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POSITIVE SOLUTIONS TO THIRD-ORDER IMPULSIVE STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS WITH DEVIATED ARGUMENTS AND ONE-DIMENSIONAL p-LAPLACIAN

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ABSTRACT. In this paper, we establish the existence of at least three positive solutions for third-order impulsive Sturm-Liouville boundary value problems with p-Laplacian, by a fixed point theorem due to Avery and Peterson. We discuss our problem both for advanced and delayed arguments. An example is included to illustrate that corresponding assumptions are satisfied.

Key words: Differential equations with advanced and delayed arguments, p-Laplacian, multiple positive solutions, the fixed point theorem due to Avery and Peterson

AMS (MOS) Subject Classification: 34B10

1. INTRODUCTION

For $J = [0,1]$, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. Put $J' = (0,1) \setminus \{t_1, t_2, \ldots, t_m\}$. Put $\mathbb{R}_+ = [0,\infty)$ and $J_k = (t_k, t_{k+1}]$, $k = 0, 1, \ldots, m-1$, $J_m = (t_m, t_{m+1})$.

Let us consider the following problem

$$\begin{cases} (\phi_p(x''(t)))' + h(t)f(t, x(t), x(\alpha(t))) = 0, & t \in J', \\ x'(t_k^+) = x'(t_k^-) + Q_k(x(t_k)), & k = 1, 2, \ldots, m, \\ \beta x(0) - \gamma x'(0) = 0, \quad \delta x(1) + \eta x'(1) = 0, & x''(0) = 0 \end{cases} \tag{1.1}$$

with $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, where ϕ_p^{-1} denotes the inverse to ϕ_p; $x'(t_k^+)$ and $x'(t_k^-)$ denote the right and left limits of x' at t_k, respectively, and

$H_1:$ $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}_+)$, $h \in C(J, \mathbb{R}_+)$, $\alpha \in C(J, J)$,
$H_2:$ $Q_k \in C(\mathbb{R}, (-\infty, 0])$ and bounded for $k = 1, 2, \ldots, m$,
$H_3:$ $\beta, \gamma, \delta, \eta \geq 0$, $\Delta \equiv \beta(\delta + \eta) + \delta\gamma > 0$,
$H_4:$ there exists $\xi \in (0,1)$ and such that $\rho \in (0,1)$ with

$$\rho = \min\left(\frac{\eta}{\delta + \eta}, \frac{\gamma + \beta\xi}{\gamma + \beta}\right),$$

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$H_5:$ there exists $\sigma \in (0,1)$ and such that $\Gamma \in (0,1)$ with

$$\Gamma = \min\left(\frac{\delta(1-\sigma)+\eta}{\delta+\eta}, \frac{\gamma}{\gamma+\beta}\right).$$

Let us introduce the space:

$$PC^1(J,\mathbb{R}) = \left\{ \begin{array}{l} x \in C(J,\mathbb{R}),\ x|_{J_k} \in C^1(J_k,\mathbb{R}),\quad k = 0,1,\dots,m \\ \text{and there exist } x'(t_k^+) \text{ for } k = 1,2,\dots,m \end{array} \right\}.$$

By a positive solution of problem (1.1) we mean a function which is positive on $(0,1)$ and satisfies problem (1.1). Throughout this paper we assume that $\alpha(t) \not\equiv t$, $t \in J$.

Recently, many authors have been interested in studying the existence of positive solutions for differential equations with boundary conditions. The existence of positive solutions to third-order differential problems is discussed, for example in papers [4], [7], [8], [11]–[14], see also paper [5]. Equations with p-Laplacian operator arise in many applications to physical and natural phenomena, see for example [2], [3], [9], [10]. Corresponding results for the existence of positive solutions to third-order Sturm-Liouville boundary value problems with p-Laplacian are only formulated in paper [12], see also paper [6]. In my paper, we discuss impulsive Sturm-Liouville boundary problems of type (1.1) with deviated arguments α. To my knowledge, it is the first paper when positive solutions have been investigated for impulsive third-order Sturm-Liouville boundary value problems with p-Laplacian and with deviating arguments α which can be both of advanced or delayed type.

