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# TOPOLOGICAL PRINCIPLES FOR ESSENTIAL TYPE MAPS

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**ABSTRACT.** In this paper we present a definition of $d$-essential and $d$-$L$-essential maps in a very general setting and we establish a homotopy property for both $d$-essential and $d$-$L$-essential maps.

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## 1. INTRODUCTION

Essential maps for single valued maps was introduced by Granas [2] and extended by Precup [6]. These notions were considered by O'Regan in [3], see also [1, 4] for particular cases. In this paper we present the notions of $d$-essential and $d$-$L$-essential maps in a very general setting and we establish a homotopy property for both $d$-essential and $d$-$L$-essential maps.

Let $X$ and $Y$ be Hausdorff topological spaces. Given a class $\mathbf{X}$ of maps, $\mathbf{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of $Y$) belonging to $\mathbf{X}$, and $\mathbf{X}_c$ the set of finite compositions of maps in $\mathbf{X}$. We let

$$\mathbf{F}(\mathbf{X}) = \{Z : Fix\, F \neq \emptyset \text{ for all } F \in \mathbf{X}(Z,Z)\}$$

where $Fix\, F$ denotes the set of fixed points of $F$.

The class $\mathbf{U}$ of maps is defined by the following properties:

(i) $\mathbf{U}$ contains the class $\mathbf{C}$ of single valued continuous functions;
(ii) each $F \in \mathbf{U}_c$ is upper semicontinuous and compact valued; and
(iii) $B^n \in \mathbf{F}(\mathbf{U}_c)$ for all $n \in \{1, 2, \dots\}$; here $B^n = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$.

We say $F \in \mathbf{U}_c^k(X,Y)$ if for any compact subset $K$ of $X$ there is a $G \in \mathbf{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Recall $\mathbf{U}_c^k$ is closed under compositions. The class $\mathbf{U}_c^k$ contains almost all the well known maps in the literature (see [5] and the references therein). It is also possible

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to consider more general maps (see [5]) and in this paper we will consider a class of maps which we will call $\mathbf{A}$.

## 2. $d$-ESSENTIAL MAPS

Let $E$ be a normal topological space and $U$ an open subset of $E$.

We will consider a class $\mathbf{A}$ of maps. The following condition will be assumed:

$$\begin{cases} \text{for Hausdorff topogical spaces } X_1, X_2 \text{ and } X_3, \\ \text{if } F \in \mathbf{A}(X_1, X_3) \text{ and } f \in \mathbf{C}(X_2, X_1), \\ \text{then } F \circ f \in \mathbf{A}(X_2, X_3). \end{cases} \tag{2.1}$$

**Definition 2.1.** We say $F \in A(\overline{U}, E)$ if $F \in \mathbf{A}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is an upper semicontinuous compact map; here $\overline{U}$ denotes the closure of $U$ in $E$ and $K(E)$ denotes the family of nonempty compact subsets of $E$.

**Definition 2.2.** We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $x \notin F(x)$ for $x \in \partial U$; here $\partial U$ denotes the boundary of $U$ in $E$.

For any map $F \in A(\overline{U}, E)$ let $F^\star = I \times F : \overline{U} \to K(\overline{U} \times E)$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^\star)^{-1}(B) \right\} \cup \{\emptyset\} \to \Omega \tag{2.2}$$

be any map with values in the nonempty set $\Omega$; here $B = \left\{ (x, x) : x \in \overline{U} \right\}$.

**Definition 2.3.** Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi \in A(\overline{U} \times [0,1], E)$, $x \notin \Psi_t(x)$ for any $x \in \partial U$ and $t \in [0,1]$, $\Psi_1 = F$ and $\Psi_0 = G$ (here $\Psi_t(x) = \Psi(x,t)$).

**Remark 2.4.** Definition 2.3 corrects a small mistake in the definition of $\cong$ in [4].

**Remark 2.5.** The results below (with (2.1) removed) also hold true if we use the following definition of $\cong$. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a upper semicontinuous compact map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $x \notin \Psi_t(x)$ for any $x \in \partial U$ and $t \in [0,1]$, $\Psi_1 = F$ and $\Psi_0 = G$.

The following condition will be assumed:

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, E). \tag{2.3}$$

**Definition 2.6.** Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to K(\overline{U} \times E)$ is $d$-essential if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ we have that $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. Otherwise $F^\star$ is $d$-inessential. It is easy to check that this means either $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ or there exists a map $J \in A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ such that $d\left((F^\star)^{-1}(B)\right) \neq d\left((J^\star)^{-1}(B)\right)$.

