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A Stochastic Differential Equation Model for Cotton Fiber Breakage

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Abstract
A stochastic differential equation model is derived for cotton fibers that are experiencing breakage. The model provides greater understanding of the fiber breakage phenomenon and the origination of different fiber-length distributions. In the stochastic model, the fibers are grouped by length. In this manner, the cotton fiber distribution can be considered as a population distribution. An Itô stochastic differential equation model is derived by carefully considering the population process and breakage possibilities over a short time interval. Comparisons between the stochastic model and Monte Carlo calculations indicate that a stochastic differential equation can accurately model fiber-length distributions. In addition, the stochastic model generalizes classic deterministic integro-differential equation models for fiber breakage.
Keywords - model, stochastic differential equation, Itô, cotton, fiber breakage

1. Introduction

In cotton thread manufacture, the cotton fiber length distribution determines many of the characteristics of the thread. Fiber length is a good indicator of spinning efficiency, yarn strength, and yarn uniformity. Fiber length distribution is affected by breakage during processing (Krifa, 2006; Meyer et al., 1966). In cotton processing, fiber breakage occurs in ginning and carding. Breakage of the fibers in cotton processing generally results in lower quality yarn.

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The development of a stochastic differential equation (SDE) model for fiber-length distributions provides more understanding of the fiber breakage phenomenon and the origination of different fiber-length distributions (Hakan, 2007). By comparing calculations of the stochastic model with fiber-length data, fiber breakage parameters can be estimated and distribution characteristics can be investigated.

In the stochastic model, the fibers are grouped by length. In this manner, the cotton fiber distribution can be considered as a population distribution. The SDE model is derived by carefully considering the population process and breakage possibilities over a short time interval using stochastic modeling techniques described, for example, in (Allen, 1999; Allen, 2003; Allen, 2007). First, a discrete stochastic model is derived where the breakage phenomenon is carefully studied for a short time interval. Then, a system of stochastic differential equations is identified whose probability distribution approximates that of the discrete stochastic model.

Comparisons of calculational results using a stochastic model for cotton fiber breakage with Monte Carlo computational results indicate that an SDE model can accurately estimate fiber-length distributions. In addition, the SDE model generalizes classic deterministic integro-differential equation models for fiber breakage described, for example, in (Meyer et al., 1966). Furthermore, the SDE model gives information on the variability in fiber-length distributions which deterministic models are unable to provide.

2. SDE Model Construction

In developing an SDE model, m populations, $\{N_k(t)\}_{k=1}^m$, of fibers having different lengths are considered as functions of time t. Some terminology associated with the stochastic model is required and is introduced as follows.

Let L = fiber length where it is assumed that $0 \le L \le L_{max}$.

Let $L_k = kh$ for $k = 0, 1, 2, \ldots, m$ where $h = L_{max}/m$.

Let $N_k(t)$ = number of fibers of length L_k for $k = 1, 2, \ldots, m$.

Let $q_k\,dt$ = fraction of fibers of length k broken in time dt.

Let $S_{k,l}$ = fraction of fragments of length L_l formed from breakage of fibers of length L_k. (Note: that $\sum_{l=1}^{k-1} S_{k,l} = 1$ and $S_{k,k-l} = S_{k,l}$.)

Let $p_{k,l}(t)dt = N_k(t)S_{k,l}q_k dt$ = probability of a fragment of length L_l being formed from breakage of a fiber of length L_k in time t to $t+dt$.

To develop the model, the changes in the fiber populations are carefully studied and tabulated for a small time interval dt. Then, for the small time interval, the mean change $E(\Delta\vec{N}(t))$ and the covariance in the change

$$E\left((\Delta\vec{N}(t) - E\Delta\vec{N}(t))(\Delta\vec{N}(t) - E\Delta\vec{N}(t))^T\right) = E((\Delta\vec{N}(t))(\Delta\vec{N}(t))^T)$$

are calculated where terms of order $(dt)^2$ are neglected. For example, consider the special case where m=8, that is, there are 8 groups of fibers. Consider a fiber in the 7th group breaking into two fibers, one in group 5 and one in group 2. The change produced is:

$$(\Delta\vec{N})^{7,5} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \text{with probability} \quad p_{7,5}(t)dt = N_7(t)S_{7,5}q_7 dt.$$

