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# Cubically Convergent a Posteriori Error Bound Method for the Inclusion of Polynomial Zeros

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Dedicated to the memory of Professor David J. Evans

**ABSTRACT.** Combining Carstensen's result from 1991 concerned with the localization of polynomial zeros and the third order iterative method for the simultaneous determination of polynomial zeros, we derive *a posteriori* error bound method with cubical convergence. The constructed method has useful property of inclusion methods to produce disks containing all simple zeros of a polynomial. Computationally verifiable initial conditions that guarantee the convergence of this method are stated. Some computational aspects and the possibility of implementation on parallel computers are considered.

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## 1. A POSTERIORI ERROR BOUND METHOD

With practical computational problems, a standard question should be "what is the error in the result?" As already pointed out by Wilkinson [33], the considerable amount of the applied procedure is to improve the approximate result and also to give error bounds for the improved approximations. The computed solution of a polynomial equation is only an approximation to the true solution, since there are errors originating from discretization or truncation and from rounding. In connection with this effect we quote Henrici's argumentation given in [11]: "Working with finite word length, we cannot hope to identify exactly a complex number such as a zero of a polynomial. We can at best exhibit a circle of arbitrary small radius that contains it." This problem can be overcome by using self-validated iterative methods that use interval arithmetic, see [1], [18]. In each iteration resulting intervals contain the desired zeros providing in this way the automatic determination of the upper error bounds given by radii or semidiagonals of inclusion approximations.

The price to be paid in order to achieve the characteristics of interval methods

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consists of the increase of numerical operations. In order to decrease the computational cost of these methods, in this paper we study a quasi-interval method that combines good properties of iterative methods with fast convergence and *a posteriori* error bounds. Simultaneous determination of both centers and radii leads to an iterative error bound method which has very convenient inclusion property at each iteration. The main attention is devoted to the construction of initial conditions for the guaranteed convergence of the proposed method and to the determination of the convergence rate of *a posteriori* error bounds.

Let $P(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$ $(a_i \in C)$ be a monic polynomial and let

$$W(z_i) = \frac{P(z_i)}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j)} \quad (i \in I_n := \{1, \ldots, n\}),$$

where $z_1, \ldots, z_n$ are distinct approximations to the simple zeros $\varsigma_1, \ldots, \varsigma_n$ of $P$. For the sake of simplicity, we will often write $W(z_i) = W_i$. The quantity $W_i$ is often called *Weierstrass' correction* since it appears in the well known Weierstrass' iterative method $\hat{z}_i = z_i - W_i$ $(i \in I_n)$ (also called the Durand-Kerner method [7], [13]) for the simultaneous computation of all simple zeros of a polynomial.

Studying the problem of determination of polynomial zeros, it is necessary to consider simultaneously several important tasks such as localization of zeros, distribution of initial approximations $z_1^{(0)}, \ldots z_n^{(0)}$, their closeness and the convergence of *a posteriori* error bounds (shorter PEB) given by the size of inclusion regions containing zeros. An extensive research carried out during the last two decades (see, e.g., [19], [21], [22], [32]) showed that successful solving the aforementioned problems can be realized by using an initial condition of the form

$$\omega^{(0)} \le c_n d^{(0)} \tag{1}$$

where

$$\omega^{(m)} = \max_{1 \le i \le n} \left| W\left(z_i^{(m)}\right) \right|, \quad d^{(m)} = \min_{\substack{1 \le i, j \le n \\ j \neq i}} \left| z_i^{(m)} - z_j^{(m)} \right| \quad (m = 0, 1, \ldots),$$

and $m = 0, 1, 2, \ldots$ is the iteration index. The quantity $c_n$ depends only on the polynomial degree $n$. When we omit the iteration index, then we write simply $\omega$ and $d$.

In what follows we will denote a disk $Z$ with center $c$ and radius $r$ by parametric notation $\{c; r\}$. Combining Carstensen's result [3] concerned with Gershgorin's disks and the localization of polynomial zeros, and the enclosure approach recently presented in [23] and [26], we can derive the following useful inclusion which has the main role in our consideration (see [26] for details).

**Theorem 1.** *Let the condition* (1) *with* $c_n < 1/(2n)$ *be valid, then the disks* $D_i$ *defined by*

$$D_i = \{z_i; \frac{|W_i|}{1-nc_n}\} = \{z_i; \rho_i\} \quad (i \in I_n)$$

*are mutually disjoint and each of them contains exactly one zero of* $P$.

