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APPLICATIONS OF OPIAL AND WIRTINGER INEQUALITIES ON ZEROS OF THIRD ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, for a third order differential equation, we will establish some new inequalities of Lyapunov type. These inequalities give implicit lower bounds on the distance between zeros of a nontrivial solution and also lower bounds for the spacing between zeros of a solution and/or its derivatives. The main results will be proved by making use of the Hölder inequality and some generalizations of Opial and Wirtinger type inequalities. Some examples are considered to illustrate the main results.

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1. INTRODUCTION

In this paper, we will establish some new Lyapunov type inequalities for the third-order differential equation

$$\left(r(t)\left|x''(t)\right|^{\gamma-1}x''\right)' + q(t)\left|x(t)\right|^{\gamma-1}x(t) = 0, \quad t \in I, \tag{1.1}$$

where we assume that I is a nontrivial interval of reals, $\gamma \geq 1$ and $r, q : I \rightarrow \mathbf{R}^{+}$ are nonnegative continuous measurable functions. By a solution of (1.1) on the interval $J \subseteq I$, we mean a nontrivial real-valued function $x \in C^2(J)$, which has the property that $r(t)\left|x''(t)\right|^{\gamma-1}x''(t) \in C^1(J)$ and satisfies equation (1.1) on J. We assume that (1.1) possesses such a nontrivial solution on I.

The nontrivial solution x of (1.1) is said to oscillate or to be oscillatory, if it has arbitrarily large zeros. By the Sturm Separation Theorem, it is known that the oscillation is an interval property, i.e., if there exists a sequence of subintervals $[\alpha_i, \beta_i]$ of $[t_0, \infty)$, as $i \rightarrow \infty$, such that for every i there exists a solution of (1.1) that has at least three zeros in $[\alpha_i, \beta_i]$. For recent oscillation and nonoscillation results for third order differential equations, we refer to the book [29].

Equation (1.1) is said to be right (left) disfocal in $[a, b]$ $(a < b)$ if the solutions of (1.1) with $x'(a) = 0$, $x(a) \neq 0$ $(x'(b) = 0$ and $x(b) \neq 0)$ do not have two zeros (counting multiplicities) in $(a, b]$ $([a, b))$. Equation (1.1) is disconjugate in $[a, b]$ if no

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nontrivial solution of (1.1) has more than two zeros (counting multiplicities). So that if equation (1.1) is disconjugate in $[a,b]$, then it is right disfocal in $[c,b]$ or left disfocal in $[a,c]$ for every $c \in (a,b)$. Equation (1.1) is said to be nonoscillatory on $[a,\infty)$ if there exists $c \in [a,\infty)$ such that this equation is disconjugate on $[c,d]$ for every $d > c$.

Lyapunov type inequalities yield implicit lower bounds on the distance between consecutive zeros of a nontrivial solution x and also give lower bounds for the distance between zeros of a solution $x(t)$ and/or its derivatives. The best known existence result of this type for the second order differential equations is due to Lyapunov [14]. This result states that: If $x(t)$ is a solution of the differential equation

$$x''(t) + q(t)x(t) = 0, \tag{1.2}$$

with $x(a) = x(b) = 0$ $(a < b)$ and $x(t) \neq 0$ for $t \in (a,b)$, then

$$\int_a^b q^+(t)dt > \frac{4}{(b-a)}, \tag{1.3}$$

where q is a real valued and continuous on a nontrivial interval of reals and $q^+ = \max\{q(t), 0\}$. Since the appearance of this inequality various proofs and generalizations or improvements have appeared in the literature for different types of equations. For contribution we refer the reader to the papers [4, 11, 13, 18, 19, 20, 21, 23, 24, 25, 26] and the references cited therein.

Our motivation for this paper comes from the papers of Parhi and Panigrahi [16, 17], Yang [31] and Cakmak [9]. In [16] the authors proved that if x is a nontrivial solution of the equation

$$x'''(t) + q(t)x(t) = 0, \tag{1.4}$$

with $x\,(a) = 0 = x\,(b)$, $x\,(t) \neq 0$, $t \in (a,b)$ and there exists a $d \in [a,b]$ such that $x''\,(d) = 0$, then

$$\int_a^b |q\,(t)|\,dt > \frac{4}{(b-a)^2}. \tag{1.5}$$

They also proved that if x is a solution of (1.4) with $x(a) = x(a') = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a,b)$ and $x''(t) \neq 0$ for $t \in (a,a')$, then

$$\int_a^b |q\,(t)|\,dt > \frac{4}{(b-a)^2}.$$

In [17] the authors used the concept of disfocality for third-order differential equations and gave a better bound than in (1.5) in some cases. In particular they proved that if x is a solution of (1.4) with $x(a) = 0 = x'(a)$, $x(b) = 0 = x'(b)$, and $x(t) \neq 0$ for $t \in (a,b)$, then

$$\int_a^b |q\,(t)|\,dt > \frac{16}{(b-a)^2}.$$

As a special case of the results proved by Yang [31] for higher order differential equations, one can deduce that if x is a solution of (1.4) with $x(a) = x(t_2) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a, t_2) \cup (t_2, b)$, then

$$\int_a^b |q(t)|\, dt > \frac{9}{2(b-a)^2}.$$

Also one can deduce for the results in [31] that if x is a solution of (1.4) with $x(a) = x(t_2) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a, t_2) \cup (t_2, b)$, then

$$\int_a^b |q(t)|\, dt > \frac{27}{2(b-a)^2}.$$

In [9] the author proved that if x is a solution of (1.4) with $x(a) = x(t_2) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a, t_2) \cup (t_2, b)$, then

$$\int_a^b |q(t)|\, dt > \frac{27}{4(b-a)^2}.$$

In this paper, we are concerned with the following two problems for the equation (1.1):

(i) obtain lower bounds for the spacing $\beta - \alpha$, where x is a solution of (1.1) satisfying $x(\alpha) = x'(\alpha) = x''(\beta) = 0$, or $x(\beta) = x'(\beta) = x''(\alpha) = 0$,

(ii) obtain lower bounds for the spacing $\beta - \alpha$, where x is a solution of (1.1) satisfying $x^{(i)}(\alpha) = 0 = x^{(i)}(\beta)$ for $i = 0, 1, 2$.

