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The Characteristic Finite Volume Element Methods for the Two-dimensional Generalized Nerve Conduction Equation

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Abstract：In the paper, a fully discrete characteristic finite volume element method is introduced and analyzed for approximating the solution of a nonlinear-hyperbolic equation in 2-space variables. Piecewise quadratic trial functions and piecewise constant test functions are used to finally obtain error estimate $O(\Delta t+h^2)$. A numerical experiment is given which showed the method is practicable.

Key words：nonlinear hyperbolic equations, characteristic finite volume element method, error estimate, numerical experiment.

1. Introduction

The finite volume elements, which can be termed as the generalized difference methods, are viewed as a new approach of numerical discretization for partial differential equation[1--2]. Since their constructions are similar to those of some finite difference methods and their convergence can be analyzed in the framework of finite element methods, the finite volume element methods enjoy not only the simplicity of difference methods but also the accuracy of finite elements. Meanwhile, the finite volume element methods maintain the (local) mass conservation law. Consequently they have been widely used in many practical computations and extensively studied in theory. On the other hand, in many cases discrete scheme derived in terms of finite volume element methods is asymmetric, it brings us many difficulties in both theoretical research and realistic computations. It is usually necessary for us to seek for some suitable technique that can transform the asymmetric scheme into symmetric one. There are many results about finite volume element methods for elliptic problems and parabolic problems[3--5].

In the process of nerve conduction, nerve conduction signal u and its variability with respect to time and space can be characterized the two-dimensional pseudohyperbolic equation[6] in Mathematics. It is a class of important nonlinear evolution equation of much current interest. There are some results about the equations[7--9]. Since generalized nerve conduction equations are a class of nonlinear evolution which can describe lots of physical phenomenons and possess

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strong physical background, thus it is important for us to develop the studies across-the-board and deeply either from the theoretical point of view or from the numerical analysis and practical point of view.

In this paper, we are concerned with numerical approximation to the two-dimensional generalized nerve conduction equation:

$$u_{tt}+b(x,u,u_t)\nabla u_t-\Delta u_t-\Delta u=f(u)u_t-g(u),x\in\Omega,t\in J. \quad (1.1)$$

$$\frac{\partial u}{\partial n}=\frac{\partial u_t}{\partial n}=0,x\in\partial\Omega,t\in J. \quad (1.2)$$

$$u(x,0)=u_0(x),u_t(x,0)=w_0(x),x\in\Omega. \quad (1.3)$$

Where $\Omega=[0,1]^2, J=[0,T], \partial\Omega$ denotes the boundary of Ω. $b(x,u,u_t)=\{b_1(x,u,u_t),b_2(x,u,u_t)\}$, u_0 and w_0 are assumed to be enough smooth functions.

We make the following physical assumption (A):

(i) $f(s)$, $g(l)$ and $b_i(x,s,l)(i=1,2)$ are bounded, and ε-continuous with respect to s and l respectively. We give the definition of ε-continuous function $f(s)$: When $|s_1-s_2|\le\varepsilon$, there exists a positive constant L, such that $|f(s_1)-f(s_2)|\le L|s_1-s_2|$.

(ii) $u\in C^2(\Omega\times J)\cap L^\infty(W^3_\infty)\cap L^2(H^3(\Omega))$, $u_t\in L^2(H^3(\Omega))\cap L^\infty(W^3_\infty)$ $u_{tt}\in L^2(H^3(\Omega))$.

In the present paper, the generalized nerve conduction equation is regarded as a model problem and characteristic direction method is applied to deal with one-order hyperbolic part of the equation in the process of scheme construction. The trial function space is chosen as the quadratic element space of Lagrangian type. Finally, we obtain the desired $O(\Delta t+h^2)$ error bound. The primary advantage of this method is that: First, it involves only two time levels and maintains the mass conservation law. Second, the estimates of u_t is obtained at the same time. Since u_t is also an important physical parameter in practice, this scheme avoids arising two times error by using the common characteristic difference method to approximate u at first, then to approximate u_t.

The rest of this paper is organized as follows: In section 2, we present a full-discrete characteristic finite volume element scheme while introducing some notations. In section 3, we give some preliminaries. The error estimates are presented in section 4. In section 5, we carry out numerical experiments to observe the performance of the proposed scheme. The letter c and C will be generic positive constants and may be different each time they are used, ε will be an arbitrarily small positive constant.

2. Some Notations And Full-Discrete Characteristic Finite Volume Element Scheme

Let $v = u_t$ in (1.1), in order to construct finite volume scheme, added $v = u_t$ to the both sides of (1.1). Then (1.1) can be written as:

$$v_t + b(x,u,v)\nabla v - \Delta v - \Delta u + u_t = H(u,v), x \in \Omega, t \in J. \quad (2.1)$$

$$u_t = v, x \in \Omega, t \in J. \quad (2.2)$$

Where $H(u,v) = (f(u)+1)v - g(u)$.

The initial and boundary condition are given by

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, x \in \partial\Omega, t \in J. \quad (2.3)$$

$$u(x,0) = u_0(x), v(x,0) = w_0(x), x \in \Omega. \quad (2.4)$$

At first, in order to attain very high accuracy we apply characteristic direction method to deal with the first two terms of (2.1):

Let $\psi(x,u,v) = \sqrt{1 + |b(x,u,v)|^2}$,here we make a convention that Γ is defined as the characteristic direct of $\psi \frac{\partial}{\partial \Gamma} = \frac{\partial}{\partial t} + b(x,u,v)\nabla$.

We have

$$\psi \frac{\partial v}{\partial \Gamma} = \frac{\partial v}{\partial t} + b(x,u,v)\nabla v.$$

Then (2.1) can be rewritten as $\psi \frac{\partial v}{\partial \Gamma} - \Delta v - \Delta u + u_t = H(u,v)$.