Let us assume that the centers $z_i$ of disks $D_i$ are calculated by an iterative method that converges under some suitable conditions, then we generate the sequences of disks $D_i^{(m)}$ $(m = 0, 1,...)$ whose radii $\rho_i^{(m)} = |W_i^{(m)}| \,/\, (1-nc_n)$ converge to 0. To provide a high computational efficiency it is necessary to apply only those methods which use quantities already calculated in the previous iterative step, in our case the corrections $W_i$ since the radii $\rho_i$ depend on Weierstrass' corrections $W_i$. For this reason, we restrict our choice to the class of derivative free methods which deal with Weierstrass' corrections, the so-called *W-class*.

In this paper we will consider the following derivative free simultaneous method:

$$z_i^{(m+1)} = \Phi\left(z_i^{(m)}\right) := z_i^{(m)} - W_i^{(m)}\Big(1 - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{W_j^{(m)}}{z_i^{(m)} - z_j^{(m)}}\Big) \quad (i \in I_n; m = 0,1,\ldots). \tag{2}$$

This method was derived in various ways by M. Prešić [29] and later by G. Milovanović [14] and Tanabe [30]. For this reason we will call this method PMT method, for brevity. Kanno, Kyurkchiev and Yamamoto [12] have shown that the method (2) can be obtained by applying classical Euler-Chebyshev's method (see Traub [31, pp. 81-84]) to the system of nonlinear equations (known as Viète's formulae)

$$(-1)^k \varphi_k(z_1,\ldots,z_n) - a_k = 0, \quad (k = 1,\ldots,n),$$

where $\varphi_i$ denotes the $k$-th elementary symmetric function:

$$\varphi_k = \sum_{1 \le j_1 \le \cdots j_k \le n} z_{j_1} z_{j_2} \cdots z_{j_k} .$$

Using a procedure for accelerating convergence of iterative processes, G. Milovanović proved [14] that the iterative method (2) has cubic convergence. It is obvious that (2) belongs to $W$-class. Another iterative methods of $W$-class are given in [2], [8], [15], [16], [24], [27], [28] and [34].

Combining the results of Theorem 1 and (2), we can state the following inclusion PMT method:

**A posteriori error bound PMT method:** *A posteriori error bound method (shorter PEB method) is defined by the sequences of disks* $\{D_i^{(m)}\}$ $(i \in I_n)$,

$$D_i^{(0)} = \{z_i^{(0)}; \frac{|W(z_i^{(0)})|}{1-nc_n}\},$$

$$D_i^{(m)} = \{z_i^{(m)}; \rho_i^{(m)}\}, \quad (i \in I_n; m = 1,2,\ldots), \tag{3}$$

$$z_i^{(m)} = \Phi\left(z_i^{(m-1)}\right) \text{ by (2)}, \ \rho_i^{(m)} = \frac{\left|W\left(z_i^{(m)}\right)\right|}{1 - nc_n},$$

*assuming that the initial condition* (1) (*with* $c_n \le 1/(2n)$) *holds.*

**Remark 1.** The sequences of disks given by (3) can be regarded as a quasi-interval method, which differs structurally from usual interval methods that deal with disks as arguments; for instance, let us present the following circular interval method of the third order which does not use the polynomial derivatives

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 + \sum\limits_{\substack{j=1 \\ j \ne i}}^{n} \frac{W_j^{(m)}}{Z_i^{(m)} - z_j^{(m)}}} \qquad (i \in I_n; m = 0,1,\ldots), \tag{4}$$

proposed by Petković in [17]. Here $Z_i^{(m)}$ is a disk with center $z_i^{(m)}$. Both methods (3) and (4) possess the crucial inclusion property: every of the produced disks contains exactly one zero in each iteration. More about interval methods for solving polynomial equations can be found in [18] and [25].

In the convergence analysis and practical realization of the PEB method (3), we encounter the following important tasks:

1) Establish computationally verifiable initial conditions that guarantee the convergence of the sequences of radii $\rho_i^{(m)}$ of the inclusion disks $D_i^{(m)}$. This very important problem has attracted a great attention during the last two decades (see [22] for details).

2) Determine the convergence order of a posteriori error bound method when the centers $z_i^{(m)}$ of disks

$$D_i^{(m)} = \left\{ z_i^{(m)}; \frac{\left|W\left(z_i^{(m)}\right)\right|}{1 - nc_n} \right\} \qquad (i \in I_n; m = 0,1,\ldots) \tag{5}$$

are calculated by the PMT iterative method (2).