In particular as a special case of our results, we will prove that if x is a solution of (1.4) with $x^{(i)}(a) = 0 = x^{(i)}(b)$ for $i = 0, 1, 2$, and $x(t) \neq 0$ for $t \in (a, b)$, then

$$\int_a^b |q(t)|^2\, dt \geq \frac{256}{(b-a)^6}. \tag{1.6}$$

The paper is organized as follows: In Section 2, we will prove the main results by employing a technique (different from the techniques employed in the above mentioned papers) depends on the applications of the Hölder inequality and some generalizations of Wirtinger Opial type inequalities. In Section 3, we will discuss some special cases of the results to derive some results for the equation (1.4) and then give some illustrative examples. To the best of the author's knowledge this technique has not been employed before for the equation (1.1) or even on the special case (1.4).

2. MAIN RESULTS

The Wirtinger type inequality and its general form have been studied in the literature in various modifications both in the continuous and in the discrete setting. It has an extensive applications on partial differential and difference equations, harmonic analysis, approximations, number theory, optimization, convex geometry, spectral theory of differential and difference operators, and others (see [27, 28]). Also the inequalities of Opial types are the most important and fundamental integral inequalities in the analysis of qualitative properties of solutions of differential equations. For more details we refer the reader to the book [2]. In the following, we present the Wirtinger type inequality due to Agarwal et al. [1] and some generalizations of Opial's inequality due to Beesack and Das [5] and Agarwal and Pang [2] that we will need in the proof of the main results.

Theorem 2.1. *For $\mathbb{I}=[\alpha,\beta]$, $\gamma\geq 1$ is a positive integer and a positive function $\lambda\in C^1(\mathbb{I})$ with either $\lambda'(t)>0$ or $\lambda'(t)<0$ on $\mathbb{I}$, we have*

$$\int_\alpha^\beta \frac{\lambda^{\gamma+1}(t)}{|\lambda'(t)|^\gamma}\left|y'(t)\right|^{\gamma+1}dt\geq\frac{1}{(\gamma+1)^{\gamma+1}}\int_\alpha^\beta |\lambda'(t)|\left|y(t)\right|^{\gamma+1}dt, \tag{2.1}$$

for any $y\in C^1(\mathbb{I})$ with $y(\alpha)=0=y(\beta)$.

Remark 1. It is clear that Theorem 2.1 is satisfied for any function y satisfies the assumptions of theorem. So if $y(t)=x'(t)$ with $x'(\alpha)=0=\lambda(\beta)$ or $x'(\beta)=0=\lambda(\alpha)$, or $x'(\alpha)=0=x'(\beta)$ and $p(t)=\lambda'(t)$ we have the following inequality which gives a relation between $x'(t)$ and $x''(t)$ on the interval $[\alpha,\beta]$.

Corollary 2.1. *For $\mathbb{I}=[\alpha,\beta]$, and $\gamma\geq 1$ is a positive integer then we have*

$$\int_\alpha^\beta |r(t)|\left|x''(t)\right|^{\gamma+1}dt\geq\frac{1}{(\gamma+1)^{\gamma+1}}\int_\alpha^\beta |p(t)|\left|x'(t)\right|^{\gamma+1}dt, \tag{2.2}$$

for any $x\in C^2(I)$ with $x'(\alpha)=0=r(\beta)$, or $x'(\beta)=0=r(\alpha)$ or $x'(\alpha)=0=x'(\beta)$, where $r(t)$ and $p(t)$ satisfy the equation

$$(r(t)(\lambda'(t))^\gamma)'-(\gamma+1)p(t)\lambda^\gamma(t)=0, \tag{2.3}$$

for any function $\lambda(t)$ satisfyies $\lambda'(t)\neq 0$.

Remark 2. For illustration, we apply the inequality (2.2) with $x'(t)=\sin t$ in the interval $[0,\pi]$. If $p(t)=1$ and $\gamma=1$ and by choosing $r(t)=t^2$, we see the equation (2.3) is satisfied when $\lambda(t)=t$. So one can see that

$$\int_0^\pi t^2\cos^2 t dt\simeq 5.9531>0.392\,70\simeq\frac{1}{4}\int_0^\pi \sin^2 t dt.$$

Note also that the equation (2.3) holds if one chooses $r(t)=p(t)=1$, where in this case

$$\lambda(t)=\exp\left(\frac{\gamma+1}{\gamma}\right)^{\frac{1}{\gamma+1}}t.$$

In the following, we present the Opial-type inequalities that we will need in the proof of the main results.

Theorem 2.2 (Beesack and Das [5]). *If xis absolutely continuous on $[\alpha,\beta]$ with $x(\alpha)=0$ (or $x(\beta)=0$), and x does not change sign in (α,β), then the following inequality holds*

$$\int_\alpha^\beta B(t)\left|x(t)\right|^m \left|x'(t)\right|^n dt \le K(m,n)\int_\alpha^\beta A(t)\left|x'(t)\right|^{m+n} dt, \tag{2.4}$$

where m,n are real numbers such that $mn>0$ and $m+n>1$, A and B are nonnegative, measurable functions on (α,β) such that

$$\int_\alpha^t (A(s))^{\frac{-1}{m+n-1}}\, ds<\infty,$$

and

$$K(m,n):=\left(\frac{n}{n+m}\right)^{\frac{n}{n+m}}\left[\int_\alpha^\beta B^{\frac{n+m}{m}}(t)A^{-\frac{n}{m}}(t)\left(\int_\alpha^t (A(s))^{\frac{-1}{m+n-1}}\, ds\right)^{m+n-1} dt\right]^{\frac{m}{m+n}}. \tag{2.5}$$

If we replace $x(\alpha)=0$ by $x(\beta)=0$, then (2.4) holds where $K(m,n)$ is replaced by

$$K(m,n):=\left(\frac{n}{n+m}\right)^{\frac{n}{n+m}}\left[\int_\alpha^\beta B^{\frac{n+m}{m}}(t)A^{-\frac{n}{m}}(t)\left(\int_t^\beta (A(s))^{\frac{-1}{m+n-1}}\, ds\right)^{m+n-1} dt\right]^{\frac{m}{m+n}}. \tag{2.6}$$

Theorem 2.3 (Agarwal and Pang [2]). *Assume that the functions p and q are nonnegative and measurable on the interval (α,β), m,n are real numbers such that $\mu/m>1$, $x(t)\in C^{(n-1)}[\alpha,\beta]$ be such that $x^{(i)}(\alpha)=0$, $0\le k\le i\le n-1$ $(n\ge 1)$, $x^{(n-1)}(t)$ absolutely continuous on (α,β) and $x^{(n)}(t)$ does not change sign on (α,β). Then*

$$\int_\alpha^\beta q(t)\left|x^{(k)}(t)\right|^l\left|x^{(n)}(t)\right|^m dt \le K_1\left[\int_\alpha^\beta \vartheta(t)\left|x^{(n)}(t)\right|^\mu dt\right]^{(l+m)/\mu}, \tag{2.7}$$

where

$$K_1:=\frac{\left(\frac{m}{l+m}\right)^{\frac{m}{\mu}}}{((n-k-1)!)^l}\left[\int_\alpha^\beta \left(q^\mu(t)\vartheta^{-m}(t)\right)^{1/(\mu-m)}\left(P_{1,k}(t)\right)^{l(\mu-1)/(\mu-m)}dt\right]^{\frac{\mu-m}{\mu}}, \tag{2.8}$$

$$P_{1,k}(t):=\int_\alpha^t (t-s)^{(n-k-1)\mu/(\mu-1)}(\vartheta(s))^{-1/(\mu-1)}ds,$$

