{(1)}(t - T_1)\| + \|u^{(1)}(t - T_1) - \widehat{x}\| \\ &\leq \|\bar{x}(T_1) - x^{(1)}(T_1)\| + 4^{-1}\epsilon_0 = 4^{-1}\epsilon_0. \end{aligned} \tag{3.70}$$

We show by induction that for any natural number q

$$\|\bar{x}\left(t + \sum_{i=1}^{q} T_i + q - 1 - T_q\right) - x^{(q)}(t)\| \leq 2\left(\sum_{i=1}^{q} 4^{-i}\epsilon_0\right), \ t \in [0, T_q], \tag{3.71}$$

$$\|\bar{x}(t) - \widehat{x}\| \leq \sum_{i=1}^{q} 2 \cdot 4^{-i}\epsilon_0, \ t \in \left[\sum_{i=1}^{q} T_i + q - 1, \sum_{i=1}^{q} T_i + q\right]. \tag{3.72}$$

By (3.70) and (3.64) equations (3.71) and (3.72) hold for $q = 1$.

Assume that q is a natural number and that (3.71) and (3.72) hold. For $t \in [0, T_{q+1}]$ it follows from (3.68), (3.58), (3.55), (3.56) and (3.72) that

$$\begin{aligned} \|\bar{x}\left(t + \sum_{i=1}^{q} T_i + q\right) - x^{(q+1)}(t)\| &\leq \|\bar{x}\left(\sum_{i=1}^{q} T_i + q\right) - x^{(q+1)}(0)\| \\ &\leq \|\bar{x}(\sum_{i=1}^{q} T_i + q) - \widehat{x}\| + \|\widehat{x} - x^{(q+1)}(0)\| \\ &\leq \|\bar{x}(\sum_{i=1}^{q} T_i + q) - \widehat{x}\| + 4^{-q-1}\epsilon_0 \leq \sum_{i=1}^{q} 2 \cdot 4^{-i}\epsilon_0 + 4^{-q-1}\epsilon_0. \end{aligned} \tag{3.73}$$

By (3.66), (3.62) (which holds for any natural number q), (3.73) and (3.72) for $t \in [\sum_{i=1}^{q+1} T_i + q, \sum_{i=1}^{q+1} T_i + q + 1]$,

$$\begin{aligned} \|\bar{x}(t) - \widehat{x}\| &\leq \|\bar{x}(t) - u^{(q+1)}\left(t - \left(\sum_{i=1}^{q+1} T_i + q\right)\right)\| + \|u^{(q+1)}\left(t - \left(\sum_{i=1}^{q+1} T_i + q\right)\right) - \widehat{x}\| \\ &\leq \|\bar{x}\left(\sum_{i=1}^{q+1} T_i + q\right) - x^{(q+1)}(T_{q+1})\| + 4^{-q-1}\epsilon_0 \\ &\leq \|\bar{x}\left(\sum_{i=1}^{q} T_i + q\right) - \widehat{x}\| + 2 \cdot 4^{-q-1}\epsilon_0 \leq \sum_{i=1}^{q+1} 2 \cdot 4^{-i}\epsilon_0. \end{aligned}$$

Thus we have shown by induction that (3.71) and (3.72) hold for any natural number q.

We show that $(\bar{x}(t), \bar{y}(t))_{t=0}^{\infty}$ is a good program. By Theorem 1.4 in order to meet this goal it is sufficient to show that $\int_0^T (w(b\bar{y}(t)) - w(b\widehat{y}))dt$ does not tend to $-\infty$ as $T \to \infty$.