**Remark 2.7.** If $F^\star$ is $d$-essential then

$$\emptyset \neq (F^\star)^{-1}(B) = \{x \in \overline{U} : (x, F(x)) \cap (x,x) \neq \emptyset\},$$

and this together with $x \notin F(x)$ for $x \in \partial U$ implies that there exists $x \in U$ with $(x,x) \in F^\star(x)$ (i.e. $x \in F(x)$).

**Theorem 2.8.** *Let $E$ be a normal topological space, $U$ an open subset of $E$, $B = \{(x,x) : x \in \overline{U}\}$, $d$ a map defined in* (2.2) *and assume* (2.1) *and* (2.3) *hold. Suppose $F \in A_{\partial U}(\overline{U}, E)$. Then the following are equivalent:*

*(i). $F^\star = I \times F : \overline{U} \to K(\overline{U} \times E)$ is $d$-inessential;*

*(ii). $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ or there exists a map $G \in A_{\partial U}(\overline{U}, E)$ with $G^\star = I \times G$ and $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ such that $d\left((F^\star)^{-1}(B)\right) \neq d\left((G^\star)^{-1}(B)\right)$.*

*Proof.* (i) implies (ii) is immediate. Next we prove (ii) implies (i). If $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ then trivially (i) is true. Next suppose there exists a map $G \in A_{\partial U}(\overline{U}, E)$ with $G^\star = I \times G$ and $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ such that $d\left((F^\star)^{-1}(B)\right) \neq d\left((G^\star)^{-1}(B)\right)$. Let $H : \overline{U} \times [0,1] \to K(E)$ be a map with $H \in A(\overline{U} \times [0,1], E)$, $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in [0,1]$, $H_0 = F$ and $H_1 = G$ (here $H_t(x) = H(x,t)$). Let $H^\star : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by

$$H^\star(x, \lambda) = (x, H(x, \lambda)).$$

Consider

$$D = \left\{x \in \overline{U} : (x,x) \in H^\star(x,t) \text{ for some } t \in [0,1]\right\}.$$

If $D = \emptyset$ then in particular $(H^\star(x,0))^{-1}(B) = \emptyset$ i.e. $(F^\star)^{-1}(B) = \emptyset$ and as a result $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$, so $F^\star$ is $d$-inessential. Next suppose $D \neq \emptyset$. Note $D$ is closed in $E$. To see this let $\{x_n\}_{n=1}^{\infty} \subseteq D$ with $x_n \to x \in \overline{U}$. Now there exists $t_n \in (0,1]$ with

$$x_n \in H(x_n, t_n) \text{ for each } n \in \{1, 2, \dots\}.$$

Without loss of generality assume $t_n \to t \in [0,1]$ so $(x_n, t_n) \to (x,t)$. Now since $H : \overline{U} \times [0,1] \to K(E)$ is a upper semicontinuous map we have $x \in H(x,t)$. As a result $(x,x) \in H^\star(x,t)$, so $D$ is closed. Also since $x \notin H_t(x)$ for $x \in \partial U$ and $t \in [0,1]$ then $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $R_\mu : \overline{U} \to K(E)$ by $R_\mu(x) = H(x, \mu(x)) = H_{\mu(x)}(x) = H \circ \tau(x)$ and let $R_\mu^\star = I \times R_\mu$; here $\tau : \overline{U} \to \overline{U} \times [0,1]$ is given by $\tau(x) = (x, \mu(x))$. Notice $R_\mu \in A(\overline{U}, E)$ (note (2.1) and $H \in A(\overline{U} \times [0,1], E)$) and notice $R_\mu|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$ since $\mu(\partial U) = 0$. Thus $R_\mu \in A_{\partial U}(\overline{U}, E)$ (note $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in [0,1]$) with $R_\mu|_{\partial U} = F|_{\partial U}$.

Note also since $\mu(D) = 1$ that

$$\begin{aligned} \left(R_\mu^\star\right)^{-1}(B) &= \left\{x \in \overline{U} : (x,x) \cap (x, H(x, \mu(x)) \neq \emptyset\right\} \\ &= \left\{x \in \overline{U} : (x,x) \cap (x, H(x,1) \neq \emptyset\right\} = (G^\star)^{-1}(B) \end{aligned}$$

so $d\left(\left(R_\mu^\star\right)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right)$. Thus $d\left((F^\star)^{-1}(B)\right) \neq d\left(\left(R_\mu^\star\right)^{-1}(B)\right)$.

