Dynamic Systems and Applications 20 (2011) 531-540

RIGHT DELTA DISCRETE FRACTIONALITY

GEORGE A. ANASTASSIOU

Department of Mathematical Sciences, University of Memphis
Memphis, TN 38152, U.S.A.,
ganastss@memphis.edu

ABSTRACT. Here we define a Caputo like right discrete delta fractional difference and we produce a right discrete delta fractional Taylor formula for the first time. We estimate the remainder. Then we produce related right discrete delta fractional Ostrowski, Poincaré and Sobolev type inequalities.

AMS (MOS) Subject Classification. Primary: 39A12, 34A25, 26A33, Secondary: 26D15, 26D20.

Keywords and phrases. delta discrete fractional calculus, delta discrete inequalities.

1. INTRODUCTION AND BACKGROUND

Here we work on the time scale $\mathbb{T} = a + \mathbb{Z}$, where $a \in \mathbb{R}$. We consider functions $f : (a + \mathbb{Z}) \to \mathbb{R}$. If a function f is defined on a subset of $a + \mathbb{Z}$, then one can extend it to all of $a + \mathbb{Z}$, by assigning zero values to f on the complement with respect to $a + \mathbb{Z}$ of that subset. Let $t \in \mathbb{R}$, $n \in \mathbb{N}$, the falling factorial is defined by $t^{(n)} = t(t-1)\cdots(t-n+1) = \prod_{i=0}^{n-1}(t-i)$, and in general $t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, where $\alpha \in \mathbb{R}$, with Γ the gamma function $\Gamma(\nu) = \int_0^\infty e^{-t}t^{\nu-1}dt$, $\nu > 0$; $t^{(0)} = 1$.

From the time scales theory [5, p. 29], [6], we know that the delta integral on $(a + \mathbb{Z})$

$$\int_{a^*}^{b^*} f(t)\,\Delta t = \sum_{t=a^*}^{b^*-1} f(t), \quad a^* < b^*, \tag{1}$$

$a^*, b^* \in (a + \mathbb{Z})$, $f : (a + \mathbb{Z}) \to \mathbb{R}$.

Let $t \in (a + \mathbb{Z})$, then the forward difference

$$\Delta f(t) := f(t+1) - f(t) = f^{\Delta}(t),$$

the delta time scale derivative, see [5, p. 5], and

$$\Delta^k f(t) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} f(t+l) = f^{\Delta^k}(t),$$

the kth order delta time scale derivative, see [5, p. 14].

Received October 1, 2011

Notice here that if f is restricted on $[a^*, b^*] \cap (a+\mathbb{Z})$, then $\Delta^k f$ runs on $[a^*, b^* - k] \cap (a+\mathbb{Z})$.

For a general time scale $\mathbb{T}$, see [5, p. 38], we define

$$h_k : \mathbb{T}^2 \to \mathbb{R},\ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ h_0(t,s) = 1,\ \forall\ s,t \in \mathbb{T}, \tag{2}$$

$$h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\,\Delta\tau,\ \forall\ s,t \in \mathbb{T}.$$

We have that the delta derivative $h_k^{\Delta}(t,s) = h_{k-1}(t,s)$, $k \in \mathbb{N}$, $t \in \mathbb{T}^k$, (for the definitions of $\mathbb{T}_k$, $\mathbb{T}^k$, see [5, p. 331, p. 2], respectively) and $h_1(t,s) = t - s$, $\forall\ s,t \in \mathbb{T}$.

Notice here that

$$(a+\mathbb{Z}) = (a+\mathbb{Z})_k = (a+\mathbb{Z})^k. \tag{3}$$

We need

Lemma 1.1. *On $(a+\mathbb{Z})$ we have*

$$h_k(t,s) = \frac{(t-s)^{(k)}}{k!},\ \forall\ k \in \mathbb{N}_0. \tag{4}$$

Lemma 1.2. *It holds on $(a+\mathbb{Z})$ that*

$$\left\{\frac{(t-s)^{(k+1)}}{(k+1)!}\right\}^{\Delta_t} = \frac{(t-s)^{(k)}}{k!},\quad k \in \mathbb{N}_0. \tag{5}$$