Now, let N denotes a positive integer such that $N\Delta t = T$, $t_n = n\Delta t$ and $\Delta t = t_{j+1} - t_j$, and for a sequence $\varphi_j (j = 1,2,......N)$, define: $\partial_t \varphi_j = \frac{\varphi_{j+1} - \varphi_j}{\Delta t}$

$\varphi_j = \varphi(t_j)$.

As far as $\psi \frac{\partial v}{\partial \Gamma}$ is concerned, we consider the standard backward difference quotient error approximation in the parameter Γ [7] along the tangent to the characteristic from $(\tilde{x}, t_n)$ to

(x,t_{n+1})

$$\psi\frac{\partial v}{\partial \Gamma}(x,t_{n+1}) \quad \psi(x,t_{n+1})\frac{v(x,t_{n+1})-v(\tilde{x},t_n)}{\sqrt{\left|x-\tilde{x}\right|^2+\Delta t^2}}=\frac{v(x,t_{n+1})-v(\tilde{x},t_n)}{\Delta t}. \quad (2.5)$$

Where $\tilde{x}=x-b(x,u_{n+1},v_{n+1})\Delta t$.

Subsequently, we need briefly explain some standard notation from this paper. Setting T_h be a quasi-uniform triangulation of $\overline{\Omega}$, T_h consists of finite number of triangular elements K_Q. Q being the barycenter of triangle. Suppose that maximum angle of each element of triangulation T_h is not greater that $\frac{\pi}{2}$, and that the ratio γ of the lengths of two sides of the maximum angle satisfies $\gamma\in[\sqrt{\frac{2}{3}},\sqrt{\frac{3}{2}}]$. The corresponding dual decomposition of T_h is denoted by T_h^*, their detailed construction(see figure 1) is as follows: (i)Construction of $K_{p_0}^*$, suppose that $p_0\in\overline{\Omega}_h$ ($\overline{\Omega}_h$ denotes the set of the vertexes of all the triangular elements), p_{0i} is a point on $\overline{p_0p_i}$ such that $\overline{p_0p_{0i}}=\frac{1}{3}\overline{p_0p_i}$, connect successively p_{0i} to obtain a polygon $K_{p_0}^*$ surrounding p_0; (ii) Construction of K_m^*, let m be midpoint of a common side of two adjacent triangular elements, a polygon K_m^* surrounding m is obtained by connecting successively $p_{20}Q_{03}Q_2Q_{23}p_{02}Q_{12}Q_1\,Q_{01}\,p_{20}$ Where Q_{01} denotes the midpoint of $\overline{p_{20}p_{21}}$, other points are also similar.

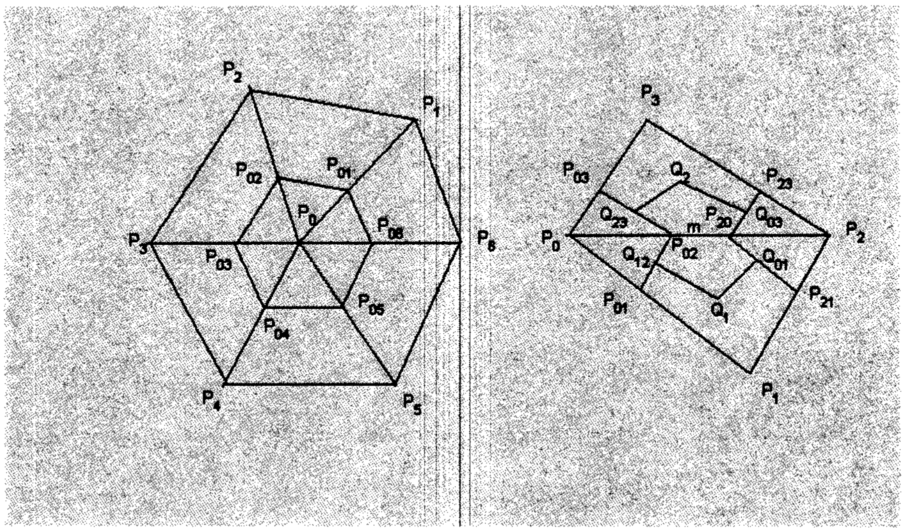
The trial function space U_h is chosen as the lagrangian quadratic element space related to T_h, the corresponding basis function are the piecewise quadratic polynomials. The test function space V_h is taken as the piecewise constant function space on T_h^*.

Let π_h: $H^1(\Omega)\to U_h$, then by the interpolation theory of Sobolev spaces [6], we obtain

$$\left\|u-\pi_h u\right\|_j\le Ch^{3-j}\left\|u\right\|_3, j=0,1,2, u\in H^3(\Omega). \quad (2.6)$$

We define the interpolation operator $\pi_h^*: U_h \to V_h$ by

$$\pi_h^* u_h = \sum_p u_h(p)\chi_p + \sum_m u_h(m)\chi_m. \qquad (2.7)$$



Left: Portion of triangulation sharing a common vertex P_0 and its control volume.

Right: Portion of two adjacent triangular elements sharing a midpoint of a common side m and its control volume.

Where χ_p and χ_m are respectively taken as the characteristic function corresponding to K_p^* and K_m^*.

Define $a_K(u,w), A_K(u,v,w)$ as follows:

$$a_K(u,w) = -\sum_{l=i,j,k} [w(p_l)\int_{e_l} \nabla u \cdot n ds + w(m_l)\int_{E_l} \nabla u \cdot n ds]. \qquad (2.8)$$

$$A_K(u,v,w) = -\sum_{l=i,j,k} [w(p_l)\int_{e_l} \nabla(u+v)\cdot n ds + w(m_l)\int_{E_l} \nabla(u+v)\cdot n ds] \quad .$$

(2.9)

Where $w \in V_h, (u,v) \in H^1(K)\times H^1(K)$.

See figure 2: $e_l = \overline{p_{ll+1}p_{ll+2}}$, $E_l = \overline{p_{l+2l+1}Q_{l+2}QQ_{l+1}p_{l+1l+2}}$, $\forall K \in T_h$.