The value of the expected change $E(\Delta\vec{N}(t))$ for the small time interval is calculated by summing the products of the changes with the respective probabilities. In general, for any m, it can be shown that the lth component of $E(\Delta\vec{N}(t))$ has the form:

$$E(\Delta\vec{N}(t))_l = \sum_{k=l+1}^{m} p_{k,l}(t)dt - \sum_{k=1}^{l-1} p_{l,k}(t)dt.$$

In addition, the covariance matrix, has the form

$$E\left((\Delta\vec{N}(t))(\Delta\vec{N}(t))^T\right) = \sum_{k=1}^{m}\sum_{l=1}^{k-1} C^{k,l} p_{k,l}(t)dt$$

where $C^{k,l}$ is the appropriate matrix that accounts for a fiber of group k breaking into a fiber of group l and group $k-l$. For example, for the special case where $m=8$ and a fiber in the 7th group breaks into two fibers, one in group 5 and one in group 2, then the corresponding term produced in the covariance matrix is:

$$C^{7,5} = (\Delta\vec{N})^{7,5}((\Delta\vec{N})^{7,5})^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, for convenience, define the expected change and the covariance matrix by:

$$\mathrm{E}(\Delta\vec{N}) = \vec{\beta}(\vec{N}(t),t)dt \text{ and } \mathrm{E}(\Delta\vec{N}\Delta\vec{N}^T) = V(\vec{N}(t),t)dt.$$

Then, the probability distribution $p(\vec{N},t)$ of the fiber-length populations with time t approximately satisfies the forward Kolmogorov equation (Allen, 1999; Allen, 2003; Allen, 2007):

$$\frac{\partial p(\vec{N},t)}{\partial t} = -\sum_{i=1}^{m} \frac{\partial}{\partial N_i}[\beta_i(\vec{N},t)p(\vec{N},t)]$$
$$+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial^2}{\partial N_i \partial N_j}\left[\sum_{k=1}^{m} v_{i,k}(\vec{N},t)v_{j,k}(\vec{N},t)p(\vec{N},t)\right].$$

The Itô SDE system corresponding to this forward Kolmogorov equation has the form:

$$d\vec{N}(t) = \vec{\beta}(\vec{N}(t),t)dt + (V(\vec{N}(t),t))^{1/2}d\vec{W}(t) \tag{1}$$

where $\vec{N}(t) = [N_1(t), N_2(t), \ldots, N_m(t)]^T$ are the fiber populations in each length group and $\vec{W}(t) = [W_1(t), \ldots, W_m(t)]^T$ is an m-dimensional Wiener process (Allen, 1999; Allen, 2003; Allen, 2007). Equation (1) is an SDE model for the fiber-length populations as a function of time t.

It is interesting to note that the deterministic part of equation (1) reduces for a large number of groups m to a well-known integro-differential equation. Consider the

jth component of $\vec{N}(t)$ in the deterministic model. Then, $N_j(t+dt)$ satisfies:

$$N_j(t+dt) = N_j(t) + E(\Delta\vec{N}(t))_j = N_j(t) + \sum_{k=j+1}^{m} p_{k,j}(t)\,dt - \sum_{k=1}^{j-1} p_{j,k}(t)\,dt.$$

This can be written as:

$$N_j(t+dt) = N_j(t) + \sum_{k=j+1}^{m} N_k(t)S_{k,j}q_k\,dt - \sum_{k=1}^{j-1} N_j(t)S_{j,k}q_j\,dt.$$

Letting $N_j(t) \approx N(L_j,t)\Delta L$, $S_{k,j} = S(L_k, L_j)\Delta L$, and $q_j = q(L_j)$, then

$$N(L_j, t+dt) \approx N(L_j,t) + \int_{L_{j+1}}^{L_{max}} S(\lambda, L_j)q(\lambda)N(\lambda,t)\,d\lambda\,dt$$
$$-\int_0^{L_{j-1}} S(L_j,\lambda)q(L_j)N(L_j,t)\,d\lambda\,dt.$$

Rearranging this expression and letting $dt \to 0$ gives:

$$\frac{\partial N(L,t)}{\partial t} = -q(L)N(L,t) + \int_L^{L_{max}} S(\lambda, L)q(\lambda)N(\lambda,t)\,d\lambda.$$

This integro-differential equation is well-known in fiber breakage studies and is derived, for example, in (Meyer et al., 1966).