The organization of this paper is as follows. In Section 2, we present some necessary lemmas connected with the case when problem (1.1) is of advanced type. In Section 3, we present some definitions and a theorem of Avery and Peterson which is useful to obtain our main results. In Section 4, we discuss the existence of at least three positive solutions to problem (1.1) with advanced argument α, by using the above mentioned Avery-Peterson theorem. At the end of this section, an example is added to verify theoretical results. In the last Section 5, we formulate sufficient conditions under which problem (1.1) with delayed argument α has at least three positive solutions.

2. SOME LEMMAS

Let us consider the following problem

$$\begin{cases} (\phi_p(u''(t)))' + y(t) = 0, & t \in J', \\ u'(t_k^+) = u'(t_k^-) + Q_k, & k = 1,2,\dots,m, \\ \beta u(0) - \gamma u'(0) = 0, & \delta u(1) + \eta u'(1), \end{cases} \tag{2.1}$$

$$u''(0) = 0. \tag{2.2}$$

Lemma 2.1. *Assume that* $\Delta \equiv \beta(\delta+\eta)+\delta\gamma \neq 0$ *and* $y \in C(J,\mathbb{R})$*. Then problem* (2.1)–(2.2) *has the unique solution given by the following formula*

$$u(t)=\int_0^1 G(t,s)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds-\sum_{i=1}^m G(t,t_i)Q_i, \quad t\in J_k, \tag{2.3}$$

for $k=0,1,\dots,m$*, where*

$$G(t,s)=\frac{1}{\Delta}\begin{cases}(\delta+\eta-\delta t)(\gamma+\beta s), & s\le t,\\ (\delta+\eta-\delta s)(\gamma+\beta t), & t\le s.\end{cases}$$

Proof. 1st step. The differential equation from (2.1) and condition (2.2) yield

$$u''(t)=-\phi_q\left(\int_0^t y(s)ds\right). \tag{2.4}$$

This shows that

$$u(t)=u(0)+u'(0)t+\sum_{i=1}^k{}' Q_i(t-t_i)-\int_0^t (t-s)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds, \;\; t\in J_k, \tag{2.5}$$

for $k=0,1,\dots,m$ with $\sum_{i=q}^s{}'\cdots=0$ if $q>s$. Indeed, (2.5) holds for $k=0$. Assume that formula (2.5) holds for a fixed integer $k=r\le m-1$. Integrating two times the differential equation in (2.4) we have

$$\begin{aligned}
u'(t)&=u'(t_{r+1}^+)-\int_{t_{r+1}}^t \phi_q\left(\int_0^s y(\tau)d\tau\right)ds, \quad t\in J_{r+1},\\
u(t)&=u(t_{r+1}^+)+(t-t_{r+1})u'(t_{r+1}^+)-\int_{t_{r+1}}^t\int_{t_{r+1}}^s \phi_q\left(\int_0^\tau y(w)dw\right)d\tau ds\\
&=u(0)+u'(0)t_{r+1}+\sum_{i=1}^r{}' Q_i(t_{r+1}-t_i)-\int_0^{t_{r+1}}(t_{r+1}-s)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds\\
&\quad+(t-t_{r+1})\left[u'(0)+\sum_{i=1}^r{}' Q_i+Q_{r+1}-\int_0^{t_{r+1}}\phi_q\left(\int_0^s y(\tau)d\tau\right)ds\right]\\
&\quad-\int_{t_{r+1}}^t (t-s)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds.
\end{aligned}$$

This relation proves that (2.4) holds for $k=r+1$. Hence, (2.5) holds, by induction.