3) Compare the computational efficiencies of the PEB method (3) and the corresponding circular interval methods (4).

4) Using numerical experiments, compare the size of inclusion disks produced by the PEB method (3) and the corresponding interval method (4).

The study of these tasks is the main goal of this paper.

## 2. CONVERGENCE OF THE PEB METHOD

In this section we consider the tasks referred to as 1) and 2) in the previous section. In the case of algebraic polynomials, initial conditions should depend only on attainable data - initial approximations, polynomial degree and polynomial coefficients. First we prove two necessary assertions given in Lemmas 1 and 2.

**Lemma 1.** *If the inequality*

$$\omega < \frac{d}{3n} \tag{6}$$

*holds, then for the iterative method* (2) *and* $i \in I_n$ *we have*

$$(i) \quad \left|1 - \sum_{j \neq i} \frac{W_j}{z_i - z_j}\right| < \frac{n-1}{3n};$$

$$(ii) \quad |\hat{z}_i - z_i| < \frac{4n-1}{3n}|W_i| < \frac{4n-1}{9n^2}d;$$

$$(iii) \quad |\hat{z}_i - z_j| > \frac{9n^2 - 4n + 1}{9n^2}d;$$

$$(iv) \quad |\hat{z}_i - \hat{z}_j| > \frac{9n^2 - 8n + 2}{9n^2}d;$$

$$(v) \quad \left|\sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1\right| < \frac{1}{6};$$

$$(vi) \quad \left|\prod_{j \neq i} \frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right| < e^{4/9}.$$

**Proof.** *Of* (*i*): Let us introduce $\sigma_i = \sum_{j \neq i} \frac{W_j}{z_i - z_j}$. Using (6) and the definition of $d$ one obtains

$$|\sigma_i| \le \sum_{j \neq i} \frac{|W_j|}{|z_i - z_j|} \le \frac{(n-1)\omega}{d} < \frac{n-1}{3n}. \tag{7}$$

Now we find

$$\left|1 - \sum_{j \neq i} \frac{W_j}{z_i - z_j}\right| \le 1 + |\sigma_i| < 1 + \frac{n-1}{3n} = \frac{4n-1}{3n}.$$

*Of* (*ii*): By (*i*) and (6) we get from (2)

$$|\hat{z}_i - z_i| = \left|W_i\left(1 - \sum_{j \neq i} \frac{W_j}{z_i - z_j}\right)\right| \le |W_i|\left|1 - \sum_{j \neq i} \frac{W_j}{z_i - z_j}\right| < \frac{4n-1}{3n}|W_i| < \frac{4n-1}{9n^2}d.$$

*Of* (*iii*): Using (*ii*) we find

$$|\hat{z}_i - z_j| \ge |z_i - z_j| - |\hat{z}_i - z_i| > d - \frac{4n-1}{9n^2}d = \frac{9n^2 - 4n + 1}{9n^2}d.$$

*Of* (*iv*): By (*ii*) one gets

$$|\hat{z}_i - \hat{z}_j| \ge |z_i - z_j| - |\hat{z}_i - z_i| - |\hat{z}_j - z_j| > d - 2 \cdot \frac{4n-1}{9n^2}d = \frac{9n^2 - 8n + 2}{9n^2}d.$$

*Of* (*v*): Let us introduce

$$T_i = \sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1.$$

From the iterative formula (2) we obtain

$$\frac{W_i}{\hat{z}_i - z_i} = -\frac{1}{1-\sum\limits_{j\neq i}\frac{W_j}{z_i - z_j}} = -\frac{1}{1-\sigma_i} \tag{8}$$

so that

$$\begin{aligned} T_i &= \frac{W_i}{\hat{z}_i - z_i} + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j} + 1 \\ &= \frac{1}{1-\sigma_i}\Big(\sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j} - \sum_{j\neq i}\frac{W_j}{z_i - z_j} - \sigma_i\sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j}\Big), \end{aligned}$$

whence

$$T_i = \frac{1}{1-\sigma_i}\Big[(z_i - \hat{z}_i)\sum_{j\neq i}\frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)} - \sigma_i\sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j}\Big]. \tag{9}$$

Using (7) we estimate

$$\frac{1}{1-|\sigma_i|} < \frac{1}{1-\frac{n-1}{3n}} = \frac{3n}{2n+1} \tag{10}$$

and

$$\frac{|\sigma_i|}{1-|\sigma_i|} < \frac{\frac{n-1}{3n}}{1-\frac{n-1}{3n}} = \frac{n-1}{2n+1}. \tag{11}$$