If we replace $x^{(i)}(\alpha)=0$ by $x^{(i)}(\beta)=0$, $0\le k\le i\le n-1$ $(n\ge 1)$, then (2.7) *holds where K_1 is replaced by K_2 which is given by*

$$K_2:=\frac{\left(\frac{m}{l+m}\right)^{\frac{m}{\mu}}}{((n-k-1)!)^l}\left[\int_\alpha^\beta \left(q^\mu(t)\vartheta^{-m}(t)\right)^{1/(\mu-m)}\left(P_{2,k}(t)\right)^{l(\mu-1)/(\mu-m)}dt\right]^{\frac{\mu-m}{\mu}}. \tag{2.9}$$

where

$$P_{2,k}(t):=\int_t^\beta (s-t)^{(n-k-1)\mu/(\mu-1)}(\vartheta(s))^{-1/(\mu-1)}ds.$$

Remark 3. For equation (1.1), we see that if $q(t) \geq 0$ and $x(t)$ be an eventually positive solution of (1.1), then $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$, or $x(t) > 0$, $x'(t) < 0$, $x''(t) > 0$. Also, if $x(t)$ is a solution of (1.1) with $x'(a) = 0$, $x(a) \neq 0$, then $x(t)$ cannot have a double zero in $(a, b]$. Indeed, if there is a $t_1 \in (a, b]$ such that $x(t_1) = 0 = x'(t_1)$ and $x(t_1) > 0$, $t \in [a, t_1)$, then $\left(r(t)\left|x''(t)\right|^{\gamma-1} x''(t)\right)' = -q(t)\left|x(t)\right|^{\gamma-1} x(t) \leq 0$, for $t \in [a, t_1)$, and hence $r(t)\left|x''(t)\right|^{\gamma-1} x''(t)$ is monotonic and decreasing. If there exists a $d \in (a, t_1)$ such that $x''(d) = 0$, then $x''(t) < 0$ for $t \in (d, t_1]$. Hence, $x'(t)$ is decreasing in $(d, t_1]$, a contradiction because $x'(t_1) = 0$. If $x(t) < 0$ for $t \in [a, t_1)$, then $x''(t)$ is increasing, and hence, $x''(t) > 0$, $t \in (d, t_1]$. Thus, $x'(t)$ is increasing. This is not possible because $x'(t_1) = 0$. Consequently, if $q(t) \geq 0$ and (1.1) is not right disfocal in $[a, b]$, then (1.1) has a solution $x(t)$ with $x'(a) = 0$, $x(a) \neq 0$ and $x(t)$ has only two simple zeros in $(a, b]$.

Now, we are ready to state and prove the main results by employing the inequalities (2.2), (2.4) and (2.7). For simplicity, we introduce the following notations:

$$
(2.10)\qquad \left.
\begin{aligned}
K_1^*(Q, r, P_{1,0}) &:= \left(\tfrac{1}{\gamma+1}\right)^{\frac{1}{\gamma+1}} \left[\int_\alpha^\beta \left(\tfrac{Q^{\gamma+1}(t)}{r(t)}\right)^{1/\gamma} (P_{1,0}(t))^\gamma dt\right]^{\frac{\gamma}{\gamma+1}}, \\
K_2^*(Q, r, P_{2,0}) &:= \gamma\left(\tfrac{1}{\gamma+1}\right)^{\frac{1}{\gamma+1}} \left[\int_\alpha^\beta \left(\tfrac{Q^{\gamma+1}(t)}{r(t)}\right)^{1/\gamma} (P_{2,0}(t))^\gamma dt\right]^{\frac{\gamma}{\gamma+1}}, \\
K_1^*(p, Q) &:= \left(\tfrac{2}{\gamma+1}\right)^{\frac{2}{\gamma+1}} \left[\int_\alpha^\beta \tfrac{Q^{\frac{\gamma+1}{\gamma-1}}(t)}{p^{\frac{2}{\gamma-1}}(t)} \left(\int_\alpha^t \left(\tfrac{1}{p(s)}\right)^{\frac{1}{\gamma}} ds\right)^\gamma dt\right]^{\frac{\gamma-1}{\gamma+1}}, \\
K_2^*(p, Q) &:= \left(\tfrac{2}{\gamma+1}\right)^{\frac{2}{\gamma+1}} \left[\int_\alpha^\beta \tfrac{Q^{\frac{\gamma+1}{\gamma-1}}(t)}{p^{\frac{2}{\gamma-1}}(t)} \left(\int_t^\beta \left(\tfrac{1}{p(s)}\right)^{\frac{1}{\gamma}} ds\right)^\gamma dt\right]^{\frac{\gamma-1}{\gamma+1}},
\end{aligned}
\right\}
$$

where

$$
P_{1,0}(t) := \int_\alpha^t \frac{(t-s)^{\gamma+1/\gamma}}{r^{1/\gamma}(s)} ds, \quad P_{2,0}(t) := \int_t^\beta \frac{(s-t)^{\gamma+1/\gamma}}{r^{1/\gamma}(s)} ds,
$$

and $r(t)$ and $p(t)$ are positive real functions satisfying the equation (2.3) for any positive function $\lambda(t)$.

Theorem 2.4. *If x is a nontrivial solution of (1.1) which satisfies $x(\alpha) = x'(\alpha) = x''(\beta) = 0$, then*

$$
(2.11)\qquad K_1^*(Q, r, P_{1,0}) + \gamma(\gamma+1)^{\gamma+1} K_1^*(p, Q) \geq 1,
$$

where in this case $Q(t) = \int_t^\beta q(s)ds$. If instead $x(\beta) = x'(\beta) = x''(\alpha) = 0$, then

$$
(2.12)\qquad K_2^*(Q, r, P_{2,0}) + \gamma(\gamma+1)^{\gamma+1} K_2^*(p, Q) \geq 1,
$$

where in this case $Q(t) = \int_\alpha^t q(s)ds$.