Proof. We have that

$$\begin{aligned}
\left\{\frac{(t-s)^{(k+1)}}{(k+1)!}\right\}^{\Delta_t} &= \frac{(t+1-s)^{(k+1)}}{(k+1)!} - \frac{(t-s)^{(k+1)}}{(k+1)!} \\
&= \frac{1}{(k+1)!}\left\{((t-s)+1)^{(k+1)} - (t-s)^{(k+1)}\right\} \\
&= \frac{1}{(k+1)!}\left\{\prod_{i=0}^{k}\left[((t-s)+1) - i\right] - \prod_{i=0}^{k}((t-s)-i)\right\} \\
&= \frac{1}{(k+1)!}\left\{\prod_{i=0}^{k}\left[((t-s)-(i-1))\right] - \prod_{i=0}^{k}((t-s)-i)\right\} \\
&= \frac{1}{(k+1)!}\{((t-s)+1)\left[(t-s)((t-s)-1)((t-s)-2)((t-s)-3)\cdots\right. \\
&\quad ((t-s)-(k-1))] - [(t-s)((t-s)-1)((t-s)-2)((t-s)-3)\cdots
\end{aligned}$$

$$\begin{aligned}
&((t-s)-(k-1))]((t-s)-k)\} \\
&= \frac{(t-s)((t-s)-1)((t-s)-2)\cdots((t-s)-(k-1))}{(k+1)!}\cdot
\end{aligned}$$

$$(((t-s)+1)-((t-s)-k))$$
$$=\frac{(t-s)((t-s)-1)((t-s)-2)\cdots((t-s)-k+1)}{k!}=\frac{(t-s)^{(k)}}{k!}.$$

□

Proof. of Lemma 1.1.

Notice that

$$h_0(t,s)=\frac{(t-s)^{(0)}}{0!}=(t-s)^{(0)}=1.$$

Assume (4) correct for k. Then

$$h_{k+1}(t,s)=\int_s^t h_k(\tau,s)\,\Delta\tau=\int_s^t\frac{(\tau-s)^{(k)}}{k!}\Delta\tau=\frac{(t-s)^{(k+1)}}{(k+1)!},$$

proving the claim. □

Let $f:(a+\mathbb{Z})\to\mathbb{R}$, $m\in\mathbb{N}$. Then by delta Taylor formula on time scales (see [4], [6]), applied on $(a+\mathbb{Z})$, see also (4), we get

$$f(t)=\sum_{k=0}^{m-1}\Delta^k f(s)\frac{(t-s)^{(k)}}{k!}+\int_s^t\frac{(t-\tau-1)^{(m-1)}}{(m-1)!}\Delta^m f(\tau)\,\Delta\tau, \tag{6}$$

$\forall\ t,s\in(a+\mathbb{Z})$.

For $s=b^*\in(a+\mathbb{Z})$, $t\in[a^*,b^*]\cap(a+\mathbb{Z})$, where $a^*<b^*$, we get that

$$f(t)=\sum_{k=0}^{m-1}\Delta^k f(b^*)\frac{(t-b^*)^{(k)}}{k!}-\int_t^{b^*}\frac{(t-s-1)^{(m-1)}}{(m-1)!}\Delta^m f(s)\,\Delta s$$
$$=\sum_{k=0}^{m-1}\Delta^k f(b^*)\frac{(t-b^*)^{(k)}}{k!}-\frac{1}{(m-1)!}\sum_{s=t}^{b^*-1}(t-s-1)^{(m-1)}\Delta^m f(s). \tag{7}$$

We call the remainder

$$R^*(t)=-\frac{1}{(m-1)!}\sum_{s=t}^{b^*-1}(t-s-1)^{(m-1)}\Delta^m f(s). \tag{8}$$

We need

Proposition 1.3. *For $s,t\in(a+\mathbb{Z})$, $m\in\mathbb{N}$, it holds*

$$(t-s-1)^{(m-1)}=(-1)^{m-1}(s+m-t-1)^{(m-1)}. \tag{9}$$

Proof. We notice that

$$(t-s-1)^{(m-1)} = \prod_{i=0}^{m-2} (t-s-1-i)$$
$$= (t-s-1)(t-s-2)(t-s-3)\cdots(t-s-(m-1))$$
$$= (-(s-t)-1)(-(s-t)-2)(-(s-t)-3)\cdots(-(s-t)-(m-1))$$
$$= (-1)^{m-1}(((s-t)+1)((s-t)+2)((s-t)+3)\cdots((s-t)+(m-1)))$$
$$= (-1)^{m-1}(s+m-t-1)^{(m-1)}.$$