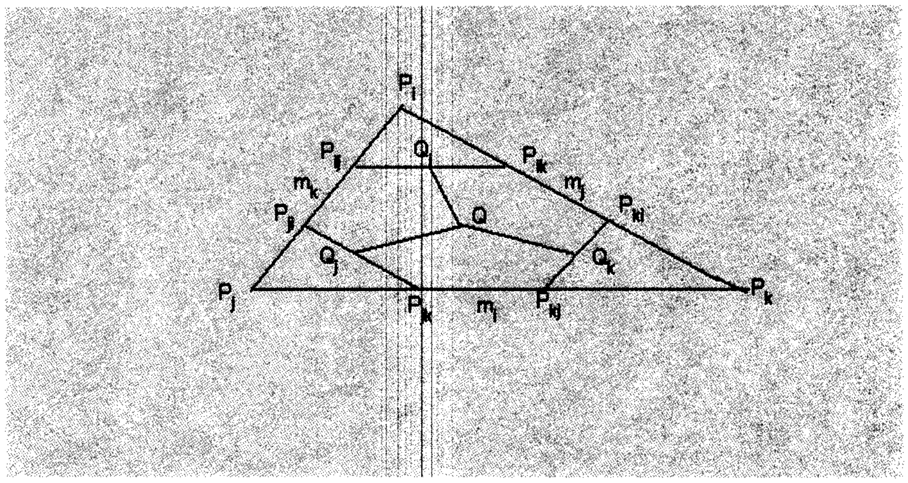
Where $i+1=j, j+1=k, k+1=i$, n is the unit outer normal vector on the boundary

.Note that $a(u,w)=\sum_K a_K(u,w), A(u,v,w)=\sum_K A_K(u,v,w)$.

Before presenting three discrete norms[10], we introduce two vectors on K related to $u_h \in U_h$.

$$(u_h)_{0,K}=\left(u_h(p_i),u_h(p_j),u_h(p_k),u_h(m_i),u_h(m_j),u_h(m_k)\right)^T. \qquad (2.10)$$

$$(u_h)_{1,K}=(u_h(p_i)-u_h(m_i),u_h(p_j)-u_h(m_j),u_h(p_k)-u_h(m_k)$$
$$u_h(m_i)-u_h(m_j),u_h(m_i)-u_h(m_k))^T. \qquad (2.11)$$



Any a triangular element K_Q.

$$\|u_h\|_{0,h}^2=\sum_K\|u_h\|_{0,h,K}^2=\sum_K\frac{m(K)}{1944}(u_h)_{0,K}^T B(u_h)_{0,K}. \qquad (2.12)$$

$$|u_h|_{1,h}^2=\sum_K|u_h|_{1,h,K}^2=\sum_K(u_h)_{1,K}^T(u_h)_{1,K}. \qquad (2.13)$$

$$\|u_h\|_{1,h}^2=\|u_h\|_{0,h}^2+|u_h|_{1,h}^2. \qquad (2.14)$$

Where $B=\left(b_{ij}\right)_{6\times6}$ is a given symmetric positive definite matrix in [11].

Finally, setting $t=t_{n+1}$, then the corresponding variational problems for (2.1) and (2.2) are:

$$\left((v_t)_{n+1}+b(x,u_{n+1},v_{n+1})\cdot\nabla v_{n+1}+(u_t)_{n+1},\chi\right)+A(u_{n+1},v_{n+1},\chi)$$
$$=(H(u_{n+1},v_{n+1},\chi),\chi). \qquad (2.15)$$

$$(\partial_t u_n, \chi) = (v_{n+1} + \partial_t u_n - (u_t)_{n+1}, \chi). \qquad (2.16)$$

Where $\forall \chi \in V_h$.

At the same time, one sees that the full discrete characteristic finite volume element schemes for (2.1) and (2.2) read as: find $U_{n+1}, V_{n+1} \in U_h$, such that

$$(\frac{V_{n+1} - V_n}{\Delta t}, \chi) + (\frac{U_{n+1} - U_n}{\Delta t}, \chi) + A(U_{n+1}, V_{n+1}, \chi) = (H(U_n, V_n), \chi) \qquad (2.17)$$

$$(\frac{U_{n+1} - U_n}{\Delta t}, \chi) = (V_{n+1}, \chi). \qquad (2.18)$$

Where $\forall \chi \in V_h$, $n = 0,1,2........N-1$.

Assure that U_0 and V_0 respectively denote some approximation of u_0 and v_0 in U_h, satisfying :

$$(U_0 - u_0, \chi) = 0, (V_0 - w_0, \chi) = 0. \qquad (2.19)$$

It should be pointed out that [7] $\hat{x} = x - b(x, U_n, V_n) \cdot \Delta t, V_n = V(\hat{x})$. If $\hat{x}$ stays out of Ω, then using mirror reflection technique, we can find the symmetric point x^* of $\hat{x}$ with respect to $\partial\Omega$. At this time, we require $V_n = V_n(x^*)$.

3. Preliminaries

Lemma 3.1 $\forall u_h \in U_h$, **there exists positive constants** c_1 **and** c_2 **independent of** h, **such that**

$$c_1 \|u_h\|_{0,h} \le \|u_h\|_0 \le c_2 \|u_h\|_{0,h}. \qquad (3.1)$$

$$c_1 |u_h|_{1,h} \le |u_h|_1 \le c_2 |u_h|_{1,h} . \qquad (3.2)$$

$$c_1 \|u_h\|_{0,h} \le \|\pi_h^* u_h\|_0 \le c_2 \|u_h\|_{0,h} . \qquad (3.3)$$

Lemma 3.2 $\forall u_h \in U_h$, there exists a positive constant α, such that,

$$a(u_h, \pi_h^* u_h) \ge \alpha |u_h|_{1,h}^2 . \qquad (3.4)$$

For the proof of the above two lemma, we can refer to [10][11].