We now claim

$$R_\mu \cong F \text{ in } A_{\partial U}(\overline{U}, E). \tag{2.4}$$

Let $Q : \overline{U} \times [0,1] \to K(E)$ be given by $Q(x,t) = H(x, t\mu(x)) = H \circ g(x,t)$ where $g : \overline{U} \times [0,1] \to \overline{U} \times [0,1]$ is given by $g(x,t) = (x, t\mu(x))$. Note $Q \in A(\overline{U} \times [0,1], E)$ (note (2.1) and $H \in A(\overline{U} \times [0,1], E)$), $Q_0 = F$ and $Q_1 = R_\mu$. Also $x \notin Q_t(x)$ for $x \in \partial U$ and $t \in [0,1]$ since if there exists $t \in [0,1]$ and $x \in \partial U$ with $x \in Q_t(x)$ then $x \in H(x, t\mu(x))$ so $x \in D$ and as a result $\mu(x) = 1$ i.e. $x \in H(x,t)$, a contradiction. Thus (2.4) holds.

Consequently $F^\star$ is $d$-inessential (take $J = R_\mu$ in the definition of $d$-inessential). □

**Remark 2.9.** From the proof above (with a minor modification in two places) we see that the result in Theorem 2.8 (with (2.1) removed) holds if the definition of $\cong$ is as in Remark 2.5.

**Remark 2.10.** We note that the map being compact in the class $A(\overline{U}, E)$ plays no role in the proof of Theorem 2.8 so we can remove it from the definition if we wish at this stage. Its when we try to construct examples of essential maps that we usually need maps to satisfy some type of compactness condition and the most popular maps in the literature are compact or condensing maps.

Now Theorem 2.8 immediately yields the following continuation theorem.

**Theorem 2.11.** *Let $E$ be a normal topological space, $U$ an open subset of $E$, $B = \{(x,x) : x \in \overline{U}\}$, $d$ a map defined in* (2.2) *and assume* (2.1) *and* (2.3) *hold. Suppose $\Phi$ and $\Psi$ are two maps in $A_{\partial U}(\overline{U}, E)$ with $\Phi^\star = I \times \Phi$ and $\Psi^\star = I \times \Psi$ and with $\Phi \cong \Psi$ in $A_{\partial U}(\overline{U}, E)$. The $\Phi^\star$ is $d$-inessential if and only if $\Psi^\star$ is $d$-inessential.*

*Proof.* Assume $\Phi^\star$ is $d$-inessential. Then (see Theorem 2.8) either $d\left((\Phi^\star)^{-1}(B)\right) = d(\emptyset)$ or there exists a map $Q \in A_{\partial U}(\overline{U}, E)$ with $Q^\star = I \times Q$ and $Q \cong \Phi$ in $A_{\partial U}(\overline{U}, E)$ such that $d\left((\Phi^\star)^{-1}(B)\right) \neq d\left((Q^\star)^{-1}(B)\right)$.

Suppose first that $d\left((\Phi^\star)^{-1}(B)\right) = d(\emptyset)$. There are two cases to consider, either $d\left((\Psi^\star)^{-1}(B)\right) \neq d(\emptyset)$ or $d\left((\Psi^\star)^{-1}(B)\right) = d(\emptyset)$.

Case (1). Suppose $d\left((\Psi^\star)^{-1}(B)\right) \neq d(\emptyset)$.

Then $d\left((\Phi^\star)^{-1}(B)\right) \neq d\left((\Psi^\star)^{-1}(B)\right)$ and we know $\Phi \cong \Psi$ in $A_{\partial U}(\overline{U}, E)$. Now Theorem 2.8 (with $F = \Psi$ and $G = \Phi$) guarantees that $\Psi^\star$ is $d$-inessential.

Case (2). Suppose $d\left((\Psi^\star)^{-1}(B)\right) = d(\emptyset)$.

Then by definition $\Psi^\star$ is $d$-inessential.