3. Computations

To test stochastic differential equation model (1), the model is compared computationally with Monte Carlo calculations. In the Monte Carlo calculations, for each small time step, each fiber is checked for breakage. If breakage occurs, the fiber is randomly divided. Considered in these calculations is the situation where breakage occurs randomly and the probability for breakage is proportional to the length of the fiber. Under this breakage assumption,

$$q_k\,dt = \mu\left(\frac{L_k}{L_{max}}\right)dt$$

where μ is a constant which determines the rate of fiber breakage and

$$S_{k,j} = \frac{h}{L_k} = \frac{1}{k}$$

where $S_{k,j}$ is the fraction of fragments of length L_j formed from breakage of fibers of length L_k.

Under this random breakage assumption, SDE model (1) simplifies to:

$$d\vec{N}(t) = B\vec{N}(t)\, dt + (V(t))^{1/2}\, d\vec{W}(t),$$

where B is a constant $m \times m$ matrix. For example, for the case where the number of fiber groups is $m = 8$, then

$$B = \begin{bmatrix} -1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & -1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & -1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & -1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & -1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \mu h / L_{max}.$$

In addition, for $m = 8$, $V(t) = \sum_{k=1}^{m} \sum_{l=1}^{k-1} C^{k,l}(t) N_k(t) \mu h / L_{max}$ where, for example,

$$C^{7,5} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In the calculations, the number of groups m was set to 50, the parameter μ was set equal to unity, and it was assumed that there were initially 100 fibers with each fiber initially 1 inch in length. The calculational results are compared in Table 1 for the Monte Carlo method and the SDE model (1) at time $t = 1.0$. The averages are given for 200 sample paths. The results indicate very good agreement between the two different procedures. In addition, in Figures 1 and 2, the average number of fibers in each group and the variances for each fiber length group are presented for the 50 fiber length groups at $t = 1.0$.

Additional computations were performed with a more realistic initial fiber-length distribution. For these calculations, it was assumed that the fibers initially were

Avg. Number of Fibers	Standard Dev. in No. of Fibers	Average Fiber Length	Standard Dev. in Fiber Length
200.5 (MC) 197.8 (SDE)	10.57 (MC) 11.47 (SDE)	0.5001 (MC) 0.5068 (SDE)	0.0263 (MC) 0.0265 (SDE)

Table 1: Monte Carlo (MC) and Stochastic Differential Equation (SDE) Calculational Results on Fiber Lengths at Time $t = 1.0$.

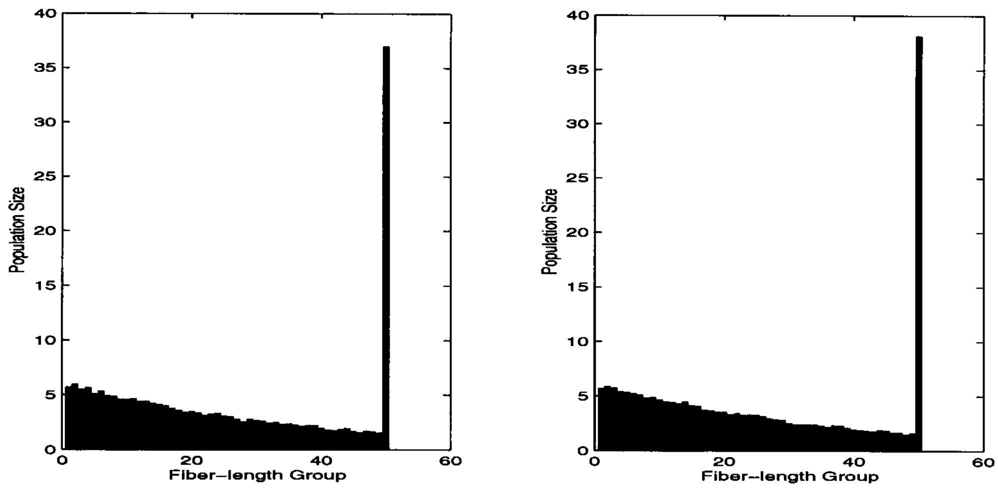

Figure 1: Average number of fibers for each group at time $t = 1.0$, SDE (left) and Monte Carlo (right), Fibers initially in group 50

distributed as $N_k(0) = 2(k-20)$ for $k = 20, 21, \ldots, 35$ and $N_k(0) = 2(50-k)$ for $k = 36, 37, \ldots, 50$ where $N_k(0)$ was the initial number of fibers of length $L_k = 0.02k$ for $k = 1, 2, \ldots, 50$. Results of an example calculation are illustrated in Figure 3. A bimodal structure is apparent in the calculated fiber-length distribution. A bimodal distribution often appears in cotton fiber-length data (Krifa, 2006). For comparison, the distribution of cotton fiber lengths collected at a carding chute is presented in Figure 4.

