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Numerical computations of the roots of the generalized twisted q-Bernoulli polynomials

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Abstract

In this paper, we introduce the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x)$, and investigate the distribution of their roots for values of the index n with the help of computer.

1. Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\bar{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$.

$$[x]_q = [x : q] = \frac{1-q^x}{1-q}, \text{ cf. [1,2,3,4,5,6].}$$

Hence, $\lim_{q \to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}),$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. For any positive integer N,

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$$

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is known to be a distribution on X. This distribution yields an integral for each non-negative integer n:

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \int_X [x]_q^n d\mu_q(x) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{k+1}{[k+1]_q},$$

where $\beta_{n,q}$ are Carlitz's q-Bernoulli numbers (see [1,2,3,4]). We define the q-Bernoulli polynomials $\beta_{n,q}(x)$ by

$$\beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_q(y).$$

Then these can be rewritten (see [2,3]) as

$$\beta_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} q^{xk} \beta_{k,q} [x]_q^{n-k} = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{xk} \frac{k+1}{[k+1]_q}.$$

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$ the p-adic q-integral was defined as

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{0 \le x < p^N} g(x) q^x.$$

Note that

$$I_1(g) = \lim_{q\to 1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{0 \le x < p^N} g(x), \text{ cf.[2,3]}.$$

We assume that $q \in \mathbb{C}$ with $|1-q|_p < 1$. Using definition, we note that $I_1(g_1) = I_1(g) + g'(x)$, where $g_1(x) = g(x+1)$. Let $T_p = \cup_{m\ge 1} C_{p^m} = \lim_{m\to\infty} C_{p^m}$, where $C_{p^m} = \{w | w^{p^m} = 1\}$ is the cyclic group of order p^m. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto w^x$. Kim [3] defined the twisted q-Bernoulli numbers using p-adic q-integral on $\mathbb{Z}_p$. If we take $g(x) = \phi_w(x) e^{tx}$, then we easily see that

$$\int_{\mathbb{Z}_p} \phi_w(x) e^{tx} d\mu_1(x) = \frac{t}{we^t - 1}.$$

The classical Bernoulli polynomials $B_k(w)$ are defined by the generating function:

$$e^{xt} \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

The Bernoulli numbers are defined through the relation: $B_k = B_k(0)$. We introduce the twisted Bernoulli polynomials $B_{n,w}(x)$

$$e^{xt} \frac{t}{we^t - 1} = \sum_{n=0}^{\infty} B_{n,w}(x) \frac{t^n}{n!}, \text{ cf. [2, 3]}.$$

Note that

$$\int_{Z_p} x^n \phi_w(x) d\mu_1(x) = B_{n,w},$$

where $B_{n,w} = B_{n,w}(0)$. For $w \in T_p$, we introduce the twisted q-Bernoulli polynomials $\beta_{n,q,w}(x)$

$$\beta_{n,q,w}(x) = \int_{Z_p} w^y [x+y]_q^n d\mu_q(y). \tag{1}$$

If $x = 0$, we denote $\beta_{n,q,w}(0) = \beta_{n,q,w}$, which are called twisted q-Bernoulli numbers. Using (1), we have (see [3])

$$\beta_{n,q,w}(x) = \frac{1}{(1-q)^{n-1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{xk} \frac{k+1}{1-q^{k+1}w}. \tag{2}$$

By (2), we obtain

$$\beta_{n,q,w} = -n \sum_{k=0}^{\infty} [k]_q^{n-1} q^k w^k - (q-1)(n+1) \sum_{k=0}^{\infty} [k]_q^n q^k w^k, \text{ cf.[3]}. \tag{3}$$

Finally, we have the distribution relation, for $n \geq 0$,

$$\beta_{n,q,w}(x) = [f]_q^{n-1} \sum_{a=0}^{f-1} w^a q^a \beta_{n,q^f,w^f}\left(\frac{a+x}{f}\right), \text{ cf.[3]}.$$

Note that

$$\lim_{q \to 1} \beta_{n,q,w}(x) = B_{n,w}(x), \quad \lim_{q \to 1} \beta_{n,q,1}(x) = B_n(x).$$