Starting from (9) and using (10) and (11) we find

$$\begin{aligned} |T_i| &\le \frac{1}{1-|\sigma_i|}|z_i - \hat{z}_i|\sum_{j\neq i}\frac{|W_j|}{|\hat{z}_i - z_j||z_i - z_j|} + \frac{|\sigma_i|}{1-|\sigma_i|}\sum_{j\neq i}\frac{|W_j|}{|\hat{z}_i - z_j|} \\ &< \frac{3n}{2n+1}\cdot\frac{4n-1}{9n^2}d\cdot\frac{(n-1)\omega}{\frac{9n^2-4n+1}{9n^2}d\cdot d} + \frac{n-1}{2n+1}\cdot\frac{(n-1)\omega}{\frac{9n^2-4n+1}{9n^2}d}. \end{aligned}$$

After short rearrangement we obtain

$$|T_i| < \frac{(n-1)(3n^2+n-1)}{(2n+1)(9n^2-4n+1)} < \frac{1}{6}. \tag{12}$$

*Of* (*vi*):By (*ii*) and (*iv*) we estimate

$$\begin{aligned} \left|\prod_{j\neq i}\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right| &\le \prod_{j\neq i}\Big(1+\frac{|\hat{z}_j - z_j|}{|\hat{z}_i - \hat{z}_j|}\Big) < \prod_{j\neq i}\Big(1+\frac{(4n-1)d/(9n^2)}{(9n^2-8n+2)d/(9n^2)}\Big)^{n-1} \\ &= \left(1+\frac{4n-1}{9n^2-8n+2}\right)^{n-1} < e^{4/9}. \qquad \square \end{aligned}$$

**Lemma 2.** *Let us consider the PEB method* (3) *based on the PMT method* (2). *If the inequality* (6) *holds, then for* $i \in I_n$ *we have*

(*i*) $|\hat{W}_i| < \frac{1}{3}|W_i|$;

(*ii*) $\hat{\omega} < \frac{\hat{d}}{3n}$;

(*iii*) $\hat{\rho}_i < \frac{8}{9d^2}[\rho_i^2 \sum_{j \neq i} \rho_j + \rho_i (\sum_{j \neq i} \rho_j)^2]$.

**Proof.** Now we use the well known result from the interpolation theory: if $z_1,...,z_n$ are distinct complex numbers, then the polynomial $P$ can be expressed by the Lagrange interpolation formula

$$P(z) = \Big(\sum_{j=1}^{n} \frac{W_j}{z - z_j} + 1\Big)\prod_{j=1}^{n}(z - z_j). \tag{13}$$

Putting $z = \hat{z}_i$ in (13) one gets

$$P(\hat{z}_i) = (\hat{z}_i - z_i)\Big(\sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1\Big)\prod_{j \neq i}(\hat{z}_i - z_j).$$

After dividing $P(\hat{z}_i)$ by $\prod_{j \neq i}(\hat{z}_i - \hat{z}_j)$, we find

$$\hat{W}_i = (\hat{z}_i - z_i)\Big(\sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1\Big)\prod_{j \neq i} \frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j} = (\hat{z}_i - z_i)T_i \prod_{j \neq i} \frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}. \tag{14}$$

Using (*ii*), (*v*) and (*vi*) of Lemma 1, we start from (14) and find

$$|\hat{W}_i| = |\hat{z}_i - z_i||T_i|\Big|\prod_{j \neq i} \frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\Big| = \frac{4n-1}{3n}|W_i|\frac{e^{4/9}}{6} < \frac{1}{3}|W_i|,$$

which proves the assertion (*i*).

From Lemma 1 (assertion (*iv*)) we observe that $\hat{d} > \frac{9n^2 - 8n + 2}{9n^2} d$. According to this and (*i*) of Lemma 2, we find

$$|\hat{W}_i| < \frac{1}{3}|W_i| < \frac{1}{3} \cdot \frac{d}{3n} < \frac{1}{9n} \cdot \frac{9n^2}{9n^2 - 8n + 2}\hat{d} < \frac{\hat{d}}{3n}.$$