4. An Equivalent SDE Model

It is interesting that the SDE model (1) is not a unique stochastic differential equation model for cotton fiber breakage. There exist alternate SDE systems that are equivalent to system (1) in the sense that they share the same sample paths. In (Allen,

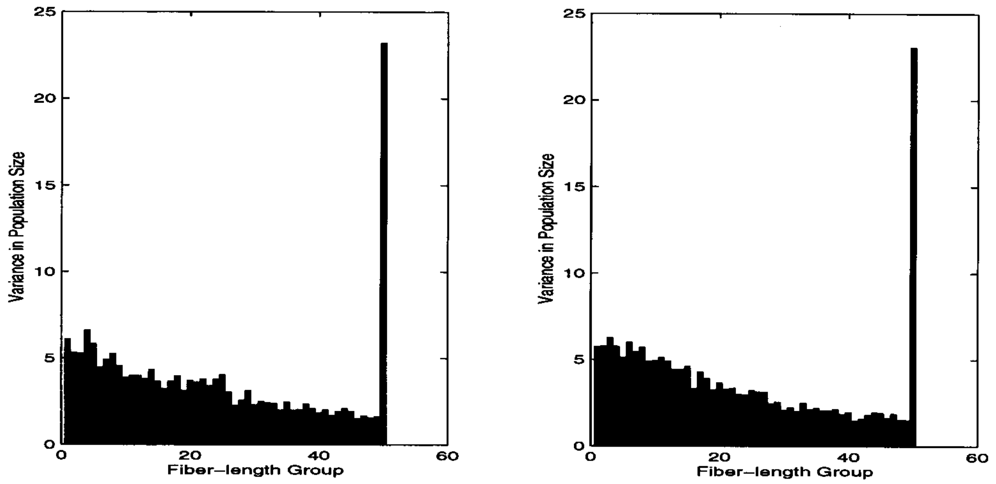

Figure 2: Variances for each group at time $t = 1.0$, SDE (left) and Monte Carlo (right), Fibers initially in group 50

2007) or (Allen et al., 2007), it is shown that if

$$V(\vec{N}(t),t) = G(\vec{N}(t),t)(G(\vec{N}(t),t))^T$$

where G is a $m \times d$ matrix with $d \geq m$ then

$$d\vec{N}(t) = \vec{\beta}(\vec{N}(t),t)\,dt + (V(\vec{N}(t),t))^{1/2}\,d\vec{W}(t)$$

is equivalent to

$$d\vec{N}(t) = \vec{\beta}(\vec{N}(t),t)\,dt + G(\vec{N}(t),t)\,d\vec{W}^*(t),$$

where $\vec{W}(t) = [W_1(t), W_2(t), \ldots, W_m(t)]^T$, $\vec{W}^*(t) = [W_1^*(t), W_2^*(t), \ldots, W_d^*(t)]^T$, and $W_i(t)$ for $i = 1, 2, \ldots, m$ and $W_i^*(t)$ for $i = 1, 2, \ldots, d$ are independent Wiener processes. Essentially, the two SDE models are equivalent if they possess the identical covariance matrix $V = E(\Delta\vec{N})(\Delta\vec{N})^T/dt$. Applying this result to model (1), an equivalent fiber-length SDE model can be written in the form:

$$d\vec{N}(t) = \vec{\beta}(\vec{N}(t),t)\,dt + \sum_{k=1}^{m}\sum_{l=1}^{k-1}(\Delta\vec{N})^{k,l}(p_{k,l}(t))^{1/2}\,dW_{k,l}^*(t), \tag{2}$$

where $p_{k,l}(t) = N_k(t)S_{k,l}q_k$, $W_{k,l}^*(t)$ for $l = 1, 2, \ldots, k-1$ and $k = 1, 2, \ldots m$ are d