By Lemma 2.1, (3.58) and (3.56) for any natural number q,

$$\int_0^{T_q} (w(by^{(q)}(t)) - w(b\widehat{y}))dt \geq -\int_0^{T_q} \delta(x^{(q)}(t), y^{(q)}(t), (x^{(q)})'(t))dt$$
$$- \|\widehat{p}\|(\|x^{(q)}(0) - x^{(q)}(T_q)\|)$$
$$\geq -\gamma_q - 2\|\widehat{p}\|\gamma_q = -\gamma_q(1 + 2\|\widehat{p}\|) \geq -4^{-q}(1 + 2\|\widehat{p}\|)\epsilon_0 \tag{3.74}$$

and in view of Lemma 2.1, (3.6) and (3.62),

$$\int_0^{1} (w(bv^{(q)}(t)) - w(b\widehat{y}))dt \geq -\int_0^{1} \delta(u^{(q)}(t), v^{(q)}(t), (u^{(q)})'(t))dt$$
$$- \|\widehat{p}\|(\|u^{(q)}(0) - u^{(q)}(1)\|)$$
$$\geq -4^{-q}\epsilon_0 - 2 \cdot 4^{-q}(\epsilon_0)\|\widehat{p}\| \geq -4^{-q}\epsilon_0(1 + 2\|\widehat{p}\|). \tag{3.75}$$

It follows from (3.74), (3.75) and the construction of the program $(\bar{x}(t), \bar{y}(t))_{t=0}^{\infty}$ (see (3.64)-(3.68)) that for all integers $q \geq 1$

$$\int_0^{\sum_{i=1}^{q} T_i + q - 1} (w(b\bar{y}(t)) - w(b\widehat{x}))dt \geq \sum_{i=1}^{q} -4^{-i}\epsilon_0(2 + 4|\widehat{p}\|) \geq -(2 + 4\|\widehat{p}\|)2\epsilon_0.$$

Therefore $(\bar{x}(t), \bar{y}(t))_{t=0}^{\infty}$ is a good program. In view of Theorem 1.6

$$\lim_{t \to \infty} \bar{x}(t) = \widehat{x}$$

and there is $S_0 > 0$ such that

$$\|\bar{x}(t) - \widehat{x}\| \leq \epsilon_0 \text{ for all } t \geq S_0. \tag{3.76}$$

By (3.76), (3.54) and (3.71) which holds for all integers $q \geq 1$,

$$\|x^{(q)}(t) - \widehat{x}\| \leq \epsilon,\ t \in [0, T_q] \text{ for all sufficiently large natural numbers } q. \tag{3.77}$$

By Theorem 1.6 there is $S_1 > 0$ such that for all $T \geq S_1$

$$\mathrm{mes}([T, T + \tau_0] :\ \|\bar{y}(t) - \widehat{x}\| > \epsilon\}) \leq \epsilon. \tag{3.78}$$

By (3.78), (3.68) (which holds for all natural numbers q) and (3.57), for all sufficiently large natural numbers q and for all $S \in [0, T_q - \tau_0]$,

$$\mathrm{mes}(\{t \in [S, S + \tau_0] :\ \|y^{(q)}(t) - \widehat{x}\| > \epsilon\} \leq \epsilon.$$

This contradicts (P3) while (3.77) contradicts (3.59). The contradiction we have reached proves Lemma 3.10. □

4. PROOF of THEOREM 1.8

By Lemma 1.3 there is $M_1 > 0$ such that for each positive number T and each program $(x(t), y(t))_{t=0}^T$ satisfying $x(0) \le Me$ the following inequality holds:

$$x(t) \le M_1\mathbf{e} \text{ for all } t \in [0, T]. \tag{4.1}$$

By Lemma 3.3 there exists $M_2 > 0$ such that for each $z_0 \in R^n_+$, each $z_1 \in R^n_+$ satisfying $az_1 \le \Gamma d^{-1}$ and each number $T > k(\Gamma)$,

$$U(z_0, z_1, 0, T) \ge Tw(b\widehat{x}) - M_2. \tag{4.2}$$

By Lemma 3.4 there exists $M_3 > 0$ such that for each number $T > 0$ and each program $(x(t), y(t))_{t=0}^T$ satisfying $x(0) \le M_1 e$ the following inequality holds:

$$\int_0^T [w(by(t)) - w(b\widehat{y})]dt \le M_3. \tag{4.3}$$

By Lemma 3.10 there exist $\epsilon_1 > 0$, $L_2 > L$ such that for each number $T \ge L_1$ and each program $(x(t), y(t))_{t=0}^T$ which satisfies