Proof. We prove (2.11). Multiplying (1.1) by x' and integrating by parts, we have

$$\begin{aligned}\int_\alpha^\beta & \left(r(t)\left|x'(t)\right|^{\gamma-1}x''(t)\right)' x'(t)dt \\ &= r(t)\left|x''(t)\right|^{\gamma-1}x'(t)x''(t)\Big|_\alpha^\beta - \int_\alpha^\beta r(t)\left|x''(t)\right|^{\gamma+1}dt \\ &= -\int_\alpha^\beta q(t)x'(t)\left|x(t)\right|^\gamma dt.\end{aligned}$$

Using the assumptions that $x'(\alpha) = x''(\beta) = 0$, and $Q(t) = \int_t^\beta q(s)ds$, we get that

$$\int_\alpha^\beta r(t)\left|x''(t)\right|^{\gamma+1}dt = \int_\alpha^\beta q(t)\left|x'(t)\right|\left|x(t)\right|^\gamma dt = -\int_\alpha^\beta Q'(t)x'(t)\left|x(t)\right|^\gamma dt. \tag{2.13}$$

Integrating by parts the right hand side, we see that

$$\begin{aligned}\int_\alpha^\beta & Q'(t)x'(t)\left|x(t)\right|^\gamma dt \\ &= Q(t)x'(t)\left|x(t)\right|^\gamma\big|_\alpha^\beta - \gamma\int_\alpha^\beta Q(t)\left|x(t)\right|^{\gamma-1}(x'(t))^2dt \\ &\quad - \int_\alpha^\beta \left|Q(t)\right|\left|x(t)\right|^\gamma\left|x''(t)\right|dt.\end{aligned}$$

Using the assumptions $Q(\beta) = 0$ and $x'(\alpha) = 0$, we see that

$$\begin{aligned}\int_\alpha^\beta Q'(t)x'(t)\left|x(t)\right|^\gamma dt &= -\gamma\int_\alpha^\beta Q(t)\left|x(t)\right|^{\gamma-1}\left|x'(t)\right|^2dt \\ &\quad - \int_\alpha^\beta \left|Q(t)\right|\left|x(t)\right|^\gamma\left|x''(t)\right|dt.\end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.13), we have

$$\int_\alpha^\beta r(t)\left|x''(t)\right|^{\gamma+1}dt \le \int_\alpha^\beta \left|Q(t)\right|\left|x(t)\right|^\gamma\left|x''(t)\right|dt + \gamma\int_\alpha^\beta \left|Q(t)\right|\left|x(t)\right|^{\gamma-1}\left|x'(t)\right|^2dt. \tag{2.15}$$

Applying the inequality (2.7) on the integral

$$\int_\alpha^\beta \left|Q(t)\right|\left|x(t)\right|^\gamma\left|x''(t)\right|dt,$$

with $q(t) = \left|Q(t)\right|$, $\vartheta(t) = r(t)$, $m = 1$, $k = 0$, $l = \gamma$, $n = 2$ and $\mu = \gamma + 1$, we get (note that $x''(t)$ is of one sign and $x(\alpha) = x'(\alpha) = 0$) that

$$\int_\alpha^\beta \left|Q(t)\right|\left|x(t)\right|^\gamma\left|x''(t)\right|dt \le K_1^*(Q, r, P_{1,0})\left[\int_\alpha^\beta r(t)\left|x''(t)\right|^{\gamma+1}dt\right], \tag{2.16}$$

where $K_1^*(Q, r, P_{1,0})$ is defined as in (2.10). Applying the inequality (2.4) on the integral

$$\gamma\int_\alpha^\beta \left|Q(t)\right|\left|x(t)\right|^{\gamma-1}\left|x'(t)\right|^2dt,$$

with $B(t) = Q(t)$, $A(t) = p(t)$, $m = \gamma - 1$ and $n = 2$, we see that

$$\gamma \int_\alpha^\beta |Q(t)| \, |x(t)|^{\gamma-1} \, |x'(t)|^2 \, dt \le \gamma K_1^*(Q, r) \int_\alpha^\beta p(t) \, |x'(t)|^{\gamma+1} \, dt, \tag{2.17}$$

where $K_1^*(Q, r)$ is defined as in (2.10) where $x(\alpha) = 0$. Applying the Wirtinger inequality (2.2) on the integral

$$\int_\alpha^\beta p(t) \, |x'(t)|^{\gamma+1} \, dt,$$

we see that

$$\int_\alpha^\beta p(t) \, |x'(t)|^{\gamma+1} \, dt \le (\gamma + 1)^{\gamma+1} \int_\alpha^\beta r(t) \, |x''(t)|^{\gamma+1} \, dt, \tag{2.18}$$

where $p(t)$ and $r(t)$ satisfying the equation (2.3) for any positive function $\lambda(t)$. Substituting (2.18) into (2.17), we have

$$\gamma \int_\alpha^\beta |Q(t)| \, |x(t)|^{\gamma-1} \, |x'(t)|^2 \, dt \le \gamma(\gamma + 1)^{\gamma+1} K_1^*(Q, r) \int_\alpha^\beta r(t) \, |x''(t)|^{\gamma+1} \, dt. \tag{2.19}$$

Substituting (2.16) and (2.19) into (2.15) and canceling the term $\left[\int_\alpha^\beta r(t) \, |x''(t)|^{\gamma+1} \, dt\right]$, we have

$$K_1^*(Q, r, P_{1,0}) + \gamma(\gamma + 1)^{\gamma+1} K_1^*(Q, r) \ge 1,$$

which is the desired inequality (2.11). The proof of (2.12) is similar using the integration by parts and the constants $K_1^*(Q, r, P_{1,0})$ and $K_1^*(Q, r)$ are replaced by $K_2^*(Q, r, P_{2,0})$ and $K_2^*(Q, r)$ which are defined as in (2.10). The proof is complete. □

From Theorem 2.4, we have the following result which gives the implicit lower bound of the spacing between β and α which satisfying $x^{(i)}(\alpha) = 0 = x^{(i)}(\beta)$ for $i = 0, 1, 2$.