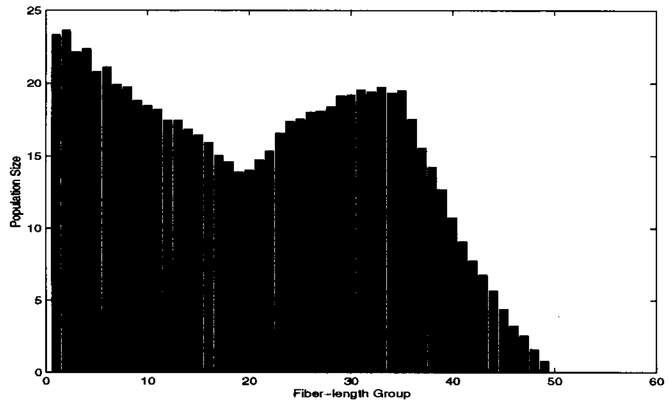

Figure 3: Calculated fiber length distribution after random breakage at time $t = 1.0$ (Using the SDE model)

independent Wiener processes, and the ith element of vector $(\Delta\vec{N})^{k,l}$ is

$$((\Delta\vec{N})^{k,l})_i = \begin{cases} -1, & \text{if } i = k \\ 1, & \text{if } i = l \text{ or } i = k - l \\ 0, & \text{otherwise.} \end{cases}$$

Notice, for SDE model (2), that $d = m(m-1)/2$ Wiener processes are required whereas SDE model (1) only requires m Wiener processes. However, model (2) does not contain a matrix square root. To show equivalence of the two SDE models, notice that

$$\begin{aligned} & E(\Delta\vec{N})(\Delta\vec{N})^T/dt = \\ & = \sum_{\hat{k}=1}^{m}\sum_{\hat{l}=1}^{\hat{k}-1}\sum_{k=1}^{m}\sum_{l=1}^{k-1} E\big[(\Delta\vec{N})^{\hat{k},\hat{l}}(p_{\hat{k},\hat{l}})^{1/2}(dW^*_{\hat{k},\hat{l}})(dW^*_{k,l})^T((\Delta\vec{N})^{k,l})^T(p_{k,l})^{1/2}\big]/dt \\ & = \sum_{k=1}^{m}\sum_{l=1}^{k-1} E((\Delta\vec{N})^{k,l}((\Delta\vec{N})^{k,l})^T p_{k,l} \\ & = \sum_{k=1}^{m}\sum_{l=1}^{k-1} C^{k,l} p_{k,l} \\ & = V. \end{aligned}$$

Finally, it is interesting to study equivalence of the two SDE models through computations. In these calculations, the situation in the previous section is considered where breakage occurs randomly and the probability for breakage is proportional to the length of the fiber. In the calculations, the number of groups m was set to 50, the parameter μ was set equal to unity, and it was assumed that there were initially 100 one-inch fibers. Two hundred sample paths were computed for each SDE model. The calculational results are compared in Table 2 for SDE model (1) and SDE model (2) at time $t = 1.0$. The results indicate good agreement between the two different SDE models.

Avg. Number of Fibers	Standard Dev. in No. of Fibers	Average Fiber Length	Standard Dev. in Fiber Length
197.8 (SDE (1))	11.47 (SDE (1))	0.5068 (SDE (1))	0.0265 (SDE (1))
196.3 (SDE (2))	10.25 (SDE (2))	0.5109 (SDE (2))	0.0271 (SDE (2))

Table 2: SDE model calculational results on fiber lengths at time $t = 1.0$.

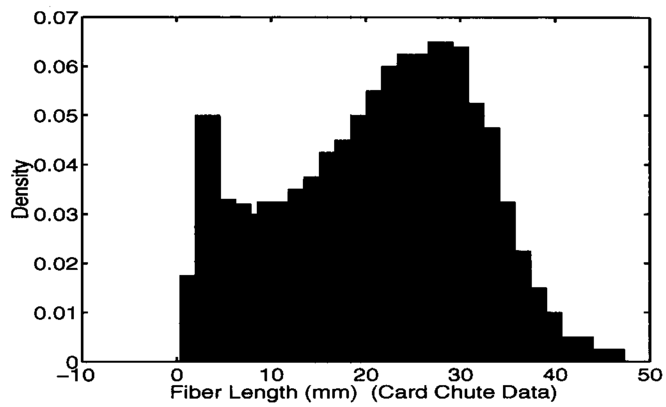

Figure 4: Cotton fiber length data (At card chute)

5. Summary

In summary, a stochastic differential equation model was developed for fibers undergoing breakage. The SDE model generalized a classic deterministic model for fiber breakage and the SDE model compared well with Monte Carlo computations for random breakage. Furthermore, calculations with the SDE model exhibited a bimodal distribution in fiber lengths which is commonly seen in data. Much work remains, however, in studying different kinds of fiber breakage possibilities and investigating cotton fiber breakage data (Hakan, 2007).

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