$$\|x(0) - \widehat{x}\| \le \epsilon_1,\ \|x(T) - \widehat{x}\| \le \epsilon_1,$$

$$\int_0^T \delta(x(t), y(t), x'(t)) \le \epsilon_1$$

we have:

$$\|x(t) - \widehat{x}\| \le \epsilon \ \text{ for all } t \in [0, T]; \tag{4.4}$$

for each $S \in [0, T - L]$,

$$\text{mes}(\{t \in [S, S + L] :\ \|y(t) - \widehat{x}\| > \epsilon\}) \le \epsilon. \tag{4.5}$$

By Lemma 3.9 there exists

$$\gamma \in (0, \min\{1, \epsilon, \epsilon_1\}) \tag{4.6}$$

such that for each number $T > 2$ and each program $(x(t), y(t))_{t=0}^T$ which satisfies

$$\|x(0) - \widehat{x}\| \le \gamma,\ \|x(T) - \widehat{x}\| \le \gamma,$$

$$\int_0^T w(by(t))dt \ge U(x(0), x(T), 0, T) - \gamma \tag{4.7}$$

the following inequality holds:

$$\int_0^T \delta(x(t), y(t), x'(t))dt \le \epsilon_1. \tag{4.8}$$

By Lemma 3.8 there exists a natural number L_2 such that for each program

$$(x(t), y(t))_{t=0}^{L_2}$$

satisfying

$$x(0) \le M_1\mathbf{e},\ \int_0^{L_2} w(by(t))dt \ge L_2 w(b\widehat{x}) - M_2 - M_3 - 1 \tag{4.9}$$

there is $t \in [0, L_2]$ such that

$$\|x(t) - \widehat{x}\| \le \gamma. \tag{4.10}$$

Set

$$l = 2L_2 + 2L_1 + L,\ Q > 4\epsilon_1^{-1}(M_3 + M_2 + 2\|\widehat{p}\|nM_2). \tag{4.11}$$

Put

$$T_* = L_2 + L_1 + k(\Gamma) + 2 + Ql. \tag{4.12}$$

Assume that

$$T > 2T_*,\ z_0,\ z_1 \in R^n_+,\ z_0 \le Me,\ az_1 \le \Gamma d^{-1} \tag{4.13}$$

and that a program $(x(t), y(t))_{t=0}^T$ satisfies

$$x(0) = z_0,\ x(T) \ge z_1,\ \int_0^T w(by(t))dt \ge U(z_0, z_1, 0, T) - \gamma. \tag{4.14}$$

In view of (4.13) and (4.14), the relation (4.1) holds. By (4.14), (4.13), the choice of M_2 (see (4.2)), (4.12) and (4.6)

$$\int_0^T w(by(t))dt \ge U(z_0, z_1, 0, T) - \gamma \ge Tw(b\widehat{x}) - M_2 - 1. \tag{4.15}$$

It follows from the choice of M_3 (see (4.3)) and (4.1) that

$$\int_{L_2}^T [w(by(t)) - w(b\widehat{x})]dt \le M_3,\ \int_0^{T-L_2} [w(by(t)) - w(b\widehat{x})]dt \le M_3. \tag{4.16}$$

By (4.16) and (4.15),

$$\int_0^{L_2} [w(by(t)) - w(b\widehat{x})]dt \ge -M_2 - 1 - M_3, \tag{4.17}$$

$$\int_{T-L_2}^T [w(by(t)) - w(b\widehat{x})]dt \ge -M_2 - 1 - M_3. \tag{4.17}$$

It follows from (4.1), (4.17) and the choice of L_2 (see (4.9), (4.10)) that there exist

$$\tau_1 \in [0, L_2],\ \tau_2 \in [T - L_2, T] \tag{4.18}$$

such that

$$\|x(\tau_i) - \widehat{x}\| \le \gamma,\ i = 1, 2. \tag{4.19}$$

If $\|x(0) - \widehat{x}\| \le \gamma$, then we put $\tau_1 = 0$ and if $\|x(T) - \widehat{x}\| \le \gamma$, then we put $\tau_2 = T$. By (4.14) and Lemma 3.6,

$$\int_{\tau_1}^{\tau_2} w(by(t))dt \ge U(x(\tau_1), x(\tau_2), \tau_1, \tau_2) - \gamma. \tag{4.20}$$