Since

$$\sum_{n=0}^{\infty} B_n(1-x) \frac{(-t)^n}{n!} = F(1-x,-t) = \frac{-t}{e^{-t}-1} e^{(1-x)(-t)}$$
$$= \frac{t}{e^t - 1} e^{xt} = F(x,t) = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

we obtain that

$$B_n(1-x) = (-1)^n B_n(x). \tag{4}$$

We prove that $B_n(x), x \in \mathbb{C}$, has $Re(x) = \frac{1}{2}$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions. The question is: what happens with the reflection symmetry (4), when one considers the twisted q-Bernoulli polynomials $\beta_{n,q,w}(x)$? We are going now to reflection at $\frac{1}{2}$ of x on the twisted q-Bernoulli polynomials $\beta_{n,q,w}(x)$. Since

$$\beta_{n,q,w}(x) = \left(\frac{1}{1-q}\right)^{n-1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{kx} \frac{k+1}{1-q^{k+1}w},$$

by simple calculation, we obtain

$$\begin{aligned}&\left(\frac{1}{1-q^{-1}}\right)^{n-1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k q^{-k(1-x)}\frac{k+1}{1-q^{-k-1}w^{-1}}\\&=\frac{(-1)^n q^n w}{(1-q)^{n-1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k q^{kx}\frac{k+1}{1-q^{k+1}w}\\&=(-1)^n q^n w\left(\frac{1}{1-q}\right)^{n-1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k q^{kx}\frac{k+1}{1-q^{k+1}w}.\end{aligned}$$

Hence we obtain the following theorem.

Theorem 1. For $n \geq 0$, we have

$$\beta_{n,q^{-1},w^{-1}}(1-x) = (-1)^n q^n w \beta_{n,q,w}(x). \tag{5}$$

Corollary 2. If $\beta_{n,q,w}(x) = 0$, then $\beta_{n,q^{-1},w^{-1}}(1-x) = 0$.

Let χ be the Dirichlet character with conductor $f \in \mathbb{N}$. Since $I_1(g_1) = I_1(g) + g'(x)$, it is obvious that

$$\int_X e^{tx}\chi(x)\phi_w(x)d\mu_1(x) = \frac{\sum_{i=0}^{f-1}\chi(i)\phi_w(i)e^{it}t}{w^f e^{ft}-1}.$$

We introduce the analogue of Bernoulli polynomials $B_{n,w,\chi}(x)$

$$\frac{\sum_{i=0}^{f-1}\chi(i)\phi_w(i)e^{it}t}{w^f e^{ft}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,w,\chi}(x)\frac{t^n}{n!}, \text{ cf.[2, 3]}.$$

If $x = 0$, we denote $B_{n,w,\chi}(0) = B_{n,w,\chi}$, which are called twisted Bernoulli numbers. Then we introduce the generalized twisted q-Bernoulli numbers as follows (see [3]):

$$\beta_{n,q,w,\chi} = \int_X \chi(x)w^x[x]_q^n d\mu_q(x). \tag{6}$$

Using (6), we obtain

$$\beta_{n,q,w,\chi} = [f]_q^{n-1}\sum_{a=0}^{f-1}\chi(a)w^a q^a \beta_{n,q^f,w^f}\left(\frac{a}{f}\right), \text{ cf.[3]}. \tag{7}$$

We note that $\lim_{q\to 1}\beta_{n,q,w,\chi} = B_{n,w,\chi}$. Next, we define the generalized twisted q-Bernoulli polynomials