Next suppose there exists a map $Q \in A_{\partial U}(\overline{U}, E)$ with $Q^\star = I \times Q$ and $Q \cong \Phi$ in $A_{\partial U}(\overline{U}, E)$ such that $d\left((\Phi^\star)^{-1}(B)\right) \neq d\left((Q^\star)^{-1}(B)\right)$. Note (since $\cong$ is an equivalence relation in $A_{\partial U}(\overline{U}, E)$) also that $Q \cong \Psi$ in $A_{\partial U}(\overline{U}, E)$. There are two cases to consider, either $d\left((Q^\star)^{-1}(B)\right) \neq d\left((\Psi^\star)^{-1}(B)\right)$ or $d\left((Q^\star)^{-1}(B)\right) = d\left((\Psi^\star)^{-1}(B)\right)$.

Case (1). Suppose $d\left((Q^\star)^{-1}(B)\right) \neq d\left((\Psi^\star)^{-1}(B)\right)$.

Then Theorem 2.8 (with $F = \Psi$ and $G = Q$) guarantees that $\Psi^\star$ is $d$-inessential.

Case (2). Suppose $d\left((Q^\star)^{-1}(B)\right) = d\left((\Psi^\star)^{-1}(B)\right)$.

Then $d\left((\Phi^\star)^{-1}(B)\right) \neq d\left((\Psi^\star)^{-1}(B)\right)$ and we know $\Phi \cong \Psi$ in $A_{\partial U}(\overline{U}, E)$. Now Theorem 2.8 (with $F = \Psi$ and $G = \Phi$) guarantees that $\Psi^\star$ is $d$-inessential.

Thus in all cases $\Psi^\star$ is $d$-inessential.

Similarly if $\Psi^\star$ is $d$-inessential then $\Phi^\star$ is $d$-inessential. □

**Remark 2.12.** The result in Theorem 2.11 (with (2.1) removed) holds if the definition of $\cong$ is as in Remark 2.5.

**Remark 2.13.** If we discuss the existence of fixed points the function $d$ is

$$d(Q) = \begin{cases} 1 \text{ if } \emptyset \neq Q \subseteq \overline{U} \\ 0 \text{ if } Q = \emptyset \end{cases}$$

whereas if we discuss degree theory the values of $d$ are usually integers which can be obtained by means of degree. Recall [3] a map $F \in A_{\partial U}(\overline{U}, E)$ is essential in $A_{\partial U}(\overline{U}, E)$ if for any map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and with $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ we have that there exists a $x \in U$ with $x \in F(x)$. Notice a map $F \in A_{\partial U}(\overline{U}, E)$ is essential in $A_{\partial U}(\overline{U}, E)$ implies that $F^\star = I \times F$ is $d_1$-essential where

$$d_1(Q) = \begin{cases} 1 \text{ if } \emptyset \neq Q \subseteq \overline{U} \\ 0 \text{ if } Q = \emptyset. \end{cases}$$

To see this suppose $F \in A_{\partial U}(\overline{U}, E)$ is essential in $A_{\partial U}(\overline{U}, E)$. Then for any $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and with $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ there exists $x \in U$ with $x \in J(x)$. Thus $(x, x) \in (x, J(x)) \equiv J^\star(x)$ and so $(J^\star)^{-1}(B) \neq \emptyset$ (in particular $(F^\star)^{-1}(B) \neq \emptyset$). Hence $d_1\left((J^\star)^{-1}(B)\right) = 1$ and $d_1\left((F^\star)^{-1}(B)\right) = 1$ so $d_1\left((J^\star)^{-1}(B)\right) = d_1\left((F^\star)^{-1}(B)\right) \neq d_1(\emptyset)$.

We note that (2.3) could be a strong assumption in Theorem 2.11 (and Theorem 2.8). However one can obtain an applicable result even if (2.3) is not assumed. To establish this we will consider new $d$-essential maps (a subset of those $d$-essential maps in Definition 2.6).

**Definition 2.14.** Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to K(\overline{U} \times E)$ is $d$-essential if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ we have that $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$.

