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STABILITY OF RATE-BASED CONGESTION CONTROL WITH COMMUNICATION DELAY

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ABSTRACT. The congestion control algorithms have an important influence on the quality of service of communication network. In this paper, a class of rate-based congestion control algorithms with communication delays is studied. Based on the Lyapunov-Razumikhin theorem, the Lyapunov stability of the algorithm is analyzed. The global attractability of the algorithm is proved by applying Barbalat Lemma. A more concise criterion to ensure the global asymptotical stability is obtained. The new result presents a simple upper delay bound, and enlarges the admissible upper delay bound. Finally, many examples are given to support the result.

Key Words Congestion control; global asymptotical stability; communication delay.

1. INTRODUCTION

With the development of network communication, the quality of service (QoS) of Internet becomes more and more important. For ensuring the QoS and the capacity of Internet, the sources of Internet apply TCP congestion control algorithms to avoid network congestion, such as TCP Reno[1] (and its variants), and the link nodes use active queue management (AQM) schemes to improve the serving capacity of Internet, such as DropTail[1], RED[2]. However, the existing congestion control algorithms which are based on "trial-and-error" methods employed on small test beds may be ill-suited for future networks [3, 4] where both communication delay and network capacity can be large, which has been proved by the fact of two times to revise the parameters of the RED algorithm[5, 6]. This has motivated the research on theoretical understanding of TCP congestion control and the search for protocols that scale properly so as to maintain stability in the presence of these variations.

From a point of view of the control theory, congestion control problems are complex and challenging because they are high-dimensional, nonlinear, and dynamic equations for the mathematical modelling of the network[7, 8, 9]. In order to improve the

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throughout and decrease the queue vibrating of Internet, there are many new Internet congestion control algorithms (ICCA) presented [10, 11, 12, 13, 14, 15]. Hollt et al[12] analyzed the TCP flows of Internet and designed a PI controller for AQM routers which can improve the serving capacity of Internet. Athuraliya et al[13] presented an ameliorated RED algorithm: REM, and an adaptive virtual queue (AVQ) algorithm for active queue management is introduced by Kunniyur and Srikant[14], which can enhance the quality of service, largely.

Based on the views of the network's optimization, Kelly et al [15] have developed a network framework with an interpretation of various congestion control mechanisms. They proposed a prime algorithm for TCP rate control and a dual algorithm for AQM scheme, which generalize the Additive Increase/Multiplicative Decrease (AIMD) congestion avoidance strategy [1] to large-scale networks. The advances in mathematical modeling of Kelly's primal algorithm have stimulated the research on the analysis of the behavior such as stability, robustness and fairness.

The convergence of Kelly's primal algorithm has been established in the absence of the communication delays in [15]. The stability of this algorithm with the communication delays has drawn much attention in the past few years. The continuous-time model and the discrete-time model of Kelly's primal algorithm with homogenous communication delays for different TCP connections have been investigated in [16, 17], respectively. For a more general case of networks with heterogeneous round-trip delays, they proposed a conjecture on the local stability of the algorithm, respectively. Recently, their conjectures have received much attention [18, 19], where Tian and Yang [19] have studied their conjectures and obtained a more general stability criterion. The new criterion in [19] is stronger than the conjecture, and enlarges the stability region of control gains and admissible communication delays.

In this paper, we study the global asymptotical stability (GAS) of Kelly's primal algorithm with the communication delay in a single link accessed by a single source. The algorithm model is described as

$$\dot{x}(t) = \begin{cases} \kappa(w - x(t-D)p(x(t-D))), & x > 0; \\ (\kappa(w - x(t-D)p(x(t-D))))^{+}, & x = 0. \end{cases} \tag{1.1}$$

where $\kappa > 0$ is the control gain of the system, D is the communication delay, $x(t)$ is the sending rate of the source at time t. The function $p(\cdot)$ is the congestion indication probability (or congestion control rate) back from the link node, which is assumed to be increasing, nonnegative, concave and not identically zero, satisfying $0 \leq p(\cdot) \leq 1$. $x(t)p(t)$ denotes the marked packets number of the source at time t, w is a desired target value of marked packets received back at source. $(f(x))^{+} = max\{f(x), 0\}$. From the description of the system (1.1), we know that the solution of Eq.(1.1) should be $x(t) \geq 0$.