$$\beta_{n,q,w,\chi}(x) = \int_X \chi(y)w^y[x+y]_q^n d\mu_q(y). \tag{8}$$

By (8), we obtain,

$$\begin{aligned}
\beta_{n,q,w,\chi}(x) &= \int_X \chi(y) w^y [x+y]_q^n d\mu_q(y) \\
&= \lim_{N\to\infty} \frac{1}{[fp^N]_q} \sum_{y=0}^{fp^N-1} \chi(y) w^n [x+y]_q^n q^y \\
&= \frac{1}{[f]_q} \lim_{N\to\infty} \frac{1}{[p^N]_{q^f}} \left(\sum_{a=0}^{f-1} \sum_{y=0}^{p^N-1} \chi(a+fy) w^{a+fy} [x+a+fy]_q^n q^{a+fy} \right) \\
&= \frac{1}{[f]_q} \lim_{N\to\infty} \frac{1}{[p^N]_{q^f}} \left(\sum_{a=0}^{f-1} \sum_{y=0}^{p^N-1} \chi(a+fy) w^{a+fy} [f]_q^n \left[\frac{x+a}{f} + y \right]_q^n q^{a+fy} \right) \\
&= \frac{[f]_q^n}{[f]_{q^f}} \sum_{a=0}^{f-1} \chi(a) w^a q^a \lim_{N\to\infty} \frac{1}{[p^N]_{q^f}} \sum_{y=0}^{p^N-1} w^{fy} \left[\frac{x+a}{f} + y \right]_{q^f}^n q^{fy} \\
&= [f]_q^{n-1} \sum_{a=0}^{f-1} \chi(a) w^a q^a \int_X (w^f)^y \left[\frac{x+a}{f} + y \right]_{q^f}^n (q^f)^y d\mu_{q^f}(y) \\
&= [f]_q^{n-1} \sum_{a=0}^{f-1} \chi(a) w^a q^a \beta_{n,q^f,w^f} \left(\frac{a+x}{f} \right).
\end{aligned}$$

Hence we have the following theorem.

Theorem 3. For $n \geq 0$, we obtain

$$\beta_{n,q,w,\chi}(x) = [f]_q^{n-1} \sum_{a=0}^{f-1} \chi(a) w^a q^a \beta_{n,q^f,w^f} \left(\frac{a+x}{f} \right). \tag{9}$$

Corollary 4. For $n \geq 0$, we have

$$\lim_{q\to 1} \beta_{n,q,w,\chi}(x) = B_{n,w,\chi}(x), \quad \beta_{n,q,w,\chi}(0) = \beta_{n,q,w,\chi}.$$

It is the aim of this paper to observe an interesting structure of the zeros of the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x)$ in complex plane. The outline of this paper is as follows. In Section 2, we display the shapes of $\beta_{n,q,w,\chi}(x)$. Finally, our numerical results for numbers of real and complex zeros of the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x)$ are displayed.

By using the results of our paper, the readers can observe the regular behaviour of the real roots of the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x)$.

2. Beautiful shapes of the generalized twisted q-Bernoulli polynomials

In this section, we display the shapes of the generalized twisted q-Bernoulli numbers $\beta_{n,q,w,\chi}$ and the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x)$ by using computer. For $n = 1, \cdots, 10$, we can draw a plot of the generalized twisted q-Bernoulli numbers $\beta_{n,q,w,\chi}$ and the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x)$ by using computer, respectively. This shows the ten plots combined into one. We display the shape of $\beta_{n,q,w,\chi}$ (Figures 1, 2). Let q be considered as an indeterminate with $|q| < 1$ in $\mathbb{C}$. We plot the zeros of the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x), x \in \mathbb{C}$. Let $w = e^{\frac{\pi i}{N}}$ in $\mathbb{C}$. Figure 1 plots lists of the twisted generalized q-Bernoulli numbers $\beta_{n,q,w,\chi}, n = 1, \cdots, 10, w = e^{\pi i}, q = \frac{1}{2}$.

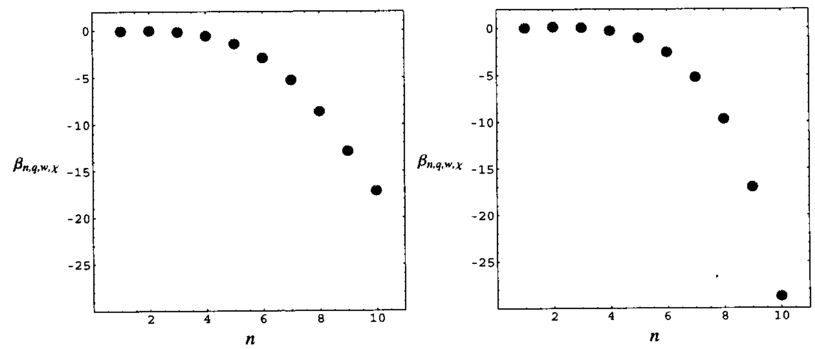

Figure 1: conductor $f = 4, 6$

Figure 2 shows shape of the $\beta_{n,q,w,\chi}, n = 1, \cdots, 10, w = e^{\pi i}, -\frac{9}{10} \le q \le \frac{9}{10}$.