**Theorem 2.15.** *Let $E$ be a normal topological space, $U$ an open subset of $E$, $B = \{(x,x) : x \in \overline{U}\}$, $d$ a map defined in (2.2) and assume (2.1) holds. Suppose $G \in A_{\partial U}(\overline{U}, E)$, $H : \overline{U} \times [0,1] \to K(E)$ with $H \in A(\overline{U} \times [0,1], E)$ and assume the following hold:*

$$H(x,0) = G(x) \text{ for } x \in \overline{U} \tag{2.5}$$

$$G^\star = I \times G : \overline{U} \to K(\overline{U} \times E) \text{ is } d\text{-essential} \tag{2.6}$$

*and*

$$x \notin H(x,t) \text{ for } x \in \partial U \text{ and } t \in (0,1]. \tag{2.7}$$

*Let $F(x) = H(x,1)$ for $x \in \overline{U}$ and $F^\star = I \times F$. Then*

$$d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right) \neq d(\emptyset).$$

**Remark 2.16.** From the proof below we see that we can remove (2.1) and remove $H : \overline{U} \times [0,1] \to K(E)$ with $H \in A(\overline{U} \times [0,1], E)$ in the statement of Theorem 2.15 if we assume $H : \overline{U} \times [0,1] \to K(E)$ is a upper semicontinuous compact map with $H(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$.

*Proof.* Let $H^\star : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by

$$H^\star(x,\lambda) = (x, H(x,\lambda)).$$

Consider

$$D = \left\{x \in \overline{U} : (x,x) \in H^\star(x,t) \text{ for some } t \in [0,1]\right\}.$$

Notice $D \neq \emptyset$ since for $t = 0$, $H^\star(x,0) = G^\star(x)$ and $G^\star$ is $d$-essential (i.e. in particular there exists $x \in U$ with $(x,x) \in (x, G(x)) = H^\star(x,0)$). Also (see Theorem 2.8) $D$ is closed in $E$. Next notice (2.7), with $G \in A_{\partial U}(\overline{U}, E)$, guarantees that $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $R_\mu : \overline{U} \to K(E)$ by $R_\mu(x) = H(x, \mu(x)) = H_{\mu(x)}(x) = H \circ \tau(x)$ and let $R_\mu^\star = I \times R_\mu$; here $\tau : \overline{U} \to \overline{U} \times [0,1]$ is given by $\tau(x) = (x, \mu(x))$. Notice (as in Theorem 2.8) that $R_\mu \in A(\overline{U}, E)$ and $R_\mu|_{\partial U} = G|_{\partial U}$ since $\mu(\partial U) = 0$. Thus $R_\mu \in A_{\partial U}(\overline{U}, E)$ with $R_\mu|_{\partial U} = G|_{\partial U}$ and since $G^\star$ is $d$-essential we have $d\left(\left(R_\mu^\star\right)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right) \neq d(\emptyset)$. Also notice (see Theorem 2.8) since $\mu(D) = 1$ that $d\left(\left(R_\mu^\star\right)^{-1}(B)\right) = d\left((F^\star)^{-1}(B)\right)$. Thus $d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right) \neq d(\emptyset)$. □

**Remark 2.17.** We now note that Theorem 2.15 holds if we use the definition of $d$-essential (see (2.6)) in Definition 2.6 instead of in Definition 2.14. Now as in the proof above we have $R_\mu \in A_{\partial U}(\overline{U}, E)$ with $R_\mu|_{\partial U} = G|_{\partial U}$. We now show $R_\mu \cong G$ in $A_{\partial U}(\overline{U}, E)$. To see this let $Q : \overline{U} \times [0,1] \to K(E)$ be given by $Q(x,t) = H(x, t\mu(x)) = H \circ g(x,t)$ where $g : \overline{U} \times [0,1] \to \overline{U} \times [0,1]$ is given by $g(x,t) = (x, t\mu(x))$.

Note $Q \in A(\overline{U} \times [0,1], E)$, $Q_1 = R_\mu$ and $Q_0 = G$. Also $x \notin Q_t(x)$ for $x \in \partial U$ and $t \in [0,1]$ since if there exists $t \in [0,1]$ and $x \in \partial U$ with $x \in Q_t(x)$ then $x \in H(x, t\mu(x))$ so $x \in D$ and as a result $\mu(x) = 1$ i.e. $x \in H(x,t)$, a contradiction. Thus $R_\mu \cong G$ in $A_{\partial U}(\overline{U}, E)$ and since $G^\star$ is $d$-essential (as in Definition 2.14) we have $d\left(\left(R_\mu^\star\right)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right) \neq d(\emptyset)$. We conclude now as in the proof of Theorem 2.15.

The result also holds if the definition of $\cong$ is as in Remark 2.5 (if we use the assumption in Remark 2.16).