Corollary 2.2. *If x is a nontrivial solution of (1.1) which satisfies*

$$x(\alpha) = x'(\alpha) = x''(\alpha) = 0,$$
$$x(\beta) = x'(\beta) = x''(\beta) = 0,$$

then (2.11) *and* (2.12) *hold.*

In the following, we will use the maximum of $|Q(t)|$ on $[\alpha, \beta]$ and establish a new formula which is different from (2.11) and (2.12). The results will be proved by making use of the Yang inequality [30] and the Cheung inequality [12]. The Yang inequality is given by

$$\int_\alpha^\beta h(t) \, |x(t)|^l \, |x'(t)|^m \, dt \le \frac{m}{m + l} (\beta - \alpha)^l \int_\alpha^\beta h(t) \, |x'(t)|^{m+l} \, dt, \tag{2.20}$$

where $l \geq 0$, $m \geq 1$, where $h(t)$ is a positive, bounded and nonincreasing function on $[\alpha, \beta]$ and $x(\alpha) = 0$, (or $x(\beta) = 0$). The Cheung inequality is given by

$$\int_{\alpha}^{\beta} h(t) \left|x(t)\right|^{l} \left|x^{(n)}(t)\right|^{m} dt \leq c_n (\beta - \alpha)^{nl} \int_{\alpha}^{\beta} h(t) \left|x^{(n)}(t)\right|^{m+l} dt, \tag{2.21}$$

where $l + m > 1$, $h(t)$ is a positive, bounded and nonincreasing function on $[\alpha, \beta]$ and $x^{(i)}(\alpha) = 0$, for $i = 0, 1, \ldots, n-1$ (or $x^{(i)}(\beta) = 0$, for $i = 0, 1, \ldots, n-1$) and

$$c_n = \frac{m^{\frac{m}{l+m}}}{(n!)^l (l+m)} \left[\frac{n(1 - \frac{1}{l+m})}{n - \frac{1}{l+m}}\right]^{l(1-\frac{1}{l+m})}.$$

Theorem 2.5. *Assume that $r(t)$ is a nonincreasing function. If x is a nontrivial solution of (1.1) which satisfies $x(\alpha) = x'(\alpha) = x''(\beta) = 0$, then*

$$\frac{(\beta - \alpha)^{2\gamma}}{r(\beta)} \max_{\alpha \leq t \leq \beta} |Q(t)| \left(\frac{1}{\gamma+1} \frac{\Gamma^2\left((\gamma+2)/2\right)}{\Gamma(\gamma+2)} + \left(\frac{2\gamma}{2\gamma+1}\right)^{\frac{\gamma^2}{\gamma+1}} \right) \geq 1. \tag{2.22}$$

where in this case $Q(t) = \int_t^{\beta} q(s)ds$. If instead $x(\beta) = x'(\beta) = x''(\alpha) = 0$, then

$$\frac{(\beta - \alpha)^{2\gamma}}{r(\beta)} \max_{\alpha \leq t \leq \beta} |Q(t)| \left(\frac{1}{\gamma+1} \frac{\Gamma^2\left((\gamma+2)/2\right)}{\Gamma(\gamma+2)} + \left(\frac{2\gamma}{2\gamma+1}\right)^{\frac{\gamma^2}{\gamma+1}} \right) \geq 1. \tag{2.23}$$

where in this case $Q(t) = \int_{\alpha}^{t} q(s)ds$.

Proof. We prove (2.22). Multiplying (1.1) by x' and proceeding as in Theorem 2.4 to obtain

$$\begin{aligned} \int_{\alpha}^{\beta} r(t) \left|x''(t)\right|^{\gamma+1} dt &\leq \int_{\alpha}^{\beta} |Q(t)| \left|x(t)\right|^{\gamma} |x''(t)| \, dt \\ &\quad + \gamma \int_{\alpha}^{\beta} Q(t) \left|x(t)\right|^{\gamma-1} \left|x'(t)\right|^{2} dt \\ &\leq \max_{\alpha \leq t \leq \beta} \left|\frac{Q(t)}{r(\beta)}\right| \int_{\alpha}^{\beta} |r(t)| \left|x(t)\right|^{\gamma} |x''(t)| \, dt \\ &\quad + \gamma \max_{\alpha \leq t \leq \beta} |Q(t)| \int_{\alpha}^{\beta} \left|x(t)\right|^{\gamma-1} \left|x'(t)\right|^{2} dt, \end{aligned} \tag{2.24}$$

where $r(t)$ is a nonincreasing function. Applying the inequality (2.21) on the integral

$$\int_{\alpha}^{\beta} |r(t)| \left|x(t)\right|^{\gamma} |x''(t)| \, dt,$$

with $h(t) = r(t)$, $m = 1$, $l = \gamma \geq 1$ and $n = 2$, we see that

$$\int_{\alpha}^{\beta} |r(t)| \left|x(t)\right|^{\gamma} |x''(t)| \, dt \leq \left(\frac{2\gamma}{2\gamma+1}\right)^{\frac{\gamma^2}{\gamma+1}} (\beta - \alpha)^{2\gamma} \left[\int_{\alpha}^{\beta} r(t) \left|x''(t)\right|^{\gamma+1} dt\right], \tag{2.25}$$

Applying the inequality (2.20) on the integral

$$\int_{\alpha}^{\beta} \left|x(t)\right|^{\gamma-1} \left|x'(t)\right|^{2} dt,$$

with $l = \gamma - 1$, $m = 2$, $p(t) = 1$ and $x(\alpha) = 0$, we see that

$$\int_\alpha^\beta |x(t)|^{\gamma-1} |x'(t)|^2 \, dt \le \frac{2}{\gamma+1} (\beta - \alpha)^{\gamma-1} \int_\alpha^\beta |x'(t)|^{\gamma+1} \, dt. \tag{2.26}$$

Applying the Wirtinger inequality

$$\frac{(b-a)^{\gamma+1}\Gamma^2\left((\gamma+2)/2\right)}{2\Gamma(\gamma+2)} \int_a^b (y'(t))^{\gamma+1} dt \ge \int_a^b y^{\gamma+1}(t) dt, \tag{2.27}$$

due to Agarwal and Pang [3] where $y(a) = y(b) = 0$, with $y(t) = x'(t)$ on the integral $\int_\alpha^\beta |x'(t)|^{\gamma+1} \, dt$, (note that $x'(\alpha) = x'(\beta) = 0$), we see that

$$\int_\alpha^\beta |x'(t)|^{\gamma+1} \, dt \le \frac{(\beta-\alpha)^{\gamma+1}\Gamma^2\left((\gamma+2)/2\right)}{2\Gamma(\gamma+2)\, r(\beta)} \int_\alpha^\beta r(t) |x''(t)|^{\gamma+1} \, dt, \tag{2.28}$$

where $r(t)$ is a nonincreasing function. Substituting (2.28) into (2.26), we have

$$\begin{aligned} &\int_\alpha^\beta |x(t)|^{\gamma-1} |x'(t)|^2 \, dt \\ &\le \frac{\gamma}{\gamma+1} \frac{(\beta-\alpha)^{\gamma+1}\Gamma^2\left((\gamma+2)/2\right)}{\Gamma(\gamma+2)\, r(\beta)} (\beta-\alpha)^{2\gamma} \int_\alpha^\beta r(t) |x''(t)|^{\gamma+1} \, dt. \end{aligned} \tag{2.29}$$