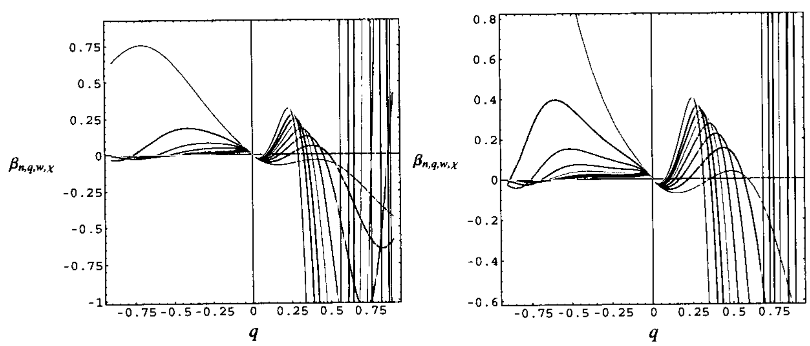

Figure 2: conductor $f = 4, 6$

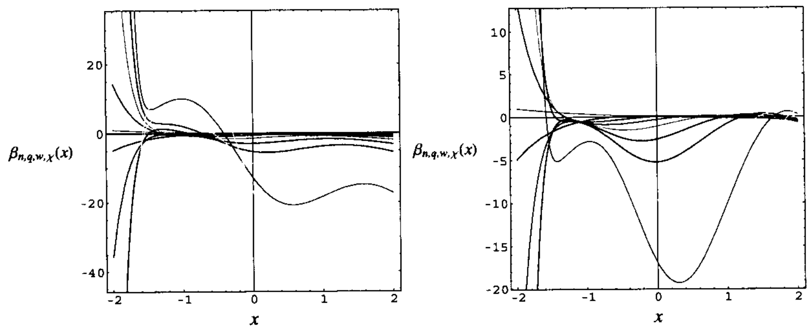

Figure 3: conductor $f = 4, 6$

We display the shapes of the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x)$. For $n = 1, 2, \cdots, 10$, we can draw a plot of $\beta_{n,q,w,\chi}(x)$, respectively. This shows the ten plots combined into one. For $n = 1, 2, \cdots, 10$, we display the shape of $\beta_{n,q,w,\chi}(x), n = 1, \cdots, 10, q = \frac{1}{2}, -2 \le x \le 2$ (Figures 3-4).

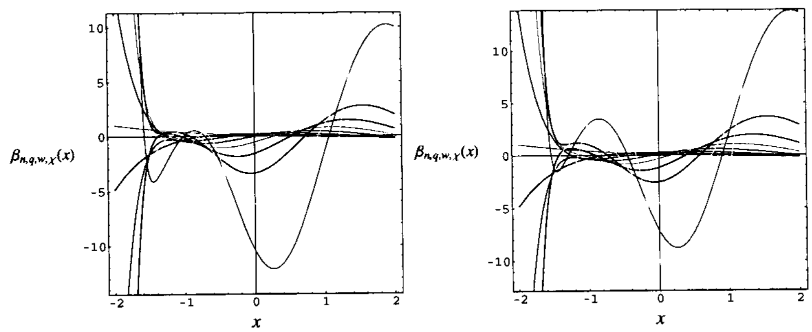

Figure 4: conductor $f = 8, 10$

3. Distribution and Structure of the zeros

In this section, we investigate the zeros of the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x)$ by using computer. We plots the zeros of the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x), x \in \mathbb{C}, q = \frac{1}{2}, f = 6$ (Figures 5-8). We plots the of the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x), x \in \mathbb{C}, q = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, w = e^{\frac{2\pi i}{3}}, f = 4$ (Figures 9-12). We plot the zeros of the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x), x \in \mathbb{C},\ q = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, w = e^{\frac{2\pi i}{6}}, f = 4.$