This proves the implication (assertion (*ii*))

$$\omega < \frac{d}{3n} \Rightarrow \hat{\omega} < \frac{\hat{d}}{3n}.$$

According to the estimates (*ii*), (*iii*) and (*iv*) of Lemma 1 and (10), from the last relation we obtain

$$|\hat{W}_i| < e^{\frac{4}{9}} \frac{4n-1}{3n} \frac{|W_i|}{1 - |\sigma_i|}\Big[|\hat{z}_i - z_i| \sum_{j \neq i} \frac{|W_j|}{|\hat{z}_i - z_j||z_i - z_j|} + \sum_{j \neq i} \frac{|W_j|}{|z_i - z_j|} \sum_{j \neq i} \frac{|W_j|}{|\hat{z}_i - z_j|}\Big]$$

$$< e^{\frac{4}{9}} \frac{4n-1}{3n}|W_i| \frac{3n}{2n+1}\Big[\frac{4n-1}{3n}|W_i| \sum_{j \neq i} \frac{|W_j|}{\frac{9n^2-4n+1}{9n^2} d \cdot d} + \sum_{j \neq i} \frac{|W_j|}{d} \sum_{j \neq i} \frac{|W_j|}{\frac{9n^2-4n+1}{9n^2} d}\Big]$$

$$< \frac{2e^{\frac{4}{9}}}{d^2}\Big[|W_i|^2 \sum_{j \neq i}|W_j| + |W_i|\Big(\sum_{j \neq i}|W_j|\Big)^2\Big].$$

for every $n \geq 3$.

Multiplying both sides of the last inequality with $1/(1 - nc_n) = 3/2$, we get

$$\hat{\rho}_i = \frac{3}{2}|\hat{W}_i| < \frac{8}{9d^2}[\rho_i^2 \sum_{j \neq i} \rho_j + \rho_i (\sum_{j \neq i} \rho_j)^2] \quad . \quad \square \tag{15}$$

The initial disks $D_i^{(0)}$ for $c_n=1/(3n)$ are given by

$$D_i^{(0)} = \left\{ z_i^{(0)}; \frac{3}{2}\left|W\left(z_i^{(0)}\right)\right| \right\} \quad (i \in I_n).$$

By (3) we define the sequences of inclusion disks

$$D_i^{(m)} = \left\{ z_i^{(m)}; \frac{3}{2}\left|W\left(z_i^{(m)}\right)\right| \right\} = \left\{ z_i^{(m)}; \rho_i^{(m)} \right\} \quad (i \in I_n; m = 1,2,\ldots), \tag{16}$$

where $z_i^{(m)}$ is calculated by the PMT iterative formula (2) and $\rho_i^{(m)} = \frac{3}{2}\left|W\left(z_i^{(m)}\right)\right|$.

**Theorem 2.** *The PEB method* (3), *based on PMT method* (2), *converges cubically if the initial condition*

$$\omega^{(0)} < \frac{d^{(0)}}{3n} \tag{17}$$

*holds.*

**Proof.** We recall that the order of convergence of an interval method in circular complex arithmetic is actually the order of convergence of the radii of inclusion disks. Therfofe, we have to prove that the sequences of a posteriori error bounds $\left\{\rho_i^{(m)}\right\}$ $(i \in I_n)$ converge cubically. The proof is by induction with the argumentation used in the proofs of Lemmas 1 and 2.

First we note that the initial condition (17) coincides with (14), which implies that all assertions of Lemmas 1 and 2 hold for the index $m = 1$. The inequality *(ii)* of Lemma 2 again reduces to the condition of the form (14) and, therefore, the assertions of Lemmas 1 and 2 hold for the next index, and so on. In fact, the implication

$$\omega^{(m)} < \frac{d^{(m)}}{3n} \Rightarrow \omega^{(m+1)} < \frac{d^{(m+1)}}{3n}$$

plays a key role because it involves the initial condition (17) which further provides the validity of all inequalities given in Lemmas 1 and 2 for each $m$=0,1,... . In particular, we have for every $i \in I_n$

$$\rho_i^{(m+1)} < \frac{8}{9(d^{(m)})^2}\left[(\rho_i^{(m)})^2 \sum_{j \neq i} \rho_j^{(m)} + \rho_i^{(m)}\left(\sum_{j \neq i} \rho_j^{(m)}\right)^2\right], \tag{18}$$

$$\frac{d^{(m)}}{d^{(m+1)}} < \frac{9n^2}{9n^2 - 8n + 2}, \tag{19}$$

$$\left|W_i^{(m+1)}\right| < \frac{1}{3}\left|W_i^{(m)}\right|, \tag{20}$$

and

$$\left|z_i^{(m+1)} - z_i^{(m)}\right| < \frac{4n-1}{3n}\left|W_i^{(m)}\right|. \tag{21}$$