We note that the GAS problem of the system (1.1) has been studied in [20, 21], which pointed out that the global asymptotic stability of system (1.1) can be ensured if the product of the control gain and the delay constant, κD, is upper bounded. However, the upper bound given in [20] is very complicated and might be inconvenient for practical application. In [21], a simpler and more explicit formula of the GAS condition was proposed as

$$\kappa D < \frac{1}{4}.$$

To prove this result, the authors of [21] used an assumption that $x(t) \geq 0$, and then all the discussion were based only on the first equation in (1.1). Although $x(t) \geq 0$ is a direct result of the second equation in (1.1), a rigorous proof of stability should be based on the switched model rather than a smooth one. In this note, we obtain a less conservative GAS criterion given by

$$\kappa D < \frac{1}{2}.$$

And more importantly, our proof does not depend on the supposition $x(t) \geq 0$. We consider the problem based on the switched model of the original Kelly's algorithm.

The rest of this paper is organized as follows. The Lyapunov stability of Kelly's prime algorithm is analyzed by applying Lyapunov-Razumikhin theorem in Section 2. In Section 3, based on the global attractability of the algorithm from Barbalat's Lemma, the criteria of the GAS is presented. An extended result for GAS is obtained in Section 4. In Section 5, a lot of simulations are used to test the results. Finally, the conclusions are showed in Section 6.

2. ANALYSIS OF LYAPUNOV STABILITY OF THE SYSTEM

In this section, we discuss the Lyapunov stability of the system (1.1) by applying Lyapunov-Razumikhin theorem. From the description of the system (1.1), we know that there exists a unique equilibrium point x^* satisfying

$$w = x^* p^*. \tag{2.1}$$

where $p^* = p(x^*)$ is the congestion rate at the equilibrium point.

Lemma 2.1. *Suppose $f(x) = \kappa(w - xp(x))$. Then $|f'(x)| \leq 2\kappa$.*

Proof. From the description of the system (1.1), all solutions $x \geq 0$. Since $p(x)$ is a increasing, concave function, satisfying

$$0 \leq p(x) \leq 1, \; p'(x) \geq 0, \; p"(x) \leq 0,$$

then

$$xp'(x) \leq \int_0^x p'(s)ds = p(x) - p(0) \leq 1,$$

we obtain

$$p(x) + xp'(x) \leq 2.$$

Note that

$$f'(x) = -\kappa(p(x) + xp'(x)),$$

therefore

$$|f'(x)| \leq 2\kappa.$$

Then, we finish the proof of Lemma 2.1. □

Throughout the paper we use the following definition of derivative

$$\dot{x}(t_0) = \dot{x}(t_0^+) = \lim_{t \to t_0^+} \frac{x(t) - x(t_0)}{t - t_0}.$$

Knowing from the system (1.1), there is $\dot{x}(t) = 0$ or $\dot{x}(t) = f(x(t-D))$, and $f(x^*) = 0$. Suppose $x(t) = x^* + \hat{x}(t)$, when $\dot{x}(t) = f(x(t-D))$, we obtain

$$|\dot{x}(t_0)| = |f(x(t-D)) - f(x^*)| \leq 2\kappa|\hat{x}(t-D)|. \tag{2.2}$$

where Eq.(2.2) is correct for $\dot{x}(t) = 0$, too. Now we can prove the following theorem.

Theorem 2.2. *When $\kappa D < \frac{1}{2}$, the system (1.1) is Lyapunov stable.*

Proof. Let $x(t) = x^* + \hat{x}(t)$, the system (1.1) can be transformed into

$$\dot{\hat{x}}(t) = \begin{cases} f(\hat{x}(t-D)), & \hat{x}(t) > -x^*; \\ (f(\hat{x}(t-D)))^+, & \hat{x}(t) = -x^*. \end{cases} \tag{2.3}$$

where

$$\begin{aligned} f(\hat{x}(t-D)) &= -\kappa\hat{x}(t-D)p(\hat{x}(t-D) + x^*) \\ &\quad -\kappa x^*(p(\hat{x}(t-D) + x^*) - p^*). \end{aligned} \tag{2.4}$$

It follows that

$$f(\hat{x}(t-D)) = -\kappa\hat{x}(t-D)[p(\hat{x}(t-D)) + x^*p'(\tilde{x})], \tag{2.5}$$

where $p(\hat{x}(t-D)) = p(\hat{x}(t-D) + x^*)$, $\tilde{x} = \alpha\hat{x}(t-D) + x^*$, $\alpha \in [0,1]$.