In view of (4.19), (4.20) and the choice of γ (see (4.6)-(4.8)),

$$\int_{\tau_1}^{\tau_2} \delta(x(t), y(t), x'(t))dt \le \epsilon_1. \tag{4.21}$$

It follows from (4.21), (4.19), the choice of ϵ_1, L_2 (see (4.4)-(4.6)), (4.18), (4.13) and (4.22) that

$$\|x(t) - \widehat{x}\| \le \epsilon,\ t \in [\tau_1, \tau_2]$$

and if a number S satisfies $\tau_1 \le S \le \tau_2 - L$, then

$$\text{mes}(\{t \in [S, S + L] :\ \|y(t) - \widehat{x}\| > \epsilon\}) \le \epsilon.$$

Theorem 1.8 is proved.

5. PROOF of THEOREM 1.9

We may assume that $\epsilon < 1/4$. By Lemma 1.3 there is $M_2 > 0$ such that for each positive number T and each program $(x(t), y(t))_{t=0}^{T}$ satisfying $x(0) \le M_0 e$ the following inequality holds:

$$x(t) \le M_2\mathbf{e} \text{ for all } t \in [0, T]. \tag{5.1}$$

By Lemma 3.3 there exists $M_3 > 0$ such that for each $z_0 \in R^n_+$, each $z_1 \in R^n_+$ satisfying $az_1 \le \Gamma d^{-1}$ and each number $T > k(\Gamma)$,

$$U(z_0, z_1, 0, T) \ge Tw(b\widehat{x}) - M_3. \tag{5.2}$$

By Lemma 3.10 there exist $\epsilon_1 \in (0, \epsilon)$, $L_1 > L$ such that for each number $T \ge L_1$ and each program $(x(t), y(t))_{t=0}^{T}$ which satisfies

$$\|x(0) - \widehat{x}\| \le \epsilon_1,\ \|x(T) - \widehat{x}\| \le \epsilon_1, \tag{5.3}$$

$$\int_0^T \delta(x(t), y(t), x'(t))dt \le 2\epsilon_1,$$

the inequality

$$\|x(t) - \widehat{x}\| \le \epsilon,\ t \in [0, T] \tag{5.4}$$

holds and for each $S \in [0, T - L]$,

$$\text{mes}(\{t \in [S, S + L] :\ \|y(t) - \widehat{x}\| > \epsilon\}) \le \epsilon. \tag{5.5}$$

By Lemma 3.8 there exists a natural number L_2 such that for each program

$$(x(t), y(t))_{t=0}^{L_2}$$

satisfying

$$x(0) \le M_2 e,\ \int_0^{L_2} w(by(t))dt \ge L_2 w(b\widehat{x}) - M_1 - M_3 - 2 - 4\|\widehat{p}\|nM_2 \tag{5.6}$$

there is $t \in [0, L_2]$ such that

$$\|x(t) - \widehat{x}\| \le \epsilon_1. \tag{5.7}$$

Fix

$$l = 2L_2 + 2L_1 + 8,\ \text{a natural number } Q > 4\epsilon_1^{-1}(M_3 + M_1 + M_2 + 2\|\widehat{p}\|nM_2), \tag{5.8}$$

$$T_* > 8L + 8L_1 + 8L_2 + k(\Gamma).$$

Assume that

$$T > T_*,\ z_0,\ z_1 \in R^n_+,\ z_0 \le M\mathbf{e},\ az_1 \le \Gamma d^{-1} \tag{5.9}$$

and that a program $(x(t), y(t))_{t=0}^T$ satisfies

$$x(0) = z_0,\ x(T) \ge z_1,\ \int_0^T w(by(t))dt \ge U(z_0, z_1, 0, T) - M_1. \tag{5.10}$$