Indeed it is

$$(s+m-t-1)^{(m-1)} = \prod_{i=0}^{m-2} (s+m-t-1-i)$$
$$= (s+m-t-1)(s+m-t-2)(s+m-t-3)\cdots(s+m-t-1-m+2)$$
$$= (s+m-t-1)(s+m-t-2)(s+m-t-3)\cdots(s-t+1).$$

□

We need

Definition 1.4 (see [3]). Let $\nu > 0$, the right fractional sum here is given by

(10) $$\left(\Delta_{b^*-1}^{-\nu} f\right)(t) := \frac{1}{\Gamma(\nu)} \sum_{s=t+\nu}^{b^*-1} (s-t-1)^{(\nu-1)} f(s),$$

$$\left(\Delta_{b^*-1}^{0} f\right)(t) := f(t),$$

where f is restricted on $[a^*, b^*] \cap (a+\mathbb{Z})$.

Notice $\left(\Delta_{b^*-1}^{-\nu} f\right)$ is defined on $\{a^*-\nu,\ a^*-\nu+1,\ a^*-\nu+2, \ldots,\ b^*-1-\nu\}$.

Here one can take $a^* = -\infty$.

We also need

Theorem 1.5 (see [7]). *Let $\mu, \nu \geq 0$. Then*

(11) $$\left(\Delta_{b^*-1-\nu}^{-\mu} \Delta_{b^*-1}^{-\nu} f\right)(t) = \left(\Delta_{b^*-1}^{-(\mu+\nu)} f\right)(t),$$

where $t \in \{a^*-(\mu+\nu),\ a^*-(\mu+\nu)+1, \ldots,\ b^*-1-(\mu+\nu)\}$.

Remark 1.6. So far we have based on (9) that

$$R^*(t) = -\frac{1}{(m-1)!}\sum_{s=t}^{b^*-1}(t-s-1)^{(m-1)}\Delta^m f(s)$$
$$= \frac{(-1)^m}{(m-1)!}\sum_{s=t}^{b^*-1}\left((s+m-t-1)^{(m-1)}\right)\Delta^m f(s)$$
$$= \frac{(-1)^m}{(m-1)!}\sum_{s=(t-m)+m}^{b^*-1}\left((s-(t-m)-1)^{(m-1)}\right)\Delta^m f(s)$$

(call $t' = t - m$)

$$= \frac{(-1)^m}{(m-1)!}\sum_{s=t'+m}^{b^*-1}\left((s-t'-1)^{(m-1)}\right)\Delta^m f(s)$$

(notice here $t' \in \{a^* - m, \ldots, b^* - 1 - m\}$)

$$\overset{(10)}{=} (-1)^m \left(\Delta_{b^*-1}^{-m}(\Delta^m f)\right)(t') = (-1)^m \left(\Delta_{b^*-1}^{-m}(\Delta^m f)\right)(t-m).$$

So we have proved

Theorem 1.7. *It holds*

$$R^*(t) = (-1)^m \left(\Delta_{b^*-1}^{-m}(\Delta^m f)\right)(t-m), \tag{12}$$

where $t \in [a^*, b^*-1] \cap (a+\mathbb{Z})$, *with* $a^* \leq b^* - 1$; $a^*, b^* \in (a+\mathbb{Z})$, $f : (a+\mathbb{Z}) \to \mathbb{R}$, a^* *could be* $-\infty$.

2. MAIN RESULTS

We give

Definition 2.1. Let $\mu > 0$, $m - 1 < \mu \leq m$, $m \in \mathbb{N}$, $m = \lceil \mu \rceil$ (ceiling of number), $\nu := m - \mu$, that is $\mu + \nu = m$.

The μ-th order delta right fractional difference (Caputo way) is given by

$$\left(\Delta_{(b^*-1)-}^{\mu} f\right)(t) := (-1)^m \left(\Delta_{(b^*-1)}^{-\nu}(\Delta^m f)\right)(t)$$
$$= \frac{(-1)^m}{\Gamma(\nu)}\sum_{s=t+\nu}^{b^*-1}(s-t-1)^{(\nu-1)}(\Delta^m f)(s), \tag{13}$$

where $t \leq b^* - 1 - \nu$, $b^* \in (a+\mathbb{Z})$, $t \in (a - \nu + \mathbb{Z})$, $a \in \mathbb{R}$.