Lemma 3.2 (the trace theorem)[12]Suppose that Ω is a bounded region with a Lipschitz continuous boundary $\partial\Omega$, then there exists a positive constant C, such that,

$$\|u\|_{L^2(\partial\Omega)} \le C\|u\|_{L^2(\Omega)}^{\frac{1}{2}} \cdot \|u\|_{H^1(\Omega)}^{\frac{1}{2}}, \forall u \in H^1(\Omega). \qquad (3.5)$$

Lemma 3.3 [13][14] $\forall u_h \in U_h$

$$\|\underline{u}_h\|_{0,h} \le C\|u_h\|_{0,h}. \qquad (3.6)$$

$$\|\pi_h^* \underline{u}_h\|_0 \le C\|u_h\|_0. \qquad (3.7)$$

$$|\underline{u}_h|_{1,h} \le C|u_h|_{1,h}. \qquad (3.8)$$

Where $\underline{u}_h \in U_h$, we introduce a one-to-one operator mapping $\underline{u}_h$ to u_h, such that for any triangular element K, a relationship between $\underline{u}_h$ and u_h always holds: $(\underline{u}_h)_{0,K} = D(u_h)_{0,K}$. $D=(d_{ij})_{6\times6}$ is a non-singular matrix defined in [13] and v_h and $\underline{u}_h$ satisfy $(v_h, \pi_h^* \underline{u}_h) = \sum_K \frac{m(K)}{1944}(v_h)_{0,K}^T B(u_h)_{0,K}$.

Proof: Using the relation of u_h and $\underline{u}_h$, the identity (2.12) and the positive definite matrix B, finally we both obtain a positive definite quadratic form related to thevector $(u_h)_{0,K}$. Thus $\|u_h\|_{0,h}$ is equivalent with $\|\underline{u}_h\|_{0,h}$, there exists a positive constant C, such that $\|\underline{u}_h\|_{0,h} \le C\|u_h\|_{0,h}$.

Then $\|\pi_h^* \underline{u}_h\|_0 \le C\|\underline{u}_h\|_{0,h}$, $\|\underline{u}_h\|_{0,h} \le C\|u_h\|_{0,h}$, the desired result follows from the two inequality. This completes the proof of (3.7).

By the elliptic condition (lemma3.2), there exists a positive constant M, such that

$$|\underline{u}_h|_{1,h,K}^2 \le \frac{1}{\alpha} a_K(\underline{u}_h, \pi_h^* \underline{u}_h) = \frac{M}{\alpha m(K)}(u_h)_{1,K}^T G(\underline{u}_h)_{1,K}.$$

Where $G=(h_{ij})_{5\times5}$ and is a matrix defined in [11], moreover $|h_{ij}| \le S \cdot m(K)$. Let S be a fixed positive constant, $m(K)$ denotes area of triangular element K. Clearly we use holder inequality to obtain

$$\frac{M}{\alpha m(K)}(u_h)_{1,K}^T G(\underline{u}_h)_{1,K} \le C|u_h|_{1,h,K}|\underline{u}_h|_{1,h,K}.$$

By the above two estimate, it is an easy matter to deduce that $|\underline{u}_h|_{1,h,K} \le C|u_h|_{1,h,K}$. This completes the proof of (3.8).

Lemma 3.4. $\forall u_h, w_h, \underline{v}_h \in U_h$

$$|A(u,w,\pi_h^* \underline{v}_h) - A(u_h,w,\pi_h^* \underline{v}_h)| \le C(h^2\|u\|_3 + |\pi_h u - u_h|_1)|v_h|_{1,h}. \quad (3.9)$$

$$|A(u,w,\pi_h^* \underline{v}_h) - A(u_h,w,\pi_h^* \underline{v}_h)| \le C(h^2\|u\|_3 + |\pi_h u - u_h|_1)|v_h|_{1,h}. \quad (3.10)$$

Proof: $A_K(u,w,\pi_h^* \underline{v}_h) - A_K(u_h,w,\pi_h^* \underline{v}_h) =$

$$-\sum_{l=i,j,k}[(\underline{v}_h(m_{l+2}) - \underline{v}_h(p_l))\cdot \int_{\overline{Q_l p_{ll+1}}} \nabla(u-u_h)\cdot n ds + (\underline{v}_h(p_l) - \underline{v}_h(m_{l+1}))\cdot$$

$$\int_{\overline{Q_l p_{ll+1}}} \nabla(u-u_h)\cdot n ds + (\underline{v}_h(m_{l+2}) - \underline{v}_h(m_{l+1}))\cdot \int_{\overline{QQ_l}} \nabla(u-u_h)\cdot n ds].$$

Applying holder inequality, the trace theorem(3.5), together with interpolation estimate and inverse property of finite element methods, we deduce that

$$\left|\int_{\overline{Q_l p_{ll+1}}} \nabla(u-u_h)\cdot n ds\right|$$

$$\le \int_{\overline{Q_l p_{ll+1}}} |\nabla(u-\pi_h u)\cdot n| ds + \int_{\overline{Q_l p_{ll+1}}} |\nabla(\pi_h u - u_h)\cdot n| ds$$

$$\le h^{\frac{1}{2}}(\|\nabla(u-\pi_h u)\|_{L^2(\overline{Q_l p_{ll+1}})} + \|\nabla(\pi_h u - u_h)\|_{L^2(\overline{Q_l p_{ll+1}})})$$

$$\le h^{\frac{1}{2}}(\|u-\pi_h u\|_{H^1(K)}^{\frac{1}{2}} \cdot \|u-\pi_h u\|_{H^2(K)}^{\frac{1}{2}}$$

$$+\|\nabla(\pi_h u - u_h)\|_{L^2(K)}^{\frac{1}{2}} + \|\nabla(\pi_h u - u_h)\|_{H^1(K)}^{\frac{1}{2}})$$

$$\le C(h^2\|u\|_{3,K} + |\pi_h u - u_h|_{1,K}).$$

It is obvious that $|\underline{v}_h(m_{l+2}) - \underline{v}_h(p_l)| \le |\underline{v}_h|_{1,h,K} \le C|v_h|_{1,h,K}$.