2nd step. Now we are going to eliminate $u(0)$ and $u'(0)$ from (2.5). To do it we use the boundary conditions from (2.1), so

$$\begin{aligned}
&\beta u(0)-\gamma u'(0)=0,\\
&\delta\left[u(0)+u'(0)+\sum_{i=1}^m Q_i(1-t_i)-\int_0^1(1-s)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds\right]\\
&+\eta\left[u'(0)+\sum_{i=1}^m Q_i-\int_0^1\phi_q\left(\int_0^s y(\tau)d\tau\right)ds\right]=0.
\end{aligned}$$

Solve this system with respect to $u(0)$ and $u'(0)$ and then substitute it to formula (2.5) to obtain the assertion. The proof is complete. □

Lemma 2.2. *Let Assumption H_3 hold. Assume that $Q_i \le 0$, $i = 1, 2, \dots, m$ and $y \in C(J, \mathbb{R}_+)$. Then the unique solution u of problem* (2.1)–(2.2) *satisfies the condition $u(t) \ge 0$ on $[0, 1]$.*

Proof. Note that

$$\begin{aligned} u(0) &= \frac{\gamma}{\Delta}\left[\int_0^1 (\delta+\eta-\delta s)\phi_q\left(\int_0^s y(\tau)d\tau\right) ds - \sum_{i=1}^m (\delta+\eta-\delta t_i)Q_i\right] \ge 0,\\ u(1) &= \frac{\eta}{\Delta}\left[\int_0^1 (\gamma+\beta s)\phi_q\left(\int_0^1 y(\tau)d\tau\right) ds - \sum_{i=1}^m (\gamma+\beta t_i)Q_i\right] \ge 0. \end{aligned}$$

Since u is concave down (by formula (2.4)), then $u(t) \ge 0,\ t \in J$. This completes the proof. □

Lemma 2.3. *Let G be given as in Lemma* 2.1 *and let Assumption H_4 hold. Then we have the following result:*

$$\frac{G(t,s)}{G(s,s)} \le 1 \quad \text{for } t, s \in [0,1], \tag{2.6}$$

$$\frac{G(t,s)}{G(s,s)} \ge \rho \quad \text{for } t \in [\xi, 1],\ \ s \in [0,1]. \tag{2.7}$$

Proof. Relation (2.6) is simply for proving. Note that

$$\frac{G(t,s)}{G(s,s)} = \frac{\delta+\eta-\delta t}{\delta+\eta-\delta s} \ge \frac{\eta}{\delta+\eta-\delta s} \ge \frac{\eta}{\delta+\eta}, \quad \text{for } s \le t,$$

$$\frac{G(t,s)}{G(s,s)} = \frac{\gamma+\beta t}{\gamma+\beta s} \ge \frac{\gamma+\beta\xi}{\gamma+\beta s} \ge \frac{\gamma+\beta\xi}{\gamma+\beta}, \quad \text{for } s \ge t$$

for $t \in [\xi, 1]$, $s \in [0, 1]$. This completes the proof. □

Lemma 2.4. *Let Assumptions H_3 and H_4 hold. Assume that $Q_i \le 0$, $i = 1, 2, \dots, m$ and $y \in C(J, \mathbb{R}_+)$. Then the unique solution u of problem* (2.1)–(2.2) *satisfies the condition*

$$\min_{[\xi,1]} u(t) \ge \rho\|u\|.$$

Proof. In view of (2.3) and (2.6), we obtain

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\phi_q\left(\int_0^s y(\tau)d\tau\right) ds - \sum_{i=1}^m G(t,t_i)Q_i\\ &\le \int_0^1 G(s,s)\phi_q\left(\int_0^s y(\tau)d\tau\right) ds - \sum_{i=1}^m G(t_i,t_i)Q_i, \end{aligned}$$

so

$$\|u\| \le \int_0^1 G(s,s)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds - \sum_{i=1}^m G(t_i,t_i)Q_i$$

This and condition (2.7) show that

$$\begin{aligned}\min_{[\xi,1]} u(t) &= \min_{[\xi,1]}\left[\int_0^1 G(t,s)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds - \sum_{i=1}^m G(t,t_i)Q_i\right] \\ &\ge \rho\left[\int_0^1 G(s,s)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds - \sum_{i=1}^m G(t_i,t_i)Q_i\right] \\ &\ge \rho\|u\|.\end{aligned}$$

This completes the proof. □

3. BACKGROUND MATERIALS AND DEFINITIONS

In this section, we provide some background materials from the theory of cones in Banach spaces.