If $\mu = m \in \mathbb{N}$, then $\left(\Delta_{(b^*-1)-}^{\mu} f\right)(t) = (-1)^m (\Delta^m f)(t)$.

Theorem 2.2. *It holds*

$$R^{*}(t)=\left(\Delta_{(b^{*}-1)-\nu}^{-\mu}\left(\Delta_{(b^{*}-1)-}^{\mu}f\right)\right)(t-m), \tag{14}$$

for $\mu>0$, $m-1<\mu\leq m$, $m\in\mathbb{N}$, $t\in[a^{*},b^{*}-1]\cap(a+\mathbb{Z})$, $a^{*}\leq b^{*}-1$; $a^{*},b^{*}\in(a+\mathbb{Z})$, $f:(a+\mathbb{Z})\to\mathbb{R}$; a^{*} *could be* $-\infty$.

Proof. Let $t\in[a^{*},b^{*}-1]\cap(a+\mathbb{Z})$, $a^{*}\leq b^{*}-1$; $a^{*},b^{*}\in(a+\mathbb{Z})$, $f:(a+\mathbb{Z})\to\mathbb{R}$. Then $t-m\in[a^{*}-m,b^{*}-1-m]\cap(a+\mathbb{Z})$. We observe that

$$\begin{aligned}\left(\Delta_{(b^{*}-1)-\nu}^{-\mu}\left(\Delta_{(b^{*}-1)-}^{\mu}f\right)\right)(t-m)&=(-1)^{m}\left(\Delta_{(b^{*}-1)-\nu}^{-\mu}\left(\Delta_{(b^{*}-1)}^{-\nu}\left(\Delta^{m}f\right)\right)\right)(t-m)\\&=(-1)^{m}\left(\Delta_{(b^{*}-1)}^{-(\mu+\nu)}\left(\Delta^{m}f\right)\right)(t-m)\\&=(-1)^{m}\left(\Delta_{b^{*}-1}^{-m}\left(\Delta^{m}f\right)\right)(t-m)\overset{(12)}{=}R^{*}(t).\end{aligned}$$

□

We have proved the following delta right discrete fractional Taylor formula.

Theorem 2.3. *Let* $a\in\mathbb{R}$, $f:(a+\mathbb{Z})\to\mathbb{R}$. *Let* $a^{*},b^{*}\in(a+\mathbb{Z})$, $a^{*}<b^{*}$. *Let* $\mu>0:m-1<\mu\leq m$, $m\in\mathbb{N}$, $(m=\lceil\mu\rceil)$, $\nu=m-\mu$, $t\in[a^{*},b^{*}-1]\cap(a+\mathbb{Z})$. *Then*

$$f(t)=\sum_{k=0}^{m-1}\Delta^{k}f(b^{*})\frac{(t-b^{*})^{(k)}}{k!}+R^{*}(t), \tag{15}$$

where

$$\begin{aligned}R^{*}(t)&=\left(\Delta_{(b^{*}-1)-\nu}^{-\mu}\left(\Delta_{(b^{*}-1)-}^{\mu}f\right)\right)(t-m)\\&=\frac{1}{\Gamma(\mu)}\sum_{s=t-m+\mu}^{(b^{*}-1-\nu)}(s-t+m-1)^{(\mu-1)}\left(\Delta_{(b^{*}-1)-}^{\mu}f\right)(s).\end{aligned} \tag{16}$$

Above a^{*} *could be* $-\infty$.

Corollary 2.4. *In the assumptions of Theorem* 2.3, *assume more that* $\Delta^{k}f(b^{*})=0$, $k=0,1,\ldots,m-1$. *Then*

$$f(t)=\frac{1}{\Gamma(\mu)}\sum_{s=t-m+\mu}^{(b^{*}-1-\nu)}(s-t+m-1)^{(\mu-1)}\left(\Delta_{(b^{*}-1)-}^{\mu}f\right)(s). \tag{17}$$

We need

Proposition 2.5 ([2])**.** *It holds*

$$\sum_{s=t+\mu}^{b^{*}-\nu}(s-t-1)^{(\mu-1)}=\frac{(b^{*}-\nu-t)^{(\mu)}}{\mu}>0. \tag{18}$$

So by (18) we get

Proposition 2.6. *We have*

$$\sum_{s=(t-m)+\mu}^{(b^*-1)-\nu}(s-(t-m)-1)^{(\mu-1)}=\frac{(b^*-1-\nu-t+m)^{(\mu)}}{\mu}>0. \tag{19}$$