Let us introduce the substitution

$$h_i^{(m)} = \frac{\rho_i^{(m)}}{d^{(m)}}\lambda_n, \quad \lambda_n = \frac{2\sqrt{2}n(n-1)}{\sqrt{9n^2 - 8n + 2}} \quad (i \in I_n). \tag{22}$$

Then the inequalities (18) become

$$h_i^{(m+1)} < \frac{8}{9\lambda_n} \cdot \frac{d^{(m)}}{d^{(m+1)}}\left[(h_i^{(m)})^2 \sum_{j \neq i} h_j^{(m)} + h_i^{(m)}\left(\sum_{j \neq i} h_j^{(m)}\right)^2\right] \quad (i \in I_n). \tag{23}$$

Taking into account (19), from (23) there follows

$$h_i^{(m+1)} < \frac{1}{(n-1)^2}[(h_i^{(m)})^2 \sum_{j\neq i} h_j^{(m)} + h_i^{(m)}( \sum_{j\neq i} h_j^{(m)})^2] \quad (i \in I_n). \tag{24}$$

Using (22) we find

$$h_i^{(0)} < \frac{\rho_i^{(0)}}{d^{(0)}}\lambda_n = \frac{\frac{3}{2}\left|W_i^{(0)}\right|}{d^{(0)}} \frac{2\sqrt{2}n(n-1)}{\sqrt{9n^2-8n+2}} < \frac{\sqrt{2}(n-1)}{\sqrt{9n^2-8n+2}} < \frac{\sqrt{2}}{3} < 1.$$

Starting from the inequality $h_i^{(0)} < 1$ $(i \in I_n)$, by successive application of (24) we find that the sequences $\{h_i^{(m)}\}$ $(i \in I_n)$ monotonically converge to 0.

By successive application of (20), (21) and the condition (17), we find the lower bound of $d^{(m)}$ :

$$\begin{aligned} d^{(m)} &\geq \left|z_i^{(m)} - z_j^{(m)}\right| \geq \left|z_i^{(m-1)} - z_j^{(m-1)}\right| - \left|z_i^{(m)} - z_i^{(m-1)}\right| - \left|z_j^{(m)} - z_j^{(m-1)}\right| \\ &> d^{(m-1)} - 2\cdot\frac{4n-1}{3n}\omega^{(m-1)} > d^{(m-2)} - 2\cdot\frac{4n-1}{3n}\omega^{(m-2)} - 2\cdot\frac{4n-1}{3n}\omega^{(m-1)} \\ &\ \ \vdots \\ &> d^{(0)} - \frac{8n-2}{3n}\left(\omega^{(0)} + \omega^{(1)} + \cdots + \omega^{(m-1)}\right) \\ &> d^{(0)} - \frac{8n-2}{3n}\omega^{(0)}\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^{m-1}\right), \end{aligned}$$

wherefrom

$$d^{(m)} > d^{(0)} - \frac{4n-1}{n}\omega^{(0)} > d^{(0)} - \frac{4n-1}{n}\cdot\frac{d^{(0)}}{3n} = \frac{3n^2-4n+1}{3n^2}d^{(0)} \geq \frac{16}{27}d^{(0)}. \tag{25}$$

Since $d^{(m)}$ is bounded, in regard to the substitution (22) we infer that the sequences $\{\rho_i^{(m)}\}$ $(i \in I_n)$ also converge to 0. Setting the inequality $d^{(m)} > \frac{3n^2-4n+1}{3n^2}d^{(0)}$ in (18) we obtain

$$\rho_i^{(m+1)} < (\frac{3n^2}{3n^2-4n+1})\frac{8}{9(d^{(0)})^2}[(\rho_i^{(m)})^2 \sum_{j\neq i}\rho_j^{(m)} + \rho_i^{(m)}( \sum_{j\neq i}\rho_j^{(m)})^2] \ (i \in I_n).$$

Let $\rho^{(m)} = \max_{1\leq i\leq n} \rho_i^{(m)}$. From the last inequality we obtain

$$\begin{aligned} \rho_i^{(m+1)} &< (\frac{3n^2}{3n^2-4n+1})^2 \frac{8}{9\left(d^{(0)}\right)^2}\left[(n-1)\left(\rho^{(m)}\right)^3 + (n-1)^2\left(\rho^{(m)}\right)^3\right] \\ &= (\frac{3n^2}{3n^2-4n+1})^2 \frac{8\left(n^2-n\right)}{9\left(d^{(0)}\right)^2}\left(\rho^{(m)}\right)^3, \end{aligned}$$