Definition 3.1. Let E be a real Banach space. A nonempty convex set $P \subset E$ is said to be a cone provided that the following are satisfied:

(i) $ku \in P$ for all $u \in P$ and all $k \ge 0$, and
(ii) $u, -u \in P$ implies $u = 0$.

Note that every cone $P \subset E$ induces an ordering in E given by $x \le y$ if $y - x \in P$.

Definition 3.2. A map Λ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\Lambda : P \to \mathbb{R}_+$ is continuous and

$$\Lambda(tx + (1-t)y) \ge t\Lambda(x) + (1-t)\Lambda(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Similarly, we say the map φ is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\varphi : P \to \mathbb{R}_+$ is continuous and

$$\varphi(tx + (1-t)y) \le t\varphi(x) + (1-t)\varphi(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Definition 3.3. An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Let φ and Θ be nonnegative continuous convex functionals on P, Λ be a nonnegative continuous concave functional on P, and Ψ be a nonnegative continuous functional on P. Then for positive numbers a, b, c and d, we define the following sets:

$$P(\varphi, d) = \{x \in P : \ \varphi(x) < d\},$$

$$P(\varphi,\Lambda,b,d)=\{x\in P:\ b\le\Lambda(x),\ \varphi(x)\le d\},$$

$$P(\varphi,\Theta,\Lambda,b,c,d)=\{x\in P:\ b\le\Lambda(x),\ \Theta(x)\le c,\ \varphi(x)\le d\},$$

and

$$R(\varphi,\Psi,a,d)=\{x\in P:\ a\le\Psi(x),\ \varphi(x)\le d\}.$$

We will use the following fixed point theorem of Avery and Peterson to establish multiple positive solutions to problem (1.1).

Theorem 3.4 (see [1])**.** *Let P be a cone in a real Banach space E. Let φ and Θ be nonnegative continuous convex functionals on P, Λ be a nonnegative continuous concave functional on P, and Ψ be a nonnegative continuous functional on P satisfying $\Psi(kx)\le k\Psi(x)$ for $0\le k\le 1$, such that for some positive numbers M and d,*

$$\Lambda(x)\le\Psi(x)\quad\textit{and}\quad\|x\|\le M\varphi(x)$$

for all $x\in\overline{P(\varphi,d)}$. Suppose

$$T:\overline{P(\varphi,d)}\to\overline{P(\varphi,d)}$$

is completely continuous and there exist positive numbers a,b and c with $a<b$ such that

$(S_1):$ *$\{x\in P(\varphi,\Theta,\Lambda,b,c,d):\Lambda(x)>b\}\neq\emptyset$ and $\Lambda(Tx)>b$ for $x\in P(\varphi,\Theta,\Lambda,b,c,d)$;*
$(S_2):$ *$\Lambda(Tx)>b$ for $x\in P(\varphi,\Lambda,b,d)$ with $\Theta(Tx)>c$,*
$(S_3):$ *$0\notin R(\varphi,\Psi,a,d)$ and $\Psi(Tx)<a$ for $x\in R(\varphi,\Psi,a,d)$ with $\Psi(x)=a$.*

Then T has at least three fixed points $x_1,x_2,x_3\in\overline{P(\varphi,d)}$, such that

$$\varphi(x_i)\le d,\quad\textit{for}\quad i=1,2,3,$$

$$b<\Lambda(x_1),\quad a<\Psi(x_2),\quad\textit{with}\quad\Lambda(x_2)<b$$

and

$$\Psi(x_3)<a.$$

4. CASE WHERE $\alpha(t)\ge t$ ON J

Let $X=C(J,\mathbb{R})$ be our Banach space with the maximum norm $\|x\|=\max_{t\in J}|x(t)|$. Let

$$P=\left\{x\in X:\ x(t)\ge 0,\ t\in J\quad\text{and}\quad\min_{[\xi,1]}x(t)\ge\rho\|x\|\right\},$$
$$\bar{P}_r=\{x\in P:\ \|x\|\le r\},$$

where ρ is defined as in Assumption H_4. Note that $P\subset X$ is a cone. Now, we define the nonnegative continuous concave functional Λ on P by

$$\Lambda(x)=\min_{[\xi,1]}|x(t)|.$$

Indeed, $\Lambda(x)\le\|x\|$. Put $\Psi(x)=\Theta(x)=\|x\|$.