By the similar technique, error estimate of other terms can easily establish, thus we further have the following result:

$$|A(u,w,\pi_h^* \underline{v}_h) - A(u_h,w,\pi_h^* \underline{v}_h)| \le C(h^2\|u\|_3 + |\pi_h u - u_h|_1)|v_h|_{1,h}.$$

An argument similar to the one in the above case implies that

$$|A(u,w,\pi_h^* \underline{v}_h) - A(u,w_h,\pi_h^* \underline{v}_h)| \le C(h^2\|w\|_3 + |\pi_h w - w_h|_1)|v_h|_{1,h}.$$

This completes the proof.

4. Error Estimate

Note that $U-u=\sigma-\eta, V-v=\xi-\theta$, where $\sigma=U-\pi_h u, \eta=u-\pi_h u$, $\xi=V-\pi_h v, \theta=v-\pi_h v$.

Substracting (2.17) from (2.15) and using (2.16) and (2.18),we have

$$(\frac{\xi_{n+1}-\xi_n}{\Delta t},\chi)=(\xi_{n+1}-\theta_{n+1},\chi)+(\frac{\hat{\xi}_n-\xi_n}{\Delta t},\chi)+(\frac{\theta_{n+1}-\hat{\theta}_n}{\Delta t},\chi)$$

$$+(\frac{\partial v_{n+1}}{\partial t}+b(x,U_n,V_n)\cdot\nabla v_{n+1}-\frac{v_{n+1}-\hat{v}_n}{\Delta t},\chi)$$

$$+((b(x,u_{n+1},v_{n+1})-b(x,U_n,V_n))\cdot\nabla v_{n+1},\chi)+(A(u_{n+1},v_{n+1},\chi)-A(U_{n+1},V_{n+1},\chi)$$

$$+(H(U_n,V_n)-H(u_{n+1},v_{n+1}),\chi)=\sum_{i=1}^{7}T_i^{n+1}(x). \tag{4.1}$$

Subtracting (2.16) from (2.18) yields

$$(\frac{\sigma_{n+1}-\sigma_n}{\Delta t},\chi)=(\frac{\eta_{n+1}-\eta_n}{\Delta t},\chi)+(\xi_{n+1}-\theta_{n+1},\chi)$$

$$+((u_t)_{n+1}-\partial_t u_n,\chi). \tag{4.2}$$

The following estimate result, which can be easily proved, will be used in our analysis.

$$\left\|\frac{w_{n+1}-w_n}{\Delta t}\right\|_0^2\le\left\|\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}w_t dt\right\|_0^2\le\frac{1}{\Delta t}\|w_t\|^2_{L^2((t_n,t_{n+1}),L^2)}. \tag{4.3}$$

Taking $\chi=\Delta t\pi_h^*\xi_{n+1}$ in (4.1) and summing from $n=0$ to $R-1$, then we will use different technique to deal with every term in the right hand side of the equality (4.1).
For the first term, it follows from (2.6) and (3.7)that

$$\sum_{n=0}^{R-1}T_1^{n+1}(\Delta t\pi_h^*\xi_{n+1})=\Delta t\sum_{n=0}^{R-1}(\xi_{n+1}-\theta_{n+1},\pi_h^*\xi_{n+1})\le C\Delta t\sum_{n=0}^{R-1}(\|\xi_{n+1}\|_0^2+\|\theta_{n+1}\|_0^2)$$

$$\le C(h^4+\Delta t\sum_{n=0}^{R-1}\|\xi_{n+1}\|_0^2). \tag{4.4}$$

It can be easily verified [15] that $\left\|\frac{\xi_k-\hat{\xi}_k}{\Delta t}\right\|_0\le C\|\xi_k\|_1$, using the initial value condition(B), together with $\Delta t=O(h^3)$ and inverse property of finite element, we find

$$\sum_{n=0}^{R-1} T_2^{n+1}(\Delta t \pi_h^* \underline{\xi}_{n+1}) = -\Delta t \sum_{n=0}^{R-1}(\frac{\xi_n - \hat{\xi}_n}{\Delta t}, \pi_h^* \underline{\xi}_{n+1}) \le C\Delta t \sum_{n=0}^{R-1}(\|\xi_n\|_1^2 + \|\xi_{n+1}\|_1^2)$$

$$\le C(\Delta t \sum_{n=0}^{R-1}\|\xi_{n+1}\|_0^2 + h\sum_{n=1}^{R-1}\|\xi_n\|_0^2 + \Delta t^2 + h^4). \qquad (4.5)$$

Deduce by an analogy for the second term estimate and use (4.3), we obtain

$$\sum_{n=0}^{R-1} T_3^{n+1}(\Delta t \pi_h^* \underline{\xi}_{n+1}) = \Delta t \sum_{n=0}^{R-1}(\frac{\theta_{n+1} - \hat{\theta}_n}{\Delta t}, \pi_h^* \underline{\xi}_{n+1}) = \Delta t \sum_{n=0}^{R-1}[(\frac{\theta_{n+1} - \theta_n}{\Delta t}, \pi_h^* \underline{\xi}_{n+1})$$

$$+(\frac{\theta_n - \hat{\theta}_n}{\Delta t}, \pi_h^* \underline{\xi}_{n+1})] \le C\Delta t \sum_{n=0}^{R-1}(\left\|\frac{\theta_{n+1} - \theta_n}{\Delta t}\right\|_0^2 + \|\theta_n\|_1^2 + \|\xi_{n+1}\|_0^2) \le C(h^4 + \Delta t \sum_{n=0}^{R-1}\|\xi_{n+1}\|_0^2) \quad (4.6)$$