which means that the sequences of PEB $\{\rho_i^{(m)}\}$ converge cubically. □

## 3. COMPUTATIONAL ASPECTS

In this section we give some practical aspects of the presented theoretical results and calculating procedures in the implementation of the proposed method. We emphasize that the PEB method (3) possesses high computational efficiency since the quantities $W_i^{(0)}$, $W_i^{(1)}$,... $(i \in I_n)$, necessary in the calculation of the centers $z_i^{(m+1)}$ (by the iterative formula (2)), are again used in the calculation of PEB $\rho_i^{(m)} = \frac{3}{2}\left|W_i^{(m)}\right|$ (taking $c_n = 1/(3n)$. Such approach causes that the PEB method (3) requires less numerical operations compared to its counterparts (4) in complex interval arithmetic. Total number of numerical operations per one iteration, reduced to real arithmetic operations, is given in Table 1 with the following abbreviations:

*AS(n)* (total number of additions and subtractions)
*M(n)* (multiplications)
*D(n)* (divisions)

| | *AS(n)* | *M(n)* | *D(n)* |
|---|---|---|---|
| (I-PMT) (2)-(3) | $15n^2 - 7n$ | $14n^2$ | $2n^2$ |
| Interval BS (4) | $23n^2 - 4n$ | $23n^2 + 2n$ | $7n^2 + 2n$ |

Table 1 The number of operations of the methods (3) and (4)

A calculating procedure can be described by the following algorithm:

**PEB algorithm:**

*Given* $z_1^{(0)},\ldots, z_n^{(0)}$ *and the tolerance parameter* $\tau$;

*Set* $m = 0$;

1° *Calculate Weierstrass' corrections* $W_1^{(m)},\ldots,W_n^{(m)}$ *at the points* $z_1^{(m)},\ldots,z_n^{(m)}$;

2° *Calculate the radii* $\rho_i^{(m)} = \frac{3}{2}\left|W_i^{(m)}\right| \quad (i = 1,\ldots,n)$;

3° *If* $\max\limits_{i} \rho_i^{(m)} < \tau$, *then STOP*

*otherwise, GO TO* 4°;

4° *Calculate the new approximations* $z_1^{(m+1)},\ldots, z_n^{(m+1)}$ *by the iterative formula* (2);

5° *Set* $m{:}=m + 1$ *and GO TO the step* 1°.

Employing PEB algorithm we realized many numerical examples and, for demonstration, we select the following one.

**Example 1.** We considered the polynomial

$$\begin{aligned} P(z) &= z^{12} - (2+5i)z^{11} - (1-10i)z^{10} + (12-25i)z^9 - 30z^8 \\ &\quad - z^4 + (2+5i)z^3 + (1-10i)z^2 - (12-25i)z + 30 \\ &= (z^8 - 1)(z^2 - 2z + 5)(z - 2i)(z - 3i). \end{aligned}$$

Starting from sufficiently close initial approximations $z_1^{(0)},\ldots,z_{12}^{(0)}$ we applied the PEB method (3) and obtained the inclusion disks $D_i^{(m)}=\{z_i^{(m)};\rho_i^{(m)}\}$ $(i\in I_{12})$. The approximations $z_i^{(m)}$ $(m\geq 1)$ were calculated by the iterative formula (2), and the corresponding inclusion method is referred to as (I-PMT). For the comparison purpose, we also tested the interval methods (4). The largest radii of the disks obtained in the first four iterations are presented in Table 2, where $A(-q)$ means $A\times 10^{-q}$. For both methods max $\rho_i^{(0)}$ has been equal to 0.287.

| Methods | max $\rho_i^{(1)}$ | max $\rho_i^{(2)}$ | max $\rho_i^{(3)}$ | max $\rho_i^{(4)}$ | CPU time (1 iter.) |
|---|---|---|---|---|---|
| (I-PMT) (2)-(3) | 4.89(-2) | 2.00(-4) | 1.41(-11) | 4.18(-33) | 6.9 msec |
| Interval BS (4) | 6.33(-1) | 2.95(-3) | 3.29(-12) | 4.44(-39) | 13 msec |

Table 2 The largest radii of disks obtained by inclusion methods (2)-(3) and (4)

In our calculation we employed multi-precision arithmetic since the tested methods converge very fast producing very small disks. From Table 2 we observe that the disks obtained in later iterations by the interval method (4) are slightly smaller than those obtained by the PEB method (2)-(3), but the PEB method requires almost twice less CPU time per iteration. A number of numerical experiments showed similar convergence behavior of the tested methods.