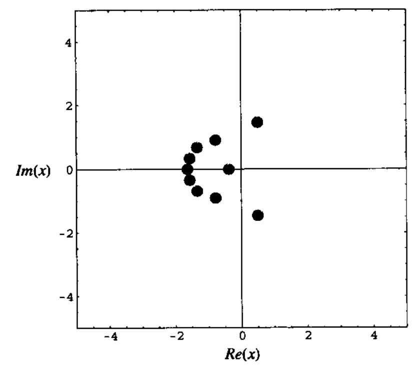

Figure 5: $\beta_{10,\frac{1}{2},w,\chi}(x), w = e^{2\pi i}$

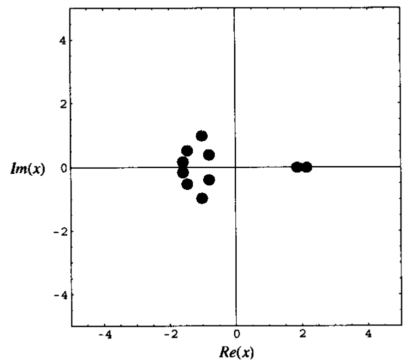

Figure 6: $\beta_{10,\frac{1}{2},w,\chi}(x), w = e^{\pi i}$

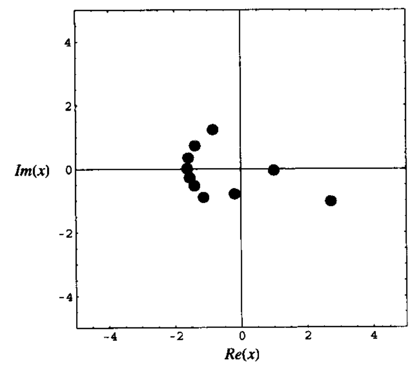

Figure 7: $\beta_{10,\frac{1}{2},w,\chi}(x), w = e^{\frac{2\pi i}{3}}$

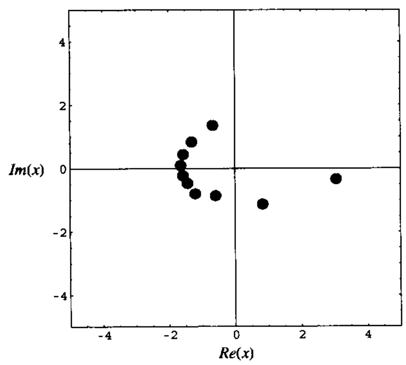

Figure 8: $\beta_{10,\frac{1}{2},w,\chi}(x), w = e^{\frac{2\pi i}{4}}$

(Figures 13-16). We plot the zeros of the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x), x \in \mathbb{C}$, $q = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, w = e^{\frac{2\pi i}{3}}, f = 4$ (Figures 17-20). We plot the zeros of the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x), x \in \mathbb{C}$, $q = \frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, w = e^{\frac{2\pi i}{3}}, f = 4$ (Figures 21-24). Our numerical results for numbers of real and complex zeros of $\beta_{n,q,w,\chi}(x)$ are displayed in Table 1 and Table 2. The result is obtained by Mathematica software.