In view of (5.10), (5.9) and the choice of M_2, the relation (5.1) holds. By (5.10), (5.9), (5.8), the choice of M_3 (see (5.2)),

$$\int_0^T w(by(t))dt \ge Tw(b\widehat{x}) - M_3 - M_1. \tag{5.11}$$

By Lemma 2.1, (5.11) and (5.1),

$$\int_0^T \delta(x(t), y(t), x'(t))dt = \int_0^T (w(b\widehat{x}) - w(by(t))dt + \widehat{p}(x(0) - x(T))$$

$$\le M_3 + M_1 + 2\|\widehat{p}\|nM_2. \tag{5.12}$$

It is not difficult to se that there is a finite sequence of numbers $\{T_i\}_{i=0}^q$ such that $T_0 = 0$, $T_i < T_{i+1}$ for each integer i satisfying $0 \le i < q$, $T_q = T$, for each integer i satisfying $0 \le i < q$

$$\int_{T_i}^{T_{i+1}} \delta(x(t), y(t), x'(t))dt = \epsilon_1, \tag{5.13}$$

$$\int_{T_{q-1}}^{T_q} \delta(x(t), y(t), x'(t))dt \le \epsilon_1. \tag{5.14}$$

By (5.13), (5.14) and (5.12),

$$q\epsilon_1 \le M_3 + M_1 + 2\|\widehat{p}\|nM_2$$

and

$$q \le \epsilon_1^{-1}(M_3 + M_1 + 2\|\widehat{p}\|nM_2). \tag{5.15}$$

By Lemma 2.1, (5.13), (5.14) and (5.1) for each $i \in \{0, \dots, q-1\}$ and each $S_1, S_2 \in [T_i, T_{i+1}]$ such that $S_1 < S_2$

$$\int_{S_1}^{S_2} (w(by(t)) - w(b\widehat{x})dt = -\int_{S_1}^{S_2} \delta(x(t), y(t), x'(t))dt + \widehat{p}(x(S_1) - x(S_2))$$

$$\ge -1 - 2\|\widehat{p}\|nM_2. \tag{5.16}$$

Set

$$J = \{i \in \{0, \dots, q-1\} :\ T_{i+1} - T_i \ge 2L_2 + 2L_1\}. \tag{5.17}$$

Let $i \in J$. By (5.17), the choice of L_2 (see (5.6) and (5.7)), (5.1) and (5.16) there are numbers t_{i1}, t_{i2} such that

$$t_{i1} \in [T_i, L_2 + T_i],\ t_{i2} \in [T_{i+1} - L_2, T_{i+1}],\ \|x(t_{ij}) - \widehat{x}\| \le \epsilon_1,\ j = 1, 2. \tag{5.18}$$

By (5.18), (5.17), (5.13), (5.14) and the choice of ϵ_1, L_1 (see (5.3)-(5.5))

$$\|x(t) - \widehat{x}\| \le \epsilon,\ t \in [t_{i1}, t_{i2}] \tag{5.19}$$

and if $S \in [t_{i1}, t_{i2} - L]$, then

$$\mathrm{mes}(\{t \in [S, S + L] :\ \|y(t) - \widehat{x}\| > \epsilon\}) \le \epsilon. \tag{5.20}$$

Put

$$\mathcal{A} = \{[T_i, T_{i+1}] :\ i \in \{0, \dots, q-1\} \setminus J\} \cup \{[T_i, t_{i1}],\ [t_{i2}, T_{i+1}] :\ i \in J\}. \tag{5.21}$$

Clearly, the length of all the intervals belonging to $\mathcal{A}$ does not exceed $2L_2 + 2L_1 < l$.

By (5.15), (5.21) and (5.8) the number of elements of $\mathcal{A}$ does not exceed

$$4q \le 4\epsilon_1^{-1}(M_3 + M_1 + 2\|\widehat{p}\| n M_2) \le Q.$$

The inequalities above and (5.19) imply the validity of Theorem 1.9.

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