Supposing the solution $\hat{x}(t) = \hat{x}(t, \varphi)$ of the system (2.3) with $\varphi \in [-2D, 0]$, and Lyapunov function

$$V(\hat{x}(t)) = \frac{1}{2}\hat{x}^2(t),$$

we know that the function $V(\hat{x}(t))$ is a continuous function in $t \in [-2D, \infty)$. Based on the Lyapunov-Razumikhin theorem [23], for any $\theta \in [-2D, 0]$, let

$$V(\hat{x}(t+\theta)) \leq V(\hat{x}(t)),$$

implying

$$|\hat{x}(t+\theta)| \leq |\hat{x}(t)|, \ \forall\, \theta \in [-2D, 0]. \tag{2.6}$$

In the following, we will compute the derivative of the Lyapunov function along the trajectories of (2.3). Firstly, when $\hat{x}(t) = -x^*$, we derive from the derivative definition,

$$\dot{V}(t) = \hat{x}(t)\dot{\hat{x}}(t) = -x^*[f(\hat{x}(t-D)]^+ \leq 0. \tag{2.7}$$

Then, when $\hat{x}(t) > -x^*$, it follows from Eq.(2.5) that

$$\begin{aligned}\dot{V}(t) &= \hat{x}(t)\dot{\hat{x}}(t) \\ &= \hat{x}(t)f(\hat{x}(t-D) \\ &= -\kappa\hat{x}(t)\hat{x}(t-D)[p(\hat{x}(t-D)) + x^*p'(\tilde{x})].\end{aligned}$$

Since

$$\hat{x}(t-D) = \hat{x}(t) - \int_{t-D}^{t} \dot{\hat{x}}(s)ds. \tag{2.8}$$

we have

$$\begin{aligned}\dot{V}(t) &= -\kappa\hat{x}^2(t)[p(\hat{x}(t-D)) + x^*p'(\tilde{x})] \\ &\quad +\kappa\hat{x}(t)[p(\hat{x}(t-D)) + x^*p'(\tilde{x})]\textstyle\int_{t-D}^{t}\dot{\hat{x}}(s)ds.\end{aligned}$$

Substituting inequation (2.2) and inequation (2.6), we obtain

$$\begin{aligned}\dot{V}(t) &\leq [p(\hat{x}(t-D)) + x^*p'(\tilde{x})] \\ &\quad (-\kappa\hat{x}^2(t) + \kappa|\hat{x}(t)|2\kappa\textstyle\int_{t-D}^{t}|\hat{x}(s-D)|ds) \\ &\leq -\kappa\hat{x}^2(t)[p(\hat{x}(t-D)) + x^*p'(\tilde{x})] \\ &\quad +2\kappa^2D\hat{x}^2(t)[p(\hat{x}(t-D)) + x^*p'(\tilde{x})].\end{aligned}$$

When $\kappa D < \frac{1}{2}$, we get

$$\begin{aligned}\dot{V}(t) &\leq -\kappa(1-2\kappa D)\hat{x}^2(t)[p(\hat{x}(t-D)) + x^*p'(\tilde{x})] \\ &\leq 0.\end{aligned} \tag{2.9}$$

Therefore, the system (2.3) is Lyapunov stable by the Lyapunov-Razumikhin theorem[2:
We finish the Proof of the theorem 2.2. □

3. ANALYSIS OF GLOBAL ASYMPTOTICAL STABILITY

In Section 2, we have known that the system (2.3) is Lyapunov stable. If the system (2.3) is the global attractable, the global asymptotical stability can be derived immediately from the definition of the GAS. In the following, we are going to prove the global attractability of the system (2.3). Now, let's firstly investigate several important results.

Lemma 3.1. *All solutions $x(t)$ of the system (1.1) will never go to infinite at a finite time t_0.*

Proof. Supposing that there exists a solution $x(t, \varphi)$ going to the infinity at a finite time t_0 with the continuous function $\varphi \in [-2D, 0]$, i.e.

$$\lim_{t \to t_0} x(t) = +\infty.$$

Since

$$\begin{aligned} x(t_0) &= x(t_0 - D) + \int_{t_0-D}^{t_0} \kappa(w - x(s-D)p(x(s-D)))ds \\ &= x(t_0 - D) + \kappa w D - \int_{t_0-D}^{t_0} \kappa x(s-D)p(x(s-D)))ds. \end{aligned}$$

note that

$$\int_{t_0-D}^{t_0} \kappa x(s-D)p(x(s-D)))ds \geq 0,$$

we yield

$$\lim_{t \to t_0 - D} x(t) = +\infty.$$

There exists a $\delta_0 > 0$ for any sufficiently large number $M_1 > 0$, when $t \in [t_0 - D - \delta_0, t_0 - D + \delta_0]$, getting

$$|x(t)| > M_1.$$

Now supposing $M_1 > x^*$, based on

$$p(M_1) \geq p(x^*) > 0,$$

it follows that

$$M_1 p(M_1) > x^* p(x^*) = w.$$

When $t \in [t_0 - \delta_0, t_0 + \delta_0]$, since

$$\begin{aligned} \dot{x}(t) &= \kappa(w - x(t-D)p(x(t-D))) \\ &< \kappa(w - M_1 p(M_1)) \\ &< 0. \end{aligned}$$

it implies $x(t)$ is strictly decrease. When $t \in [t_0 - \delta_0, t_0 + \delta_0]$, we obtain

$$x(t_0) < x(t_0 - \delta_0).$$

i.e. $x(t_0)$ is a bounded number, which is in contradiction with $\lim_{t \to t_0} x(t) = +\infty$. Then, we finish the proof of the Lemma 3.1. □