Substituting (2.25) and (2.29) into (2.15) and canceling the term

$$\left[\int_\alpha^\beta r(t) |x''(t)|^{\gamma+1} \, dt\right],$$

we have

$$\frac{(\beta-\alpha)^{2\gamma}}{r(\beta)} \max_{\alpha \le t \le \beta} |Q(t)| \left(\frac{1}{\gamma+1} \frac{\Gamma^2\left((\gamma+2)/2\right)}{\Gamma(\gamma+2)} + \left(\frac{2\gamma}{2\gamma+1}\right)^{\frac{\gamma^2}{\gamma+1}} \right) \ge 1,$$

which is the desired inequality (2.22). The proof of (2.23) is similar. The proof is complete. □

In the following, we apply an inequality due to Boyd [6] and the Hölder inequality to obtain results similar to Theorem 2.1. The Boyd inequality states that: if $x \in C^1[a,b]$ with $x(a) = 0$ (or $x(b) = 0$), then

$$\int_a^b |x(t)|^\nu |x'(t)|^\eta \, dt \le N(\nu,\eta,s)(b-a)^\nu \left(\int_a^b |x'(t)|^s \, dt\right)^{\frac{\nu+\eta}{s}}, \tag{2.30}$$

where $\nu > 0$, $s > 1$, $0 \le \eta < s$,

(2.31)

$$N(\nu,\eta,s) := \frac{(s-\eta)\,\nu^\nu}{(s-1)(\nu+\eta)\left(I(\nu,\eta,s)\right)^\nu} \sigma^{\nu+\eta-s}, \;\; \sigma := \left\{\frac{\nu(s-1)+(s-\eta)}{(s-1)(\nu+\eta)}\right\}^{\frac{1}{s}},$$

and

$$I(\nu,\eta,s) := \int_0^1 \left\{1 + \frac{s(\eta-1)}{s-\eta} t\right\}^{-(\nu+\eta+s\nu)/s\nu} [1+(\eta-1)t] t^{1/\nu-1} dt.$$

Note that the inequality (2.30) has immediate application to the case where $x(a) = x(b) = 0$. Choose $c = (a+b)/2$ and apply (2.30) to $[a,c]$ and $[c,b]$ and then add we obtain

$$\int_a^b |x(t)|^\nu |x'(t)|^\eta \, dt \leq N(\nu,\eta,s) \left(\frac{b-a}{2}\right)^\nu \left(\int_a^b |x'(t)|^s \, dt\right)^{\frac{\nu+\eta}{s}}, \tag{2.32}$$

where $N(\nu,\eta,s)$ is defined as in (2.31).

Theorem 2.6. *Assume that $r(t)$ is a nonincreasing function. If x is a nontrivial solution of* (1.1) *which satisfies $x^{(i)}(\alpha) = x^{(i)}(\beta) = 0$ for $i = 0,1,2$, then*

$$(\beta-\alpha)^{(2\gamma+1)(\gamma+1)} \int_\alpha^\beta |q(t)|^{\gamma+1} \, dt \geq \frac{\left(2\Gamma\left(\gamma+2\right) r(\beta)\right)^{\gamma+1}}{\left(\Gamma\left(\frac{\gamma+2}{2}\right)\right)^{2(\gamma+1)} N^\gamma(\gamma+1, \frac{\gamma+1}{\gamma}, \gamma+1)}. \tag{2.33}$$

Proof. Proceeding as in Theorem 2.4 by multiplying (1.1) by x', integrating by parts and using the assumptions that $x''(\alpha) = x''(\beta) = 0$, to get that

$$\int_\alpha^\beta r(t) \left|x''(t)\right|^{\gamma+1} dt = \int_\alpha^\beta q(t) \left|x'(t)\right| |x(t)|^\gamma \, dt. \tag{2.34}$$

Applying the Hölder inequality

$$\int_a^b |f(t)g(t)| \, dt \leq \left[\int_a^b |f(t)|^l \, dt\right]^{\frac{1}{l}} \left[\int_a^b |g(t)|^k \, dt\right]^{\frac{1}{k}},$$

on the term $\int_\alpha^\beta q(t) \left|x'(t)\right| |x(t)|^\gamma \, dt$ with $l = \gamma+1$ and $k = (\gamma+1)/\gamma$, we see that

$$\begin{aligned} \int_\alpha^\beta |q(t)| \, |x(t)|^\gamma \left|x'(t)\right| dt \;\; &\leq \;\; \left(\int_\alpha^\beta |q(t)|^{\gamma+1} \, dt\right)^{\frac{1}{\gamma+1}} \\ &\times \left(\int_\alpha^\beta |x(t)|^{(\gamma+1)} \left|x'(t)\right|^{\frac{\gamma+1}{\gamma}} dt\right)^{\frac{\gamma}{\gamma+1}}. \end{aligned} \tag{2.35}$$

Applying the Boyd inequality (2.32) with $\nu = (\gamma+1)$, $\eta = (\gamma+1)/\gamma$ and $s = \gamma+1$, we obtain

$$\begin{aligned} &\int_\alpha^\beta |x(t)|^{(\gamma+1)} \left|x'(t)\right|^{\frac{\gamma+1}{\gamma}} dt \\ &\leq N(\gamma+1, \frac{\gamma+1}{\gamma}, \gamma+1)(\frac{\beta-\alpha}{2})^{(\gamma+1)} \left(\int_a^b \left|x'(t)\right|^{\gamma+1} dt\right)^{\frac{(\gamma+1)}{\gamma}}. \end{aligned} \tag{2.36}$$

Substituting into (2.35), we have

$$\begin{aligned} &\int_\alpha^\beta |q(t)| \left|x'(t)\right| |x(t)|^\gamma \, dt \\ &\leq \left(N(\gamma+1, \frac{\gamma+1}{\gamma}, \gamma+1)(\frac{\beta-\alpha}{2})^{(\gamma+1)}\right)^{\frac{\gamma}{\gamma+1}} \left(\int_\alpha^\beta |q(t)|^{\gamma+1} \, dt\right)^{\frac{1}{\gamma+1}} \\ &\times \int_a^b \left|x'(t)\right|^{\gamma+1} dt. \end{aligned} \tag{2.37}$$