**Remark 2.18.** As one would expect we note that if in Theorem 2.15 the map $F \in A_{\partial U}(\overline{U}, E)$ satisfies the condition

$$(2.8)\qquad \begin{cases} \text{for any map } J \in A_{\partial U}(\overline{U}, E) \text{ with } J|_{\partial U} = F|_{\partial U} \text{ there exists a} \\ \text{map } Q : \overline{U} \times [0,1] \to K(E) \text{ with } Q \in A(\overline{U} \times [0,1], E) \text{ and} \\ \text{with } x \notin Q_t(x) \text{ for any } x \in \partial U \text{ and } t \in [0,1] \text{ (here } Q_t(x) = Q(x,t)) \\ \text{and } Q_0 = G \text{ and } Q_1 = J, \end{cases}$$

then $F^\star$ is $d$-essential.

To see this let $J \in A_{\partial U}(\overline{U}, E)$ be any map with $J|_{\partial U} = F|_{\partial U}$. We must show if $J^\star = I \times J$ then

$$d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset). \tag{2.9}$$

Let $Q$ be the map described in (2.8) for the map $J$. Now Theorem 2.15 (with $F$, $G$ and $H$) implies

$$d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right) \neq d(\emptyset). \tag{2.10}$$

Also Theorem 2.15 (with $J$, $G$ and $Q$) implies

$$d\left((J^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right) \neq d(\emptyset). \tag{2.11}$$

Now (2.10) and (2.11) give (2.9) so $F^\star$ is $d$-essential.

From the above we also note that (2.8) could be replaced by (use Remark 2.16) the condition

$$(2.12)\qquad \begin{cases} \text{for any map } J \in A_{\partial U}(\overline{U}, E) \text{ with } J|_{\partial U} = F|_{\partial U} \text{ there exists a upper} \\ \text{semicontinuous compact map } Q : \overline{U} \times [0,1] \to K(E) \text{ with} \\ Q(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E) \text{ for any continuous map } \eta : \overline{U} \to [0,1] \\ \text{with } \eta(\partial U) = 0, x \notin Q_t(x) \text{ for any } x \in \partial U \text{ and } t \in [0,1] \\ \text{and } Q_0 = G \text{ and } Q_1 = J. \end{cases}$$

**Remark 2.19.** Remark 2.13 holds in this situation as well (with the obvious new essential map definition).

We now show that the ideas in this section can be applied to other natural situations. Let $E$ be a normal topological vector space, $Y$ a topological vector space, and $U$ an open subset of $E$. Also let $L : dom\, L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of $E$. Finally $T : E \to Y$ will be a linear, continuous single valued map with $L + T : dom\, L \to Y$ an isomorphism (i.e. a linear homeomorphism); for convenience we say $T \in H_L(E, Y)$.

A map $F : \overline{U} \to 2^Y$ is said to be a $(L, T)$ upper semicontinuous compact map if $(L + T)^{-1}F : \overline{U} \to K(X)$ is a upper semicontinuous compact map.

**Definition 2.20.** Let $F : \overline{U} \to 2^Y$. We say $F \in A(\overline{U}, Y; L, T)$ if $(L + T)^{-1}F \in A(\overline{U}, E)$.

**Definition 2.21.** We say $F \in A_{\partial U}(\overline{U}, Y; L, T)$ if $F \in A(\overline{U}, Y; L, T)$ with $Lx \notin F(x)$ for $x \in \partial U \cap dom\, L$.

For any map $F \in A(\overline{U}, Y; L, T)$ let $F^\star = I \times (L+T)^{-1}[F+T] : \overline{U} \to K(\overline{U} \times E)$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{(F^\star)^{-1}(B)\right\} \cup \{\emptyset\} \to \Omega \tag{2.13}$$

be any map with values in the nonempty set $\Omega$; here $B = \left\{(x, x) : x \in \overline{U}\right\}$.

**Definition 2.22.** Let $F, G \in A_{\partial U}(\overline{U}, Y; L, T)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$ if there exists a map $\Psi : \overline{U} \times [0, 1] \to 2^Y$ with $\Psi \in A(\overline{U} \times [0, 1], Y; L, T)$, $Lx \notin \Psi_t(x)$ for any $x \in \partial U \cap dom\, L$ and $t \in [0, 1]$, $\Psi_1 = F$ and $\Psi_0 = G$ (here $\Psi_t(x) = \Psi(x, t)$).