Theorem 4.1. *Let Assumptions H_1–H_4 hold and let $\alpha(t) \geq t$, $t \in J$. In addition, we assume that there exist positive constants Q, a, b, d, $a < b$, and such that*

$$\mu \geq \int_0^1 G(s,s)\phi_q\left(\int_0^s h(\tau)d\tau\right)ds + Q,$$

$$0 < L < \rho\int_\xi^1 G(s,s)\phi_q\left(\int_0^s h(\tau)d\tau\right)ds$$

and

(A_1) : $-\sum_{i=1}^m G(t_i,t_i)Q_i(u_i)) \leq \frac{Q}{\mu}d$ *for* $u_i \in [0,d]$ *and* $f(t,u,v) \leq \phi_p(\frac{d}{\mu})$ *for* $(t,u,v) \in J \times [0,d] \times [0,d]$,

(A_2) : $f(t,u,v) \geq \phi_p\left(\frac{b}{L}\right)$ *for* $(t,u,v) \in [\xi,1] \times [b,\frac{b}{\rho}] \times [b,\frac{b}{\rho}]$,

(A_3) : $-\sum_{i=1}^m G(t_i,t_i)Q_i(u_i)) \leq \frac{Q}{\mu}a$ *for* $u_i \in [0,a]$ *and* $f(t,u,v) \leq \phi_p(\frac{a}{\mu})$ *for* $(t,u,v) \in J \times [0,a] \times [0,a]$.

Then, problem (1.1) *has at least three positive solutions x_1, x_2, x_3 satisfying $\|x_i\| \leq d$, $i = 1,2,3$ and*

$$b \leq \Lambda(x_1), \quad a < \|x_2\| \quad \text{with} \quad \Lambda(x_2) < b$$

and $\|x_3\| < a$.

Proof. Put $(Fx)(t) = h(t)f(t,x(t),x(\alpha(t)))$. Now we define an operator T by

$$(Tx)(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds - \sum_{i=1}^m G(t,t_i)Q_i(x(t_i)), \quad t \in J_k$$

for $k = 0,1,\ldots,m$, where G is defined as in Lemma 2.1.

Indeed, $T : X \to X$. Problem (1.1) has a solution x if and only if x solves the operator equation $x = Tx$.

Note that

$$(Tx)'' \leq 0.$$

It shows that Tx is concave down. Moreover

$$\begin{aligned}(Tx)(0) &= \frac{\gamma}{\Delta}\left[\int_0^1 (\delta+\eta-\delta s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds - \sum_{i=1}^m(\delta+\eta-\delta t_i)Q_i(x(t_i))\right] \\ &\geq 0, \\ (Tx)(1) &= \frac{\eta}{\Delta}\left[\int_0^1 (\gamma+\beta s)\phi_q\left(\int_0^1 (Tx)(\tau)d\tau\right)ds - \sum_{i=1}^m(\gamma+\beta t_i)Q_i(x(t_i))\right] \\ &\geq 0.\end{aligned}$$

This and the fact that Tx is concave down show that $Tx(t) \geq 0$, $t \in J$. Now, in view of condition (2.6), we see that

$$\|Tx\| = \max_{t\in J}\left[\int_0^1 G(t,s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds - \sum_{i=1}^m G(t,t_i)Q_i(x(t_i))\right]$$

$$\leq \int_0^1 G(s,s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds - \sum_{i=1}^m G(t_i,t_i)Q_i(x(t_i)).$$

Moreover, condition (2.7) yields

$$\begin{aligned}\min_{[\xi,1]}(Tx)(t) &= \min_{[\xi,1]}\left[\int_0^1 G(t,s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds - \sum_{i=1}^m G(t,t_i)Q_i(x(t_i))\right]\\ &\geq \rho\left[\int_0^1 G(s,s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds - \sum_{i=1}^m G(t_i,t_i)Q_i(x(t_i))\right],\end{aligned}$$

so

$$\min_{[\xi,1]}(Tx)(t) \geq \rho\|Tx\|.$$

This proves that $TP \subset P$.