Set $\psi(x, U_n, V_n) = (1 + |b(x, U_n, V_n)|^2)^{\frac{1}{2}}$, the characteristic direction corresponding to $\frac{\partial}{\partial t} + b(x, U_n, V_n)\cdot\nabla$ is denoted by $\Gamma(x, U_n, V_n)$, we can get

$$\psi \frac{\partial v_{n+1}}{\partial \Gamma} = \frac{\partial v_{n+1}}{\partial t} + b(x, U_n, V_n)\cdot \nabla v_{n+1}.$$

In terms of a taylor expansion with an integral remainder, we then arrive at

$$\sum_{n=0}^{R-1} T_4^{n+1}(\Delta t \pi_h^* \underline{\xi}_{n+1}) = \Delta t \sum_{n=0}^{R-1}(\psi \frac{\partial v_{n+1}}{\partial \Gamma} - \frac{v_{n+1} - \hat{v}_n}{\Delta t}, \pi_h^* \underline{\xi}_{n+1}) \le C\Delta t \sum_{n=0}^{R-1}(\|\xi_{n+1}\|_0^2$$

$$+\left\|\frac{1}{\Delta t}\int_{(\hat{x}, t_n)}^{(x, t_{n+1})}[|x(\Gamma) - \hat{x}|^2 + (t(\Gamma) - t_n)^2 \frac{\partial^3 v}{\partial \Gamma^3}]\right\|_0^2 \le C(\Delta t^4 \left\|\frac{\partial^3 v}{\partial \Gamma^3}\right\|_{L^2(0,T,L^2(\Omega))}^2 + \Delta t \sum_{n=0}^{R-1}\|\xi_{n+1}\|_0^2)$$

$$\le C(\Delta t^2 + \Delta t \sum_{n=0}^{R-1}\|\xi_{n+1}\|_0^2). \qquad (4.7)$$

In the following further analysis, we need two induction hypothesis (C) as follows:

(1) There exists a positive constant M, such that $\sup_{0<n<R}\|V_n\|_{0,\infty} \le M$.

(2) $\forall \varepsilon_0 > 0$, when Δt and h are sufficiently small, we have.

$$\sup_{0<n<R-1}\|w_{n+1} - W_n\|_{0,\infty} \le \varepsilon_0.$$

Where $w = u$ (or v), $W = U$ (or V), we will employ mathematical induction to proof the above two results.

For $n = 0$, integrating inverse property and interpolation estimate of finite element methods, together with $\Delta t = O(h^3)$ and initial value condition(B), as a result, we can obtain

$$\|V_0\|_{0,\infty} \le \|v_0\|_{0,\infty} + \|v_0 - \pi_h v_0\|_{0,\infty} + \|\pi_h v_0 - V_0\|_{0,\infty} \le C_1 + C_2 h^2 + C_3 h \le M.$$

$$\|w_1 - W_0\|_{0,\infty} \le \|w_1 - w_0\|_{0,\infty} + \|w_0 - W_0\|_{0,\infty} \le C_1 \Delta t^{1/2} + \|w_0 - \pi_h w_0\|_{0,\infty} + \|\pi_h w_0 - W_0\|_{0,\infty}$$

$\le C_1 h + C_2 h^2 + C_3 h \le \varepsilon_0$.

Assume that $\sup\limits_{0\le n\le R-1} \|V_n\|_{0,\infty} \le M$, $\sup\limits_{0\le n\le R-2} \|w_{n+1} - W_n\|_{0,\infty} \le \varepsilon_0$. then we give a proof for

$$\sup_{0\le n\le R} \|V_n\|_{0,\infty} \le M,\ \sup_{0\le n\le R-1} \|w_{n+1} - W_n\|_{0,\infty} \le \varepsilon_0.$$

while taking into account error estimate of the other terms.

Noticing that $b_i(x,l,s)$ are ε-continuous with respect to l and s respectively, it leads to

$$|b(x,u_{n+1},v_{n+1}) - b(x,U_n,V_n)| \le |b(x,u_{n+1},v_{n+1}) - b(x,U_n,v_{n+1})| + |b(x,U_n,v_{n+1}) - b(x,U_n,V_n)|$$

$$\le L_1 |u_{n+1} - U_n| + L_2 |v_{n+1} - V_n|.$$

Which together with (4.3) and (2.6) and initial value condition(B) implies

$$\sum_{n=0}^{R-1} T_5^{n+1}(\Delta t \pi_h^* \underline{\xi}_{n+1}) = \Delta t \sum_{n=0}^{R-1} ((b(x,u_{n+1},v_{n+1}) - b(x,U_n,V_n)) \cdot \nabla v_{n+1}, \pi_h^* \underline{\xi}_{n+1})$$

$$\le C\Delta t \sum_{n=0}^{R-1} (\|u_{n+1} - U_n\|_0^2 + \|v_{n+1} - V_n\|_0^2 + \|\xi_{n+1}\|_0^2)$$

$$\le C\Delta t \sum_{n=0}^{R-1} (\|u_{n+1} - u_n\|_0^2 + \|u_n - U_n\|_0^2 + \|v_{n+1} - v_n\|_0^2 + \|v_n - V_n\|_0^2 + \|\xi_{n+1}\|_0^2)$$

$$\le C(\Delta t^2 + h^4 + \Delta t \sum_{n=0}^{R-2} \|\sigma_{n+1}\|_0^2 + \Delta t \sum_{n=0}^{R-1} \|\xi_{n+1}\|_0^2). \tag{4.8}$$

Similar to lemma(3.5), we have

$$|A(u_{n+1}, v_{n+1}, \pi_h^* \underline{\xi}_{n+1}) - A(U_{n+1}, V_{n+1}, \pi_h^* \underline{\xi}_{n+1})|$$

$$\le |A(u_{n+1}, v_{n+1}, \pi_h^* \underline{\xi}_{n+1}) - A(U_{n+1}, v_{n+1}, \pi_h^* \underline{\xi}_{n+1})| + |A(U_{n+1}, v_{n+1}, \pi_h^* \underline{\xi}_{n+1}) - A(U_{n+1}, V_{n+1}, \pi_h^* \underline{\xi}_{n+1})|$$

$$\le C(h^2 \|u_{n+1}\|_3 + h^2 \|v_{n+1}\|_3 + |\sigma_{n+1}|_1 + |\xi_{n+1}|_1) \cdot |\xi_{n+1}|_{1,h}.$$