**Remark 2.23.** The results below (with (2.1) removed) also hold true if we use the following definition of $\cong$. Let $F, G \in A_{\partial U}(\overline{U}, Y; L, T)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$ if there exists a $(L, T)$ upper semicontinuous compact map $\Psi : \overline{U} \times [0, 1] \to 2^Y$ with $(L + T)^{-1}\Psi(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $Lx \notin \Psi_t(x)$ for any $x \in \partial U \cap dom\, L$ and $t \in [0, 1]$, $\Psi_1 = F$ and $\Psi_0 = G$.

The following condition will be assumed:

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, Y; L, T). \tag{2.14}$$

**Definition 2.24.** Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^\star = I \times (L + T)^{-1}[F + T]$. We say $F^\star : \overline{U} \to K(\overline{U} \times E)$ is $d$-$L$-essential if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J^\star = I \times (L + T)^{-1}[J + T]$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ we have that $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. Otherwise $F^\star$ is $d$-$L$-inessential. It is easy to check that this means either $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ or there exists a map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J^\star = I \times (L + T)^{-1}[J + T]$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ such that $d\left((F^\star)^{-1}(B)\right) \neq d\left((J^\star)^{-1}(B)\right)$.

**Remark 2.25.** If $F^\star$ is $d$-$L$-essential then

$$\emptyset \neq (F^\star)^{-1}(B) = \{x \in \overline{U} : (x, (L+T)^{-1}[F+T](x)) \cap (x,x) \neq \emptyset\},$$

and this together with $Lx \notin F(x)$ for $x \in \partial U \cap dom\, L$ implies that there exists $x \in U \cap dom\, L$ with $(x,x) \in F^\star(x)$ (i.e. $Lx \in F(x)$).

**Theorem 2.26.** *Let $E$ be a normal topological vector space, $Y$ a topological vector space, $U$ an open subset of $E$, $L : dom\, L \subseteq E \to Y$ a linear single valued map, $T \in H_L(E,Y)$, $d$ a map defined in* (2.13) *and assume* (2.1) *and* (2.14) *hold. Suppose $F \in A_{\partial U}(\overline{U}, Y; L, T)$. Then the following are equivalent:*

*(i). $F^\star = I \times (L+T)^{-1}[F+T] : \overline{U} \to K(\overline{U} \times E)$ is $d$-$L$-inessential;*

*(ii). $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ or there exists a map $G \in A_{\partial U}(\overline{U}, Y; L, T)$ with $G^\star = I \times (L+T)^{-1}[G+T]$ and $G \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ such that $d\left((F^\star)^{-1}(B)\right) \neq d\left((G^\star)^{-1}(B)\right)$.*

*Proof.* (i) implies (ii) is immediate. Next we prove (ii) implies (i). If $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ then trivially (i) is true. Next suppose there exists a map $G \in A_{\partial U}(\overline{U}, Y; L, T)$ with $G^\star = I \times (L+T)^{-1}[G+T]$ and $G \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ such that $d\left((F^\star)^{-1}(B)\right) \neq d\left((G^\star)^{-1}(B)\right)$. Let $H : \overline{U} \times [0,1] \to 2^Y$ be a map with $H \in A(\overline{U} \times [0,1], Y; L, T)$, $Lx \notin H_t(x)$ for any $x \in \partial U \cap dom\, L$ and $t \in [0,1]$, $H_0 = F$ and $H_1 = G$ (here $H_t(x) = H(x,t)$). Let $H^\star : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by

$$H^\star(x,\lambda) = (x\,,\ (L+T)^{-1}[H+T](x,\lambda)).$$

Consider

$$D = \left\{x \in \overline{U} : (x,x) \in H^\star(x,t) \text{ for some } t \in [0,1]\right\}.$$

Notice that it is immediate that

$$D = \left\{x \in \overline{U} \cap dom\, L : (x, Lx) \in (x, H(x,t)) \text{ for some } t \in [0,1]\right\}.$$