Applying the Wirtinger inequality (2.27) with $y(t) = x'(t)$ on the integral $\int_\alpha^\beta |x'(t)|^{\gamma+1}\,dt$, (note that $x'(\alpha) = x'(\beta) = 0$), we see that

$$\int_\alpha^\beta |x'(t)|^{\gamma+1}\,dt \le \frac{(\beta-\alpha)^{\gamma+1}\Gamma^2\left((\gamma+2)/2\right)}{2\Gamma\left(\gamma+2\right)r(\beta)} \int_\alpha^\beta r(t)\left|x''(t)\right|^{\gamma+1}dt, \tag{2.38}$$

where $r(t)$ is a nonincreasing function. Substituting (2.38) into (2.37), we have

$$\begin{aligned}
&\int_\alpha^\beta |q(t)|\,|x'(t)|\,|x(t)|^\gamma\,dt \\
&\le \left(N(\gamma+1,\frac{\gamma+1}{\gamma},\gamma+1)(\beta-\alpha)^{(\gamma+1)}\left(\frac{1}{2}\right)^{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \\
&\quad \times \frac{(\beta-\alpha)^{\gamma+1}\Gamma^2\left((\gamma+2)/2\right)}{2\Gamma\left(\gamma+2\right)r(\beta)}\left(\int_\alpha^\beta |q(t)|^{\gamma+1}\,dt\right)^{\frac{1}{\gamma+1}} \\
&\quad \times \int_\alpha^\beta r(t)\left|x''(t)\right|^{\gamma+1}dt.
\end{aligned} \tag{2.39}$$

Substituting (2.39) into (2.34), we obtain

$$(\beta-\alpha)^{(2\gamma+1)(\gamma+1)}\int_\alpha^\beta |q(t)|^{\gamma+1}\,dt \ge \frac{\left(\frac{2^{\gamma+1}\Gamma(\gamma+2)r(\beta)}{\Gamma^2((\gamma+2)/2)}\right)^{\gamma+1}}{N^\gamma(\gamma+1,\frac{\gamma+1}{\gamma},\gamma+1)},$$

which is the desired inequality (2.33). The proof is complete. □

The inequality (2.30) has immediate application when $\eta = s$, to the case where $x(a) = x(b) = 0$. In this case the equation (2.30) becomes

$$\int_a^b |x(t)|^\nu\,|x'(t)|^\eta\,dt \le L(\nu,\eta)(\frac{b-a}{2})^\nu\left(\int_a^b |x'(t)|^\eta\,dt\right)^{\frac{\nu+\eta}{\eta}}, \tag{2.40}$$

where

$$L(\nu,\eta) := \frac{\eta\nu^\eta}{\nu+\eta}\left(\frac{\nu}{\nu+\eta}\right)^{\frac{\nu}{\eta}}\left(\frac{\Gamma\left(\frac{\eta+1}{\eta}+\frac{1}{\nu}\right)}{\Gamma\left(\frac{\eta+1}{\eta}\right)\Gamma\left(\frac{1}{\nu}\right)}\right)^\nu, \tag{2.41}$$

and Γ is the Gamma function. Follows the proof of Theorem 2.6 and applying the inequality (2.40) instead of the inequality (2.30), we have the following result.

Theorem 2.7. *If x is a nontrivial solution of* (1.1) *which satisfies $x^{(i)}(\alpha) = x^{(i)}(\beta) = 0$ for $i = 0, 1, 2$, then*

$$(\beta-\alpha)^{(2\gamma+1)(\gamma+1)}\int_\alpha^\beta |q(t)|^{\gamma+1}\,dt \ge \frac{\left(2^{\gamma+1}\Gamma\left(\gamma+2\right)r(\beta)\right)^{\gamma+1}}{\left(\Gamma\left(\frac{\gamma+2}{2}\right)\right)^{2(\gamma+1)}L^\gamma(\gamma+1,\gamma+1)}, \tag{2.42}$$

where L is defined as in (2.41).

3. DISCUSSIONS AND EXAMPLES

In this section, we present some special cases of the results obtained in Section 2 and also give some illustrative examples. We begin by Theorem 2.1 and consider the case when $r(t) = 1$, and $p(t) = 1$. Using the definitions of the functions $P_{1,0}$ and $P_{2,0}$, and putting $r(t) = 1$, we see after integration that

$$\begin{aligned} P_{1,0}(t) &:= \int_\alpha^t (t-s)^{(\gamma+1)/\gamma} ds = \frac{\gamma}{2\gamma+1}(t-\alpha)^{(2\gamma+1)/\gamma}, \\ P_{2,0}(t) &:= \int_t^\beta (t-s)^{(\gamma+1)/\gamma} ds = \frac{\gamma}{2\gamma+1}(\beta-t)^{(2\gamma+1)/\gamma}. \end{aligned}$$

This gives us that

$$K_1^*(Q,1,P_{1,0}) = u_1, \quad K_2^*(Q,1,P_{2,0}) = u_2, \quad K_1^*(1,Q) = u_1^*, \quad K_2^*(1,Q) = u_2^*,$$

where

$$\begin{aligned} u_1 &:= \left(\frac{\gamma}{2\gamma+1}\right)^{\frac{\gamma^2}{\gamma+1}} \left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma+1}} \left[\int_\alpha^\beta Q^{\frac{\gamma+1}{\gamma}}(t)(t-\alpha)^{(2\gamma+1)}dt\right]^{\frac{\gamma}{\gamma+1}}, \\ u_2 &:= \left(\frac{\gamma}{2\gamma+1}\right)^{\frac{\gamma^2}{\gamma+1}} \left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma+1}} \left[\int_\alpha^\beta Q^{\frac{\gamma+1}{\gamma}}(t)(\beta-t)^{(2\gamma+1)}dt\right]^{\frac{\gamma}{\gamma+1}}, \\ u_1^* &:= \left(\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma+1}} \left[\int_\alpha^\beta Q^{\frac{\gamma+1}{\gamma-1}}(t)\,(t-\alpha)^\gamma\, dt\right]^{\frac{\gamma-1}{\gamma+1}}, \\ u_2^* &:= \left(\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma+1}} \left[\int_\alpha^\beta Q^{\frac{\gamma+1}{\gamma-1}}(t)\,(\beta-t)^\gamma\, dt\right]^{\frac{\gamma-1}{\gamma+1}}. \end{aligned}$$

This leads to the following results for the equation

$$\left(|x''(t)|^{\gamma-1} x''\right)' + q(t)\,|x(t)|^{\gamma-1}\,x(t) = 0, \quad \alpha \le t \le \beta. \tag{3.1}$$

Theorem 3.1. *If x is a nontrivial solution of* (3.1) *which satisfies $x(\alpha) = x'(\alpha) = x''(\beta) = 0$, then*

$$u_1 + \gamma(\gamma+1)^{\gamma+1}u_1^* \ge 1, \quad Q(t) = \int_t^\beta q(s)ds. \tag{3.2}$$

If instead $x(\beta) = x'(\beta) = x''(\alpha) = 0$, then

$$u_2 + \gamma(\gamma+1)^{\gamma+1}u_2^* \ge 1, \quad Q(t) = \int_\alpha^t q(s)ds. \tag{3.3}$$

Remark 4. By using the maximum of $|Q|$ on $[\alpha, \beta]$ in (3.2) and (3.3) and integrating, we have the following result.