We use lemma(3.1), $\Delta t = O(h^3)$ and inverse property of inite element to conclude that

$$\sum_{n=0}^{R-1} T_6^{n+1}(\Delta t \pi_h^* \underline{\xi}_{n+1}) = \Delta t \sum_{n=0}^{R-1} (A(u_{n+1}, v_{n+1}, \pi_h^* \underline{\xi}_{n+1}) - A(U_{n+1}, V_{n+1}, \pi_h^* \underline{\xi}_{n+1}))$$

$$\le C(h^4 + h \sum_{n=0}^{R-1} \|\sigma_{n+1}\|_0^2 + h \sum_{n=0}^{R-1} \|\xi_{n+1}\|_0^2). \tag{4.9}$$

For sufficiently small Δt and h, a combination of riangluar inequality and assumption (A) and induction ypothesis(C) results in

$$|H(U_n,V_n) - H(u_{n+1},v_{n+1})| \le |V_n| \cdot |f(U_n) - f(u_{n+1})| + (|f(u_{n+1})| + 1) \cdot |V_n - v_{n+1}| + |g(U_n) - g(u_{n+1})|$$

$$\le (L_1 \|V_n\|_{0,\infty} + L_2)(|U_n - u_n| + |u_{n+1} - u_n|) + (|f(u_{n+1})| + 1)(|V_n - v_n| + |v_{n+1} - v_n|)$$

$$\le C(|U_n - u_n| + |u_{n+1} - u_n| + |V_n - v_n| + |v_{n+1} - v_n|).$$

This together with initial value condition (B) yields

$$\sum_{n=0}^{R-1} T_7^{n+1}(\Delta t \pi_h^* \xi_{n+1}) = \Delta t \sum_{n=0}^{R-1} (H(U_n, V_n) - H(u_{n+1}, v_{n+1}), \pi_h^* \xi_{n+1})$$

$$\le C_1[\Delta t \sum_{n=0}^{R-1} (\|\sigma_n\|_0^2 + \|\eta_n\|_0^2 + \|\xi_n\|_0^2 + \|\theta_n\|_0^2) + \Delta t^2] + C_2 \Delta t \sum_{n=0}^{R-1} \|\xi_{n+1}\|_0^2$$

$$\le C(h^4 + \Delta t^2 + \Delta t \sum_{n=0}^{R-2} \|\sigma_{n+1}\|_0^2 + \Delta t \sum_{n=0}^{R-1} \|\xi_{n+1}\|_0^2). \tag{4.10}$$

Now, we turn to the error estimate in the left-hand terms of (4.1), recalling the discrete norms of (2.12) and using $ab \le \frac{1}{2}(a^2 + b^2)$, we obtain

$$\sum_{n=0}^{R-1} (\frac{\xi_{n+1} - \xi_n}{\Delta t}, \Delta t \pi_h^* \xi_{n+1}) \ge \sum_{n=0}^{R-1} [\|\xi_{n+1}\|_{0,h}^2 - \frac{1}{2}(\|\xi_n\|_{0,h}^2 + \|\xi_{n+1}\|_{0,h}^2)]$$

$$\ge \frac{1}{2} \sum_{n=0}^{R-1} (\|\xi_{n+1}\|_{0,h}^2 - \|\xi_n\|_{0,h}^2). \tag{4.11}$$

At the same time, choosing $\chi = \Delta t \pi_h^* \sigma_{n+1}$ in (4.2), we show the error estimate of two-hand sides of (4.2) and sum from $n = 0$ to $R-1$.

We proceed in analogy with (4.11) and conclude that

$$\sum_{n=0}^{R-1} (\frac{\sigma_{n+1} - \sigma_n}{\Delta t}, \Delta t \pi_h^* \sigma_{n+1}) \ge \frac{1}{2} \sum_{n=0}^{R-1} (\|\sigma_{n+1}\|_{0,h}^2 - \|\sigma_n\|_{0,h}^2). \tag{4.12}$$

By lemma (2.6) and (4.3), we have

$$\sum_{n=0}^{R-1} (\frac{\eta_{n+1} - \eta_n}{\Delta t}, \Delta t \pi_h^* \sigma_{n+1}) \le C \Delta t \sum_{n=0}^{R-1} (\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \|\eta_t\|_0^2 dt + \|\sigma_{n+1}\|_0^2)$$

$$\le C(h^4 + \Delta t \sum_{n=0}^{R-1} \|\sigma_{n+1}\|_0^2). \tag{4.13}$$

The following proof parallels to that of (4.4)

$$\sum_{n=0}^{R-1} (\xi_{n+1} - \theta_{n+1}, \Delta t \pi_h^* \sigma_{n+1}) \le C(h^4 + \Delta t \sum_{n=0}^{R-1} \|\sigma_{n+1}\|_0^2 + \Delta t \sum_{n=0}^{R-1} \|\xi_{n+1}\|_0^2). \tag{4.14}$$

Let $r_{n+1} = (u_t)_{n+1} - \frac{u_{n+1} - u_n}{\Delta t}$, it is easy to see [6] that $r_{n+1} = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (t - t_n) u_{tt} dt$

We have the following results

$$\|r_{n+1}\|_0^2 \le \frac{1}{\Delta t^2} (\int_{t_n}^{t_{n+1}} |t - t_n| \cdot \|u_{tt}\|_0)^2 \le \Delta t \int_{t_n}^{t_{n+1}} \|u_{tt}\|_0^2 dt$$