We give the estimate

Theorem 2.7. *All as in Theorem* 2.3. *Then*

$$\begin{aligned}\left|f(t)-\sum_{k=0}^{m-1}\Delta^k f(b^*)\frac{(t-b^*)^{(k)}}{k!}\right|&=|R^*(t)|\\ &\leq\frac{(b^*-1-\nu-t+m)^{(\mu)}}{\Gamma(\mu+1)}\max_{s\in\{t-m+\mu,\dots,b^*-1-\nu\}}\left|\left(\Delta^{\mu}_{(b^*-1)-}f\right)(s)\right|.\end{aligned} \tag{20}$$

Proof. By (16) we get

$$\begin{aligned}|R^*(t)|&\leq\frac{1}{\Gamma(\mu)}\left(\sum_{s=t-m+\mu}^{b^*-1-\nu}(s-(t-m)-1)^{(\mu-1)}\right)\\ &\quad\times\max_{s\in\{t-m+\mu,\dots,b^*-1-\nu\}}\left|\left(\Delta^{\mu}_{(b^*-1)-}f\right)(s)\right|\\ &\overset{(19)}{=}\frac{(b^*-1-\nu-t+m)^{(\mu)}}{\Gamma(\mu+1)}\max_{s\in\{t-m+\mu,\dots,b^*-1-\nu\}}\left|\left(\Delta^{\mu}_{(b^*-1)-}f\right)(s)\right|.\end{aligned}$$

□

We need

Lemma 2.8 ([1, p. 580])**.** *Let $a>\nu$, $a,\nu>-1$, $a,\nu\in\mathbb{R}$, $a\leq b$. Then*

$$\sum_{r=a}^{b}r^{(\nu)}=\left(\frac{(b+1)^{(\nu+1)}-a^{(\nu+1)}}{\nu+1}\right). \tag{21}$$

We give a related Ostrowski inequality.

Theorem 2.9. *Let $a\in\mathbb{R}$, $f:(a+\mathbb{Z})\to\mathbb{R}$. Let $a^*,b^*\in(a+\mathbb{Z})$, $b^*-a^*\geq 2$. Let $\mu>0:m-1<\mu\leq m$, $m\in\mathbb{N}$, $\nu=m-\mu$, $t\in[a^*,b^*-1]\cap(a+\mathbb{Z})$. Assume $\Delta^k f(b^*)=0$, $k=1,\dots,m-1$. Then*

$$\begin{aligned}&\left|\frac{1}{b^*-a^*}\sum_{t=a^*}^{b^*-1}f(t)-f(b^*)\right|\\ &\quad\leq\frac{(b^*-a^*+\mu)^{(\mu+1)}}{(b^*-a^*)\Gamma(\mu+2)}\max_{s\in\{a^*-m+\mu,\dots,b^*-1-\nu\}}\left|\left(\Delta^{\mu}_{(b^*-1)-}f\right)(s)\right|.\end{aligned} \tag{22}$$

Proof. Using (15) and (16), since $\Delta^k f(b^*) = 0$, $k = 1, \ldots, m-1$, we get $f(t) - f(b^*) = R^*(t)$, $t \in [a^*, b^* - 1] \cap (a + \mathbb{Z})$. Then we observe

$$E_1 := \left| \frac{1}{b^* - a^*} \sum_{t=a^*}^{b^*-1} f(t) - f(b^*) \right| = \frac{1}{b^* - a^*} \left| \sum_{t=a^*}^{b^*-1} (f(t) - f(b^*)) \right|$$

$$\leq \frac{1}{b^* - a^*} \sum_{t=a^*}^{b^*-1} |f(t) - f(b^*)| = \frac{1}{b^* - a^*} \sum_{t=a^*}^{b^*-1} |R^*(t)|$$

$$\overset{(20)}{\leq} \frac{1}{(b^* - a^*)\Gamma(\mu+1)} \left(\sum_{t=a^*}^{b^*-1} (b^* - 1 - \nu - t + m)^{(\mu)} \right)$$

$$\times \max_{s \in \{a^* - m + \mu, \ldots, b^* - 1 - \nu\}} \left| \left(\Delta^{\mu}_{(b^*-1)-} f \right)(s) \right| =: (*).$$

Call $r = b^* - 1 - \nu - t + m$. Since $a^* \leq t \leq b^* - 1$, then $-a^* \geq -t \geq 1 - b^*$ and $b^* - 1 - \nu - a^* + m \geq b^* - 1 - \nu - t + m \geq b^* - 1 - \nu + 1 - b^* + m = m - \nu = \mu$.