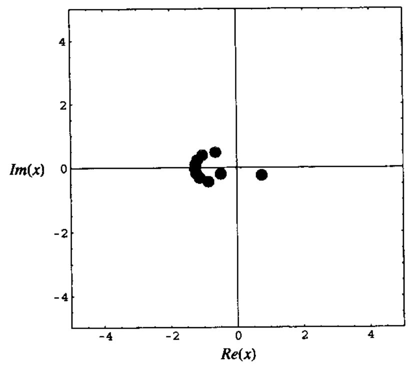

Figure 9: $\beta_{10,\frac{1}{5},w,\chi}(x), w = e^{\frac{2\pi i}{3}}$

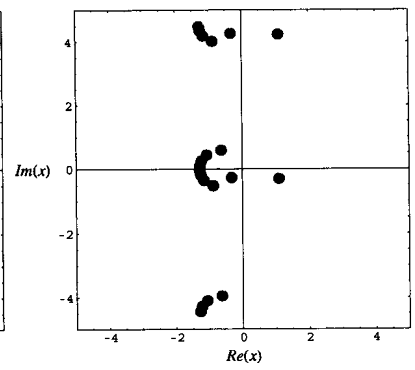

Figure 10: $\beta_{10,\frac{1}{4},w,\chi}(x), w = e^{\frac{2\pi i}{3}}$

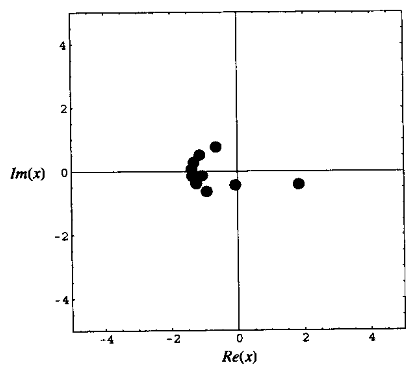

Figure 11: $\beta_{10,\frac{1}{3},w,\chi}(x), w = e^{\frac{2\pi i}{3}}$

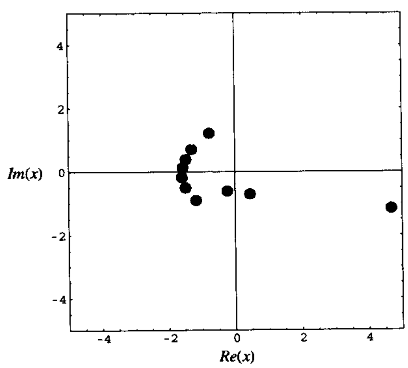

Figure 12: $\beta_{10,\frac{1}{2},w,\chi}(x), w = e^{\frac{2\pi i}{3}}$

Table 1. Numbers of real and complex zeros of $\beta_{n,\frac{1}{2},w,\chi}(x), f = 4$

	$w = e^{2\pi i}$		$w = e^{\pi i}$	
degree n	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	0	2
3	1	2	1	2
4	2	2	0	4
5	1	4	1	4
6	2	4	2	4
7	1	6	1	6
8	2	6	2	6
9	3	6	1	8

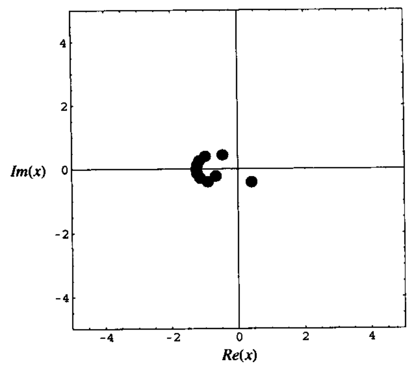

Figure 13: $\beta_{10,\frac{1}{5},w,\chi}(x), w = e^{\frac{2\pi i}{6}}$

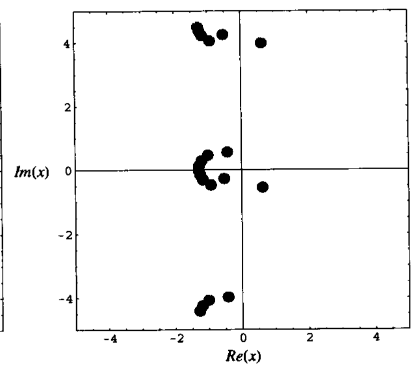

Figure 14: $\beta_{10,\frac{1}{4},w,\chi}(x), w = e^{\frac{2\pi i}{6}}$

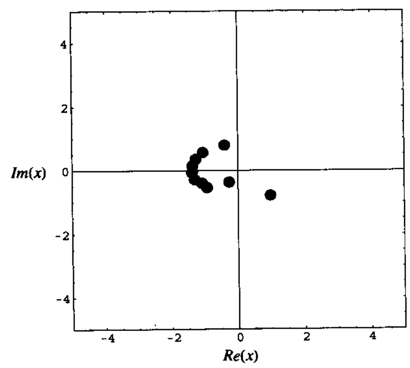

Figure 15: $\beta_{10,\frac{1}{3},w,\chi}(x), w = e^{\frac{2\pi i}{6}}$

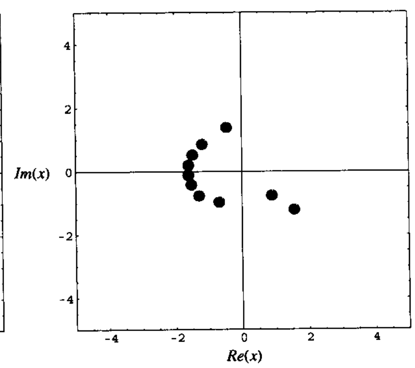

Figure 16: $\beta_{10,\frac{1}{2},w,\chi}(x), w = e^{\frac{2\pi i}{6}}$

Table 2. Numbers of real and complex zeros of $\beta_{n,\frac{1}{2},w,\chi}(x), f = 4$