Now we prove that the operator $T : P \to P$ is completely continuous. Let $x \in \bar{P}_r$. Then $|x| \leq r$. Note that h and f are continuous so h is bounded on J and f is bounded on $J \times [0,r] \times [0,r]$. It means that there exists a constant $K > 0$ such $\|Tx\| \leq K$. This proves that $T\bar{P}$ is uniformly bounded. Moreover, $(Tx)'$ is uniformly bounded too. On the other hand for $t_1, t_2 \in J$ there exists a constant $L_1 > 0$ such that

$$|(Tx)(t_1) - (Tx)(t_2)| = \left|\int_{t_2}^{t_1} (Tx)'(s)ds\right| \leq L_1|t_1 - t_2|.$$

This shows that $T\bar{P}$ is equicontinuous on J, so T is completely continuous.

Let $x \in \overline{P(\varphi, d)}$, so $0 \leq x(t) \leq d$, $t \in J$, and $\|x\| \leq d$. This means that also $0 \leq x(\alpha(t)) \leq d$, $t \in J$ because $0 \leq t \leq \alpha(t) \leq 1$ on J. By (2.6) and Assumption (A_1), we see that

$$\begin{aligned}\varphi(Tx) &= \|Tx\| = \max_{t\in J}|(Tx)(t)| = \max_{t\in J}(Tx)(t) =\\ &= \max_{t\in J}\left[\int_0^1 G(t,s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds - \sum_{i=1}^m G(t,t_i)Q_i(x(t_i))\right]\\ &\leq \int_0^1 G(s,s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds - \sum_{i=1}^m G(t_i,t_i)Q_i(x(t_i))\\ &\leq \int_0^1 G(s,s)\phi_q\left(\int_0^s \phi_p\left(\frac{d}{\mu}\right)h(\tau)d\tau\right)ds + \frac{Q}{\mu}d\\ &= \frac{d}{\mu}\left[\int_0^1 G(s,s)\phi_q\left(\int_0^s h(\tau)d\tau\right)ds + Q\right]\\ &\leq d.\end{aligned}$$

This proves that $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$.

Now we need to show that condition (S_1) is satisfied. Take

$$x(t) = \frac{1}{2}\left(b + \frac{b}{\rho}\right), \quad t \in J.$$

Then

$$\|x\| = \frac{b(\rho+1)}{2\rho} < \frac{b}{\rho} \quad \text{so} \quad \Lambda(x) = \min_{[\xi,1]} x(t) = \frac{b(\rho+1)}{2\rho} > b = \frac{b}{\rho}\rho > \rho\|x\|.$$

It proves that

$$\{x \in P(\varphi, \Theta, \Lambda, b, \frac{b}{\rho}, d) : \ b < \Lambda(x)\} \neq \emptyset.$$

Let $b \leq x(t) \leq \frac{b}{\rho}$ for $t \in [\xi, 1]$. Then $\xi \leq t \leq \alpha(t) \leq 1$ on $[\xi, 1]$. It yields $b \leq x(\alpha(t)) \leq \frac{b}{\rho}$ on $[\xi, 1]$. Note that

$$\begin{aligned}
\Lambda(Tx) &= \min_{[\xi,1]}(Tx)(t) \\
&= \min_{[\xi,1]}\left[\int_0^1 G(t,s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds - \sum_{i=1}^m G(t,t_i)Q_i(x(t_i))\right] \\
&\geq \rho\int_0^1 G(s,s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds \\
&\geq \rho\int_\xi^1 G(s,s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds \\
&\geq \rho\int_\xi^1 G(s,s)\phi_q\left(\int_0^s \phi_p\left(\frac{b}{L}\right)h(\tau)d\tau\right)ds \\
&\geq \frac{b}{L}\rho\int_\xi^1 G(s,s)\phi_q\left(\int_0^s h(\tau)d\tau\right)ds \\
&> b.
\end{aligned}$$

This proves that condition (S_1) holds.