Corollary 3.1. *If x is a nontrivial solution of* (3.1) *which satisfies $x(\alpha) = x'(\alpha) = x''(\beta) = 0$, then*

$$1 < \left(\frac{(\beta-\alpha)^{2\gamma}}{\gamma+1} \left(\frac{1}{2}\right)^{\frac{\gamma}{\gamma+1}} \left(\frac{\gamma}{2\gamma+1}\right)^{\frac{\gamma^2}{\gamma+1}} + \gamma(\gamma+1)^{\gamma} 2^{\frac{2}{\gamma+1}} (\beta-\alpha)^{\gamma-1} \right)$$

$$\times \max_{\alpha \le t \le \beta} \left| \int_t^{\beta} q(s)ds \right|. \tag{3.4}$$

If instead $x(\beta) = x'(\beta) = x''(\alpha) = 0$, then

$$1 < \left(\frac{(\beta-\alpha)^{\gamma}}{\gamma+1} \left(\frac{1}{2}\right)^{\frac{\gamma}{\gamma+1}} \left(\frac{\gamma}{2\gamma+1}\right)^{\frac{\gamma^2}{\gamma+1}} + \gamma(\gamma+1)^{\gamma} 2^{\frac{2}{\gamma+1}} (\beta-\alpha)^{\gamma-1} \right)$$

$$\times \max_{\alpha \le t \le \beta} \left| \int_{\alpha}^{t} q(s)ds \right|. \tag{3.5}$$

Remark 5. When $\gamma = 1$, the conditions (3.4) and (3.5) reduce to

$$\max_{\alpha \le t \le \beta} \left| \int_t^{\beta} q(s)ds \right| > \frac{1}{\frac{\sqrt{44}}{88}(\beta-\alpha)^2 + 4},$$

when $x(\alpha) = x'(\alpha) = x''(\beta) = 0$, and

$$\max_{\alpha \le t \le \beta} \left| \int_{\alpha}^{t} q(s)ds \right| > \frac{1}{\frac{\sqrt{44}}{88}(\beta-\alpha)^2 + 4},$$

when $x(\beta) = x'(\beta) = x''(\alpha) = 0$, where $x(t)$ is a solution of the equation

$$x'''(t) + q(t)x(t) = 0, \quad \alpha \le t \le \beta. \tag{3.6}$$

From Theorem 2.6, when $\gamma = 1$ and $r(t) = 1$, we have the following result.

Corollary 3.2. *If x is a solution of* (3.6) *which satisfies $x(\alpha) = x'(\alpha) = x''(\beta) = 0$, then*

$$\max_{\alpha \le t \le \beta} \left| \int_t^{\beta} q(s)ds \right| \ge \frac{1}{\left(\frac{1}{16}\pi + \frac{1}{3}\sqrt{2}\sqrt{3}\right)(\beta-\alpha)^2}.$$

If instead $x(\beta) = x'(\beta) = x''(\alpha) = 0$, then

$$\max_{\alpha \le t \le \beta} \left| \int_{\alpha}^{t} q(s)ds \right| \ge \frac{1}{\left(\frac{1}{16}\pi + \frac{1}{3}\sqrt{2}\sqrt{3}\right)(\beta-\alpha)^2}.$$

where in this case $Q(t) = \int_{\alpha}^{t} q(s)ds$.

As a special case of Theorem 2.7, if $r(t) = 1$ and $\gamma = 1$ and $\gamma = 2$, we have the following result respectively.

Corollary 3.3. *If x is a nontrivial solution of* (3.6) *which satisfies $x^{(i)}(\alpha) = x^{(i)}(\beta) = 0$ for $i = 0, 1, 2$, then*

$$\int_{\alpha}^{\beta} |q(t)|^2 \, dt \ge \frac{256}{(\beta-\alpha)^6}. \tag{3.7}$$

Corollary 3.4. *If x is a nontrivial solution of* (1.1) *which satisfies $x^{(i)}(\alpha) = x^{(i)}(\beta) = 0$, for $i = 0, 1, 2,$ then*

$$\int_{\alpha}^{\beta} |q(t)|^3 \, dt \geq \frac{4194\,304}{19\,683} \frac{\pi^{12}}{\left(\Gamma\left(\frac{2}{3}\right)\right)^{18}} \frac{1}{(\beta - \alpha)^{15}}. \tag{3.8}$$

One can also use Theorems 2.2–2.6 to get similar results and due to the limited space the details are let to the reader. The following examples illustrate the results.

Example 1. Consider the equation

$$x'''(t) + \lambda \cos^2(t)\, x(t) = 0, \quad 0 \leq t \leq \pi, \tag{3.9}$$

where $\lambda \geq 1$ is a positive constant. If x is a solution of (3.9) with $x^{(i)}(0) = x^{(i)}(\pi) = 0$ for $i = 0, 1, 2,$ then

$$\lambda^2 \int_0^{\pi} \cos^4 t dt = \lambda^2 \frac{3\pi}{8} \geq \frac{256}{\pi^6}, \qquad \text{for } \lambda \geq 1,$$

and (3.7) holds if $\lambda \geq 1$.

Example 2. Consider the equation

$$\left((x''(t))^2\right)' + \lambda \cos^2(t)\, x^2(t) = 0, \quad 0 \leq t \leq \pi, \tag{3.10}$$

where $\lambda \geq 1$ is a positive constant. If x is a solution of (3.10) with $x^{(i)}(0) = x^{(i)}(\pi) = 0$ for $i = 0, 1, 2,$ then

$$\lambda^3 \int_0^{\pi} \cos^6 t dt = \lambda^3 \frac{5}{16}\pi \geq \frac{4194\,304}{19\,683\pi^3} \frac{1}{\left(\Gamma\left(\frac{2}{3}\right)\right)^{18}}, \ \text{for } \lambda \geq 1,$$

and (3.8) holds if $\lambda \geq 1$. Note that

$$\frac{5}{16}\pi - \frac{4194\,304}{19\,683} \frac{\pi^{12}}{\left(\Gamma\left(\frac{2}{3}\right)\right)^{18}} \frac{1}{\pi^{15}} = 0.952\,42 > 0.$$

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