	$w = e^{\frac{2\pi i}{3}}$		$w = e^{\frac{2\pi i}{4}}$	
degree n	real zeros	complex zeros	real zeros	complex zeros
1	0	1	0	1
2	0	2	0	2
3	0	3	0	3
4	0	4	0	4
5	0	5	0	5
6	0	6	0	6
7	0	7	0	7
8	0	8	0	8
9	0	9	0	9

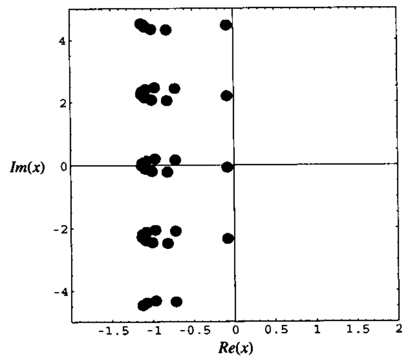

Figure 17: $\beta_{10,\frac{1}{16},w,\chi}(x), w = e^{\frac{2\pi i}{3}}$

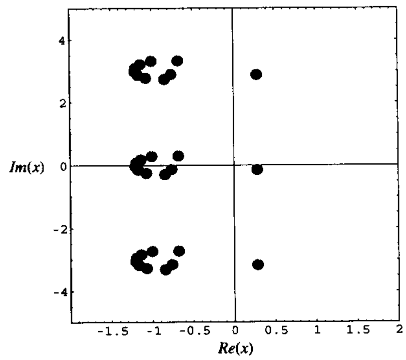

Figure 18: $\beta_{10,\frac{1}{8},w,\chi}(x), w = e^{\frac{2\pi i}{3}}$

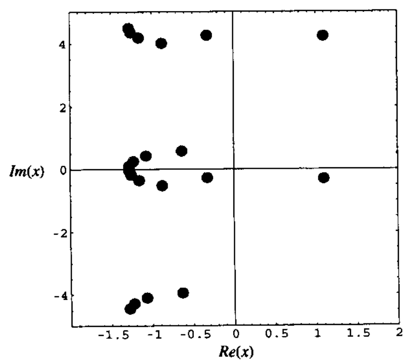

Figure 19: $\beta_{10,\frac{1}{4},w,\chi}(x), w = e^{\frac{2\pi i}{3}}$

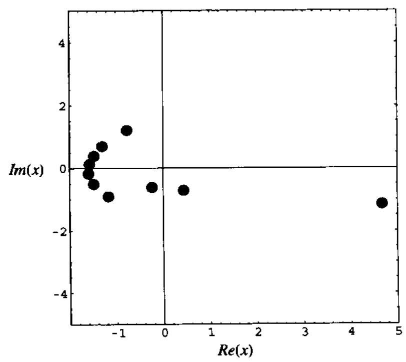

Figure 20: $\beta_{10,\frac{1}{2},w,\chi}(x), w = e^{\frac{2\pi i}{3}}$

We calculated an approximate solution satisfying $\beta_{n,\frac{1}{2},w,\chi}(x), x \in \mathbb{R}$. The results are given in Table 3 and Table 4.