Now we need to prove that condition (S_2) is satisfied. Take $x \in P(\varphi, \Lambda, b, d)$ and $\|Tx\| > \frac{b}{\rho}$. Then

$$\Lambda(Tx) = \min_{[\xi,1]}(Tx)(t) \geq \rho\|Tx\| > \rho\frac{b}{\rho} = b,$$

so condition (S_2) holds.

Indeed, $\varphi(0) = 0 < a$, so $0 \notin R(\varphi, \Psi, a, d)$. Suppose that $x \in R(\varphi, \Psi, a, d)$ with $\Psi(x) = \|x\| = a$. Then

$$\begin{aligned}
\Psi(Tx) = \|Tx\| &= \max_{t\in J}(Tx)(t) \\
&= \max_{t\in J}\left[\int_0^1 G(t,s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds - \sum_{i=1}^m G(t,t_i)Q_i(x(t_i))\right] \\
&\leq \int_0^1 G(s,s)\phi_q\left(\int_0^s (Fx)(\tau)d\tau\right)ds - \sum_{i=1}^m G(t_i,t_i)Q_i(x(t_i)) \\
&\leq \int_0^1 G(s,s)\phi_q\left(\int_0^s \phi_p\left(\frac{a}{\mu}\right)h(\tau)d\tau\right)ds + \frac{Q}{\mu}a \\
&= \frac{a}{\mu}\left[\int_0^1 G(s,s)\phi_q\left(\int_0^s h(\tau)d\tau\right)ds + Q\right] \\
&\leq a.
\end{aligned}$$

This shows that condition (S_3) is satisfied.

Since all the conditions of Theorem 3.4 are satisfied, problem (1.1) has at least three positive solutions x_1, x_2, x_3 such that $\|x_i\| \le d$ for $i = 1, 2, 3$,

$$b \le \min_{[\xi,1]} x_1(t), \quad a < \|x_2\| \text{ with } \min_{[\xi,1]} x_2(t) < b$$

and $\|x_3\| < a$. This ends the proof. □

Example 4.2. For $p = 3$, we consider the following problem

$$\begin{cases} [\phi_p(x''(t))]' + htf(t, x(\alpha(t)) = 0, \quad t \in J' = (0, 1), \\ x'(t_1^+) = x'(t_1^-) + Q_1, \ \ t_1 = \frac{1}{2}, \\ x(0) - x'(0) = 0, \quad x(1) + x'(1) = 0, \ \ x''(0) = 0, \end{cases} \tag{4.1}$$

where $h \ge 2\left(\frac{72}{13}\right)^2$ and

$$f(t, u) = \begin{cases} \dfrac{u^2}{\mu^2}, & u \in [0, 1], \\ \dfrac{1}{\mu^2} + \left(\dfrac{4}{L^2} - \dfrac{1}{\mu^2}\right)(u - 1), & u \in [1, 2], \\ \dfrac{4}{L^2}, & u \in [2, 4], \\ \dfrac{4}{L^2} + \left(\dfrac{d^2}{\mu^2} - \dfrac{4}{L^2}\right)\left(\dfrac{u-4}{d-4}\right), & u \in [4, d], \\ \dfrac{d^2}{\mu^2}, & u \ge d. \end{cases}$$