If $D = \emptyset$ then in particular $(H^\star(x,0))^{-1}(B) = \emptyset$ i.e. $(F^\star)^{-1}(B) = \emptyset$ and as a result $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$, so $F^\star$ is $d$-$L$-inessential. Next suppose $D \neq \emptyset$. Note (see the argument in Theorem 2.8) that $D$ is closed in $E$. Also since $Lx \notin H_t(x)$ for $x \in \partial U \cap dom\, L$ and $t \in [0,1]$ then $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $R_\mu : \overline{U} \to 2^Y$ by $R_\mu(x) = H(x, \mu(x)) = H_{\mu(x)}(x) = H \circ \tau(x)$ and let $R_\mu^\star = I \times (L+T)^{-1}[R_\mu + T]$; here $\tau : \overline{U} \to \overline{U} \times [0,1]$ is given by $\tau(x) = (x, \mu(x))$. Notice $R_\mu \in A(\overline{U}, Y; L, T)$ (note (2.1) and $H \in A(\overline{U} \times [0,1], Y; L, T)$) and notice $R_\mu|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$ since $\mu(\partial U) = 0$. Thus $R_\mu \in A_{\partial U}(\overline{U}, Y; L, T)$ (note $Lx \notin H_t(x)$ for any $x \in \partial U \cap dom\, L$ and $t \in [0,1]$) with $R_\mu|_{\partial U} = F|_{\partial U}$.

Note also since $\mu(D)=1$ that

$$\begin{aligned}\left(R^\star_\mu\right)^{-1}(B) &= \left\{x\in\overline{U}:(x,x)\cap(x,(L+T)^{-1}[H+T](x,\mu(x))\neq\emptyset\right\}\\ &= \left\{x\in\overline{U}:(x,x)\cap(x,(L+T)^{-1}[H+T](x,1)\neq\emptyset\right\}=(G^\star)^{-1}(B)\end{aligned}$$

so $d\left(\left(R^\star_\mu\right)^{-1}(B)\right)=d\left((G^\star)^{-1}(B)\right)$. Thus $d\left((F^\star)^{-1}(B)\right)\neq d\left(\left(R^\star_\mu\right)^{-1}(B)\right)$.

We now claim

$$R_\mu \cong F \text{ in } A_{\partial U}(\overline{U},Y;L,T). \tag{2.15}$$

Let $Q:\overline{U}\times[0,1]\to 2^Y$ be given by $Q(x,t)=H(x,t\mu(x))=H\circ g(x,t)$ where $g:\overline{U}\times[0,1]\to\overline{U}\times[0,1]$ is given by $g(x,t)=(x,t\mu(x))$. Note $Q\in A(\overline{U}\times[0,1],Y;L,T)$ (note (2.1) and $H\in A(\overline{U}\times[0,1],Y;L,T)$), $Q_0=F$ and $Q_1=R_\mu$. Also $Lx\notin Q_t(x)$ for $x\in\partial U\cap dom\,L$ and $t\in[0,1]$ since if there exists $t\in[0,1]$ and $x\in\partial U\cap dom\,L$ with $Lx\in Q_t(x)$ then $Lx\in H(x,t\mu(x))$ so $x\in D$ and as a result $\mu(x)=1$ i.e. $Lx\in H(x,t)$, a contradiction. Thus (2.15) holds.

Consequently $F^\star$ is $d$-$L$-inessential (take $J=R_\mu$ in the definition of $d$-$L$-inessential). □

**Remark 2.27.** From the proof above we see that the result in Theorem 2.26 (with (2.1) removed) holds if the definition of $\cong$ is as in Remark 2.23.

Essentially the same reasoning as in Theorem 2.11 establishes the following result.

**Theorem 2.28.** *Let $E$ be a normal topological vector space, $Y$ a topological vector space, $U$ an open subset of $E$, $L:dom\,L\subseteq E\to Y$ a linear single valued map, $T\in H_L(E,Y)$, $d$ a map defined in* (2.13) *and assume* (2.1) *and* (2.14) *hold. Suppose $\Phi$ and $\Psi$ are two maps in $A_{\partial U}(\overline{U},Y;L,T)$ with $\Phi^\star=I\times(L+T)^{-1}[\Phi+T]$ and $\Psi^\star=I\times(L+T)^{-1}[\Psi+T]$ and with $\Phi\cong\Psi$ in $A_{\partial U}(\overline{U},Y;L,T)$. The $\Phi^\star$ is d-L-inessential if and only if $\Psi^\star$ is d-L-inessential.*

**Remark 2.29.** There is also an analogue of Theorem 2.15 for the maps in $A_{\partial U}(\overline{U},Y;L,T)$; we leave the details to the reader.

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