Consequently,

$$\sum_{n=0}^{R-1}((u_t)_{n+1}-\partial_t u_n, \Delta t \pi_h^* \underline{\sigma}_{n+1}) \le C(\Delta t^2 + \Delta t \sum_{n=0}^{R-1} \|\sigma_{n+1}\|_0^2). \tag{4.15}$$

By virtue of (4.12)-(4.15), we have

$$\|\sigma_R\|_0^2 \le C(h^4 + \Delta t^2 + \|\sigma_0\|_0^2 + \Delta t \sum_{n=0}^{R-1} \|\sigma_{n+1}\|_0^2 + \Delta t \sum_{n=0}^{R-1} \|\xi_{n+1}\|_0^2). \tag{4.16}$$

Collecting (4.4)-(4.11) and (4.16), let Δt and h be sufficiently small, we can obtain

$$\|\sigma_R\|_0^2 + \|\xi_R\|_0^2 \le C(h^4 + \Delta t^2 + \|\sigma_0\|_0^2 + \|\xi_0\|_0^2 + (\Delta t + h)\sum_{n=0}^{R-1} \|\sigma_{n+1}\|_0^2 + (\Delta t + h)\sum_{n=0}^{R-1} \|\xi_{n+1}\|_0^2) \tag{4.17}$$

Consequently, the discrete Gronwall inequality argument and initial value condition (B) produce

$$\|\sigma_R\|_0^2 + \|\xi_R\|_0^2 \le C(h^4 + \Delta t^2). \tag{4.18}$$

Combining (4.18) and (2.6) lead to

$$\sup_{0\le n\le N} (\|u_n - U_n\|_0^2 + \|v_n - V_n\|_0^2) \le C(\Delta t^2 + h^4). \tag{4.19}$$

Where positive integer R is not greater than N.

It remains to check the induction hypothesis (C),

For $n = R$,

$$\|V_R\|_{0,\infty} \le \|v_R\|_{0,\infty} + \|v_R - V_R\|_{0,\infty} \le C_1 + C_2 h \le M.$$

We also see that for $n = R-1$,

$$\|W_{R-1} - w_R\|_{0,\infty} \le \|w_R - w_{R-1}\|_{0,\infty} + \|w_{R-1} - W_{R-1}\|_{0,\infty} \le \varepsilon_0.$$

Theorem 4.1 : Let u, v be the solutions to problem(2.1)-(2.4), $\{U_k\}_{k=0}^N$, $\{V_k\}_{k=0}^N$ to the characteristic finite volume element scheme (2.17)-(2.19). Suppose that initial value U_0 and V_0 satisfy the conditions(B), i.e.,

$$\|U_0 - \pi_h u_0\|_0 + \|V_0 - \pi_h v_0\|_0 \le C(\Delta t + h^2),$$

if partition parameters satisfy $\Delta t = O(h^3)$, provided initial assumption(A) is satisfied, then for sufficiently small Δt and h, the following error estimate holds

$$\sup_{0\le k\le N} (\|u_k - U_k\|_0 + \|v_k - V_k\|_0) \le C(\Delta t + h^2). \tag{4.20}$$

5. Numerical Experiment

The characteristic finite volume element method is used to approximate the following nonlinear hyperbolic equation:

$$u_{tt} - \Delta u_t - \Delta u = r(x,t), x \in \Omega = (0,\pi)\times(0,\pi), t \in (0,\frac{1}{2}].$$

$$u(x,0) = \sin 2x_1 \sin x_2, u_t(x,0) = -\sin 2x_1 \sin x_2, x \in \Omega.$$

$$u = 0, u_t = 0, x \in \partial\Omega, t \in (0,\frac{1}{2}].$$

Where the true solution $u = e^{-t} \sin 2x_1 \sin x_2$, $r(x,t) = u = e^{-t} \sin 2x_1 \sin x_2$. To obtain numerical solution to this problem, we place over $\overline{\Omega} = [0,\pi]\times[0,\pi]$ $6\times 6 = 36$ uniform squares, ending up with a square mesh; Then we further decompose it into right triangulation by drawing the dragonal of each small square. $\pi/6$ and 0.1 respectively denote space mesh size h and time step size Δt.

Two generalize difference methods are used to solve that problem.

(1) The characteristic finite volume element method on triangular meshes denoted by CTFVM.

(2) The bilinear finite volume element method along characteristics on quadrilateral networks denoted by CQFVM.

The numerical results and corresponding true solution (TS) are partly given in Table 1, the maximum absolute error(MAE)($\max_{i=1}^{M}|u_i - U_i|$) and the average absolute error(AAE) ($(\sum_{i=1}^{M}|u_i - U_i|)/M$) is also provided in Table 2. It is easy to see that the finite volume element method on the triangular mesh behaves better than the one on the quadrilateral mesh. Although the accuracy on trianglar grid is greatly improved, the algorithm is slightly more complicated than the case that the mesh is quadrilateral.

Table 1. The comparison between CTFVM and CQFVM with $t = 0.3$.

(x_i, y_j)	CTFVM(U_h)	CQFVM(U_h)	TS(u)
$(\pi/6, \pi/3)$	0.5580333	0.5692364	0.5556136
$(\pi/6, 2\pi/3)$	0.5591276	0.5692375	0.5556137
$(\pi/3, \pi/3)$	0.5570505	0.5692383	0.5556137
$(\pi/3, \pi/2)$	0.6456243	0.6572987	0.6415675
$(2\pi/3, \pi/3)$	-0.5586519	-0.5692382	-0.5556135
$(2\pi/3, \pi/2)$	-0.6456393	-0.6573002	-0.6415672

$(5\pi/6, \pi/3)$	-0.5580313	-0.5692373	-0.5556138
$(5\pi/6, 2\pi/3)$	-0.5596563	-0.5692372	-0.5556139

Table 2. The comparison of MAE and AAE between CTFVM and CQFVM

	MAE	AAE
CTGDM (u)	2.0317098E-02	2.4786643E-03
CQGDM (u)	2.7010024E-02	9.4673503E-03

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