We note that the PEB method (3) always produces inclusion disks, while the ordinary interval methods (like (4)) can encounter the inversion of zero-disk, which breaks the process. Furthermore, the PEB method automatically generates initial disks. This is another advantage of PEB methods.

**Parallel implementation.** The error bound method (3) is very convenient for the implementation on parallel computers since it runs in several identical versions providing that a great deal of computation can be executed simultaneously. More details about the implementation of simultaneous methods on parallel processing computers can be found, e.g., in [4], [5], [6], [9], [20].

Let

$$\mathbf{W}^{(m)}=\left(W_1^{(m)},\ldots,W_n^{(m)}\right),\ \boldsymbol{\rho}^{(m)}=\left(\rho_1^{(m)},\ldots,\rho_n^{(m)}\right),\ \mathbf{z}^{(m)}=\left(z_1^{(m)},\ldots,z_n^{(m)}\right)$$

denote vectors in the $m$-th iterative step, where $\rho_i^{(m)}=\frac{3}{2}\left|W\left(z_i^{(m)}\right)\right|$ and $z_i^{(m)}$ is calculated by the iterative formula (2). The model of parallel impleme-ntation is as follows: It is assumed that the number of processors $k$ $(\leq n)$ is given in advance. All processors $P_1,\ldots,P_k$ find the starting vector $\mathbf{z}^{(0)}$ using some suitable globally convergent method based on a subdivided procedure and the inclusion annulus $\{z: r\leq |z|\leq R\}$ which contains all zeros, where

$$r=\frac{1}{2}\min_{1\leq k\leq n}\left|\frac{a_n}{a_{n-k}}\right|^{1/k},\quad R=2\max_{1\leq k\leq n}\left|\frac{a_k}{a_0}\right|^{1/k}$$

(see [10, Theorem 6.4b, Corollary 6.4k]).

The next steps of the algorithm consist in sharing the calculation of $W_i^{(m)}$, $\rho_i^{(m)}$, $z_i^{(m+1)}$ among the processors and in updating their data through a broadcast procedures (shorter $BCAST(\mathbf{W}^{(m)},\boldsymbol{\rho}^{(m)})$, $BCAST(\mathbf{z}^{(m+1)})$). As in [5], let $I_1,\ldots,I_k$ be disjunctive partitions of the set $\{1,\ldots, n\}$ where $\cap I_j = \{1,\ldots,n\}$. Good load balancing between the processors is provided choosing the index sets $I_1,\ldots,I_k$ in such a way that the number of their components $\iddots(I_j)$ $(j = 1,\ldots,k)$ is determined as $\iddots(I_j)\le\left[\frac{n}{k}\right]$. For all $i\in I_j$ the processor $P_j$ $(j = 1,\ldots,k)$ computes $W_i^{(m)}$ , $\rho_i^{(m)}$ and, if necessary, $z_i^{(m+1)}$ and then it transmits these values to all other processors using a broadcast procedure. The program terminates when a suitable stopping criterion is satisfied, say, if for a given tolerance $\tau$ the inequality

$$\max_{1\le i\le n}\left|\rho_i^{(m)}\right| < \tau$$

holds. A program written in pseudocode for a parallel implementation of the PEB method (3) is given below:

**Program** A POSTERIORI ERROR BOUND METHOD
**begin**
  **for** all $j = 1,\ldots, k$ **do** determination of the approximations $\mathbf{z}^{(0)}$;
  $m$ :=0
  $C$ :=false
  **do**
    **for** all $j = 1,\ldots, k$ **do in parallel**
    **begin**
      Compute $W_i^{(m)}$, $i\in I_j$;
      Compute $\rho_i^{(m)} = \frac{3}{2}\left|W_i^{(m)}\right|$, $i\in I_j$;
      Communication: $\mathrm{BCAST}\left(\mathbf{W}^{(m)},\boldsymbol{\rho}^{(m)}\right)$;
    **end**
    **if** $\max_{1\le i\le n}\left|\rho_i^{(m)}\right| < \tau$ ; $C$:=true
    **else**
      $m := m + 1$
      **for** all $j = 1,\ldots, k$ **do in parallel**
      **begin**
        Compute $z_i^{(m)}$, $i\in I_j$, by (2);
        Communication: BCAST($\mathbf{z}^{(m)}$);
      **end**
    **endif**
  **until** $C$
  OUTPUT $\mathbf{z}^{(m)},\boldsymbol{\rho}^{(m)}$
**end**

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