Indeed, $\Delta = 3$. Let $\xi \in (0, 1)$, $a = 1$, $b = 2$, $d > 4$, $Q_1 \in \left[-\frac{4}{9}, 0\right)$. Then $\rho = \frac{1}{2}$. Take $Q = 1$. Hence

$$L < \rho \int_\xi^1 G(s, s)\phi_q\left(\int_0^s h(\tau)d\tau\right) ds = \frac{1}{6\sqrt{2}}\sqrt{h}\int_\xi^1 (2 + s - s^2)s\, ds < \frac{13}{72\sqrt{2}}\sqrt{h},$$

$$\mu \ge \int_0^1 G(s, s)\phi_q\left(\int_0^s h(\tau)d\tau\right) ds + Q = \frac{\sqrt{h}}{3\sqrt{2}}\frac{13}{12} + Q \ge 2$$

and

$$-G(t_1, t_1)Q_1 = -\frac{3}{4}Q_1 \le \frac{1}{3}.$$

It means that for $\mu = 3$, $0 < L < \frac{13}{72\sqrt{2}}\sqrt{h}$, all the assumptions of Theorem 4.1 hold, so problem (4.1) has at least three positive solutions.

5. CASE WHERE $\alpha(t) \le t$ ON J

Similarly as Lemmas 2.3 and 2.4, we can prove the following results.

Lemma 5.1. *Let G be given as in Lemma 2.1 and let Assumption H_5 hold. Then we have the following result:*

$$\frac{G(t, s)}{G(s, s)} \ge \Gamma \quad \textit{for } t \in [0, \sigma], \ \ s \in [0, 1].$$

Lemma 5.2. *Let Assumptions H_3 and H_5 hold. Assume that $Q_i \leq 0$, $i = 1, 2, \ldots, m$ and $y \in C(J, \mathbb{R}_+)$. Then the unique solution u of problem* (2.1)–(2.2) *satisfies the condition*

$$\min_{[0,\,\sigma]} u(t) \geq \Gamma \|u\|.$$

Let $X = C(J, \mathbb{R})$ be our Banach space with the maximum norm $\|x\| = \max_{t \in J} |x(t)|$. Let

$$P = \left\{ x \in X :\ x(t) \geq 0,\ t \in J \quad \text{and} \quad \min_{[0,\,\sigma]} x(t) \geq \Gamma \|x\| \right\},$$
$$\bar{P}_r = \{x \in P :\ \|x\| \leq r\},$$

where Γ is defined as in Assumption H_5. We define the nonnegative continuous concave functional Λ on P by

$$\Lambda(x) = \min_{[0,\,\sigma]} |x(t)|.$$

Note that $\Lambda(x) \leq \|x\|$. Put $\Psi(x) = \Theta(x) = \|x\|$.

Theorem 5.3. *Let Assumptions H_1–H_3, H_5 hold and let $\alpha(t) \leq t$, $t \in J$. In addition, we assume that there exist positive constants Q, a, b, d, $a < b$, and such that*

$$\mu \geq \int_0^1 G(s,s)\phi_q\left(\int_0^s h(\tau)d\tau\right)ds + Q,$$
$$0 < L < \Gamma \int_0^\sigma G(s,s)\phi_q\left(\int_0^s h(\tau)d\tau\right)ds$$

and

$(A_4):$ $-\sum_{i=1}^m G(t_i, t_i)Q_i(u_i)) \leq \frac{Q}{\mu}d$ *for* $u_i \in [0, d]$ *and* $f(t, u, v) \leq \phi_p(\frac{d}{\mu})$ *for* $(t, u, v) \in J \times [0, d] \times [0, d]$,

$(A_5):$ $f(t, u, v) \geq \phi_p\left(\frac{b}{L}\right)$ *for* $(t, u, v) \in [0, \sigma] \times [b, \frac{b}{\Gamma}] \times [b, \frac{b}{\Gamma}]$,

$(A_6):$ $-\sum_{i=1}^m G(t_i, t_i)Q_i(u_i)) \leq \frac{Q}{\mu}a$ *for* $u_i \in [0, a]$ *and* $f(t, u, v) \leq \phi_p(\frac{a}{\mu})$ *for* $(t, u, v) \in J \times [0, a] \times [0, a]$.

Then, problem (1.1) *has at least three positive solutions x_1, x_2, x_3 satisfying $\|x_i\| \leq d$, $i = 1, 2, 3$ and*

$$b \leq \Lambda(x_1), \quad a < \|x_2\| \quad \textit{with} \quad \Lambda(x_2) < b$$

and $\|x_3\| < a$.

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