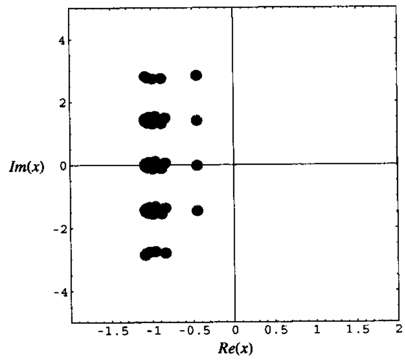

Figure 21: $\beta_{10,\frac{1}{81},w,\chi}(x), f = 4$

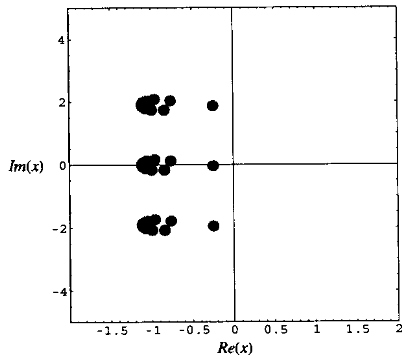

Figure 22: $\beta_{10,\frac{1}{27},w,\chi}(x), f = 4$

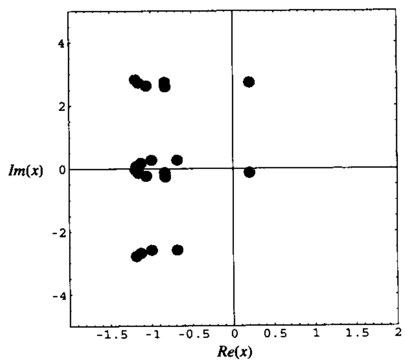

Figure 23: $\beta_{10,\frac{1}{9},w,\chi}(x), f = 4$

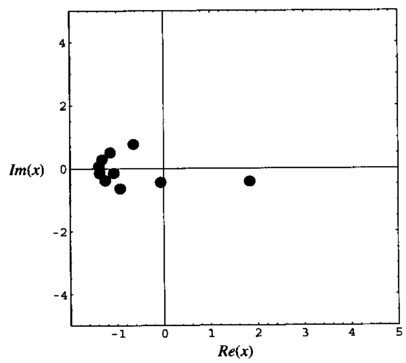

Figure 24: $\beta_{10,\frac{1}{3},w,\chi}(x), f = 4$

Table 3. Approximate solutions of $\beta_{n,\frac{1}{2},w,\chi}(x) = 0, f = 4, w = e^{2\pi i}, x \in \mathbb{R}$

degree n	x
1	-0.792007
2	-0.904731, -0.687456
3	-0.409796
4	-1.35708, -0.150012
5	0.0746417
6	-1.43378, 0.270571

Table 4. Approximate solutions of $\beta_{n,\frac{1}{2},w,\chi}(x) = 0, f = 4, w = e^{\pi i}, x \in \mathbb{R}$

degree n	x
1	-0.212994
2	$\times$
3	-1.27925
4	$\times$
5	-1.09694
6	$-1.49157, \quad -0.895819$

Our numerical results of the twisted q-Bernoulli numbers $\beta_{n,\frac{1}{4},w,\chi}, f = 4$ calculated with 15-digit precision are displayed in Table 5.

Table 5. Generalized twisted q-Bernoulli numbers $\beta_{n,\frac{1}{3},w,\chi}$

degree n	$w = e^{2\pi i}$	$w = e^{\pi i}$
1	0.281437399862669	-0.0617074845502403
2	0.250292755825797	0.00491941236435879
3	0.225116670600604	0.0695924565502794
4	0.206951486895994	0.131061246290292
5	0.196992134068178	0.187568586051981
6	0.196598322090949	0.236674439293792
7	0.207303516110806	0.275023394085619
8	0.230818355315520	0.298038392642383
9	0.269025213685214	0.299518390187122
10	0.323959293992920	0.271111064094877

4. Directions for Further Research

In general, how many roots does $\beta_{n,q,w,\chi}(x)$ have ? Find the numbers of complex zeros of the $\beta_{n,q,w,\chi}(x)$, the equation of envelope curves bounding the real zeros lying on the plane, and the equation of a trajectory curve running through the complex zeros on any one of the arcs. The question is: what happens with the reflection symmetry (4), when one considers the generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x)$? In any case, these calculations are too complicated to compute by hand, we have to use computer. In fact, computer software provides abundant opportunities to enhance the way we research mathematics, applied mathematics and physics. By using software, we can explore concepts much more easily than in the past. The ability to create and manipulate figures on the computer screen enables students to quickly visualize and produce many problems, examine properties of the figures, look for patterns, and make conjectures. The authors have no doubt that investigation along this line will lead to a new approach employing numerical methods in the field of research of generalized twisted q-Bernoulli polynomials $\beta_{n,q,w,\chi}(x)$ to appear in mathematics and physics. The reader may refer to [5, 6] for the details

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