Therefore

$$\mu \leq r \leq b^* - a^* - 1 + \mu.$$

We would like to calculate

$$\sum_{t=a^*}^{b^*-1} (b^* - 1 - \nu - t + m)^{(\mu)} = \sum_{r=\mu}^{b^* - a^* - 1 + \mu} r^{(\mu)}$$

$$= \mu^{(\mu)} + \sum_{r=\mu+1}^{b^* - a^* - 1 + \mu} r^{(\mu)} = \Gamma(\mu+1) + \sum_{r=\mu+1}^{b^* - a^* - 1 + \mu} r^{(\mu)}$$

(since $b^* - a^* \geq 2$ we get that $\mu + 1 \leq b^* - a^* - 1 + \mu$, so that we can apply (21))

$$= \Gamma(\mu+1) + \frac{(b^* - a^* + \mu)^{(\mu+1)} - (\mu+1)^{(\mu+1)}}{\mu+1}$$

$$= \Gamma(\mu+1) + \frac{(b^* - a^* + \mu)^{(\mu+1)}}{\mu+1} - \frac{\Gamma(\mu+2)}{\mu+1}$$

$$= \Gamma(\mu+1) + \frac{(b^* - a^* + \mu)^{(\mu+1)}}{\mu+1} - \Gamma(\mu+1) = \frac{(b^* - a^* + \mu)^{(\mu+1)}}{\mu+1}.$$

Consequently it holds

$$\sum_{t=a^*}^{b^*-1} (b^* - 1 - \nu - t + m)^{(\mu)} = \frac{(b^* - a^* + \mu)^{(\mu+1)}}{\mu+1}.$$

So we have

$$(*) = \frac{(b^* - a^* + \mu)^{(\mu+1)}}{(b^* - a^*)\Gamma(\mu+2)} \max_{s \in \{a^* - m + \mu, \ldots, b^* - 1 - \nu\}} \left| \left(\Delta^{\mu}_{(b^*-1)-} f \right)(s) \right| =: E_2.$$

Hence $E_1 \leq E_2$, proving the claim. ∎

A related Poincaré type inequality follows

Theorem 2.10. *All as in Corollary* 2.4. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\sum_{t=a^*}^{b^*-1} |f(t)|^q \leq \frac{1}{(\Gamma(\mu))^q} \left(\sum_{t=a^*}^{b^*-1} \left(\sum_{s=t-m+\mu}^{b^*-1-\nu} (s-t+m-1)^{p(\mu-1)} \right)^{\frac{q}{p}} \right)$$

$$\times \left(\sum_{s=a^*-m+\mu}^{b^*-1-\nu} \left| \left(\Delta^{\mu}_{(b^*-1)-} f \right)(s) \right|^q \right). \tag{23}$$

Proof. Notice here $s-t+m \geq \mu > 0$ and $s-t+m-\mu \geq 0$ and $s-t+m-\mu+1 > 0$, so that

$$(s-t+m-1)^{(\mu-1)} = \frac{\Gamma(s-t+m)}{\Gamma(s-t+m-\mu+1)} > 0.$$

By (17) we get

$$|f(t)| \leq \frac{1}{\Gamma(\mu)} \sum_{s=t-m+\mu}^{b^*-1-\nu} (s-t+m-1)^{(\mu-1)} \left| \left(\Delta^{\mu}_{(b^*-1)-} f \right)(s) \right|$$

$$\leq \frac{1}{\Gamma(\mu)} \left(\sum_{s=t-m+\mu}^{b^*-1-\nu} (s-t+m-1)^{p(\mu-1)} \right)^{\frac{1}{p}} \left(\sum_{s=t-m+\mu}^{b^*-1-\nu} \left| \left(\Delta^{\mu}_{(b^*-1)-} f \right)(s) \right|^q \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\Gamma(\mu)} \left(\sum_{s=t-m+\mu}^{b^*-1-\nu} (s-t+m-1)^{p(\mu-1)} \right)^{\frac{1}{p}} \left(\sum_{s=a^*-m+\mu}^{b^*-1-\nu} \left| \left(\Delta^{\mu}_{(b^*-1)-} f \right)(s) \right|^q \right)^{\frac{1}{q}}.$$

Therefore we found

$$|f(t)| \leq \frac{1}{\Gamma(\mu)} \left(\sum_{s=t-m+\mu}^{b^*-1-\nu} (s-t+m-1)^{p(\mu-1)} \right)^{\frac{1}{p}}$$

$$\times \left(\sum_{s=a^*-m+\mu}^{b^*-1-\nu} \left| \left(\Delta^{\mu}_{(b^*-1)-} f \right)(s) \right|^q \right)^{\frac{1}{q}}. \tag{24}$$

Hence

$$|f(t)|^q \leq \frac{1}{(\Gamma(\mu))^q} \left(\sum_{s=t-m+\mu}^{b^*-1-\nu} (s-t+m-1)^{p(\mu-1)} \right)^{\frac{q}{p}} \cdot$$

$$\left(\sum_{s=a^*-m+\mu}^{b^*-1-\nu} \left| \left(\Delta^{\mu}_{(b^*-1)-} f \right)(s) \right|^q \right). \tag{25}$$

Applying $\sum_{t=a^*}^{b^*-1}$ to both sides of (25) we derive (23). □

We finish with a related Sobolev type inequality

Theorem 2.11. *All as in Theorem* 2.10. *Let* $r \geq 1$. *Then*

$$\left(\sum_{t=a^*}^{b^*-1} |f(t)|^r \right)^{\frac{1}{r}} \leq \frac{1}{\Gamma(\mu)} \left(\sum_{t=a^*}^{b^*-1} \left(\sum_{s=t-m+\mu}^{b^*-1-\nu} (s-t+m-1)^{p(\mu-1)} \right)^{\frac{r}{p}} \right)^{\frac{1}{r}} \cdot$$

$$\left(\sum_{s=a^*-m+\mu}^{b^*-1-\nu}\left|\left(\Delta^{\mu}_{(b^*-1)-}f\right)(s)\right|^q\right)^{\frac{1}{q}}. \tag{26}$$

Proof. Raising (24) to the power r we obtain

$$|f(t)|^r \leq \frac{1}{(\Gamma(\mu))^r}\left(\sum_{s=t-m+\mu}^{b^*-1-\nu}(s-t+m-1)^{p(\mu-1)}\right)^{\frac{r}{p}}\cdot$$

$$\left(\sum_{s=a^*-m+\mu}^{b^*-1-\nu}\left|\left(\Delta^{\mu}_{(b^*-1)-}f\right)(s)\right|^q\right)^{\frac{r}{q}}.$$

Hence it holds

$$\sum_{t=a^*}^{b^*-1}|f(t)|^r \leq \frac{1}{(\Gamma(\mu))^r}\left(\sum_{t=a^*}^{b^*-1}\left(\sum_{s=t-m+\mu}^{b^*-1-\nu}(s-t+m-1)^{p(\mu-1)}\right)^{\frac{r}{p}}\right)\cdot$$

$$\left(\sum_{s=a^*-m+\mu}^{b^*-1-\nu}\left|\left(\Delta^{\mu}_{(b^*-1)-}f\right)(s)\right|^q\right)^{\frac{r}{q}}. \tag{27}$$

Raising (27) to the power $\frac{1}{r}$ we derive (26). □

REFERENCES

[1] G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Heidelberg, 2011.

[2] G.A. Anastassiou, *Right nabla discrete fractional calculus*, submitted, 2011.

[3] N. Bastos, R. Ferreira, D. Torres, *Discrete-time fractional variational problems*, Signal Precessing, 91(2011), 513–524.

[4] M. Bohner, G. Guseinov, *The convolution on time scales*, Abstract and Applied Analysis, Article ID 58373, 24(2007).

[5] M. Bohner, A. Peterson, *Dynamic equations on time scales: An Introduction with Applications*, Birkhaüser, Boston, 2001.

[6] R. Higgins, A. Peterson, *Cauchy functions and Taylor's formula for Time scales* $\mathbb{T}$, in B. Aulbach, S. Elaydi, G. Ladas (eds.), Proc. Sixth Internat. Conf. on Difference equations, New Progress in Difference Equations, Augsburg, Germany, 2001, pp. 299–308, Chapman & Hall / CRC (2004).

[7] R. Ferreira, D. Tores, *Fractional h-difference equations arising from the calculus of variations*, Applicable Analysis and Discrete Mathematics, 5 (2011), 110–121.