Lemma 3.2. *All solutions $x(t)$ of the system (1.1) does not escape to infinite when $t \to \infty$.*

Proof. The proof is similar to that for Lemma 3.1. By contradiction we suppose that $\lim_{t \to \infty} x(t) = +\infty$. Then, for any sufficiently large number M_2 there exists a T_2 such that $x(t) > M_2$ for all $t \geq T_2$. Without loss of generality, we can assume that $M_2 > x^*$. Since $p(x)$ is an increasing function, we have

$$p(M_2) \geq p(x^*) = p^* > 0,$$

it follows that

$$M_2 p(M_2) > x^* p(x^*) = w.$$

Therefore, when $t \geq T_2$, we have

$$\dot{x}(t+D) = \kappa(w - x(t)p(x(t))) < 0.$$

This implies that $x(t)$ is strictly decreasing when $t \geq T_2 + D$, and, hence,

$$x(t) < x(T_2 + D), \quad \forall t > T_2 + D,$$

which contradicts the assumption $\lim_{t\to\infty} x(t) = +\infty$. The lemma 3.2 is thus proved. □

The proof of Lemma 3.1 and Lemma 3.2 shows that when $x(t)$ is increasing, it is going to achieve a maximum at some time. After that time, it will decrease. For convenience of further discussion we denote

$$\begin{aligned} T_0 &= \inf\{t > D : \dot{x}(t) < 0\}, \\ T &= \inf\{t > T_0 : \dot{x}(t) > 0\}, \\ T_1 &= \inf\{t > T : \dot{x}(t) \leq 0\} \\ T_2 &= \inf\{t > T_1 : \dot{x}(t) \geq 0\}. \end{aligned}$$

Lemma 3.3. *There exists a positive number M such that for any $t > T$, $x(t)$ satisfies $0 < x(t) \leq M$.*

Proof. As $x(t)$ is differentiable, and we have excluded the possibility of escaping to infinity for $x(t)$, to prove this lemma it suffices to show all the extreme values of $x(t)$ are greater than zero and upper bounded by M.

Now, let us consider all the staying points of $x(t)$. Suppose $x(t)$ reaches a staying point at $t = t_1$, i.e.,

$$\dot{x}(t_1) = \kappa(w - x(t_1 - D)p(x(t_1 - D))) = 0,$$

then, by the uniqueness of the equilibrium we have

$$x(t_1 - D) = x^*.$$

There are three possibilities for the derivative of $x(t_1 - D)$ at the time $t_1 - D$, namely, $\dot{x}(t_1 - D) > 0$, $\dot{x}(t_1 - D) < 0$, or $\dot{x}(t_1 - D) = 0$. We discuss these three cases below.

(1) $\dot{x}(t_1 - D) > 0$. Without loss of generality we assume that $t_1 - D \in (T, T_1)$. Next we prove that $x(t)$ achieves a maximum at t_1 in this case. Indeed, when $t_1 - D < t < t_1$ and $\max(T, t_1 - 2D) < t - D < t_1 - D$, since $x(t - D) < x(t_1 - D)$, we obtain

$$\dot{x}(t) > \dot{x}(t_1) = 0.$$

When $t_1 < t < t_1 + D$ and $t_1 - D < t - D < \min(t_1, T_1)$, since $x(t - D) > x(t_1 - D)$, we get

$$\dot{x}(t) < \dot{x}(t_1) = 0.$$

Therefore, $x(t)$ achieves a maximum at $t = t_1$ (see Figure 1).

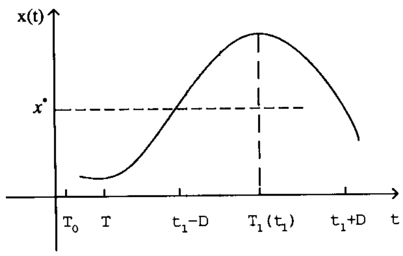

FIGURE 1. The case when $\dot{x}(t_1 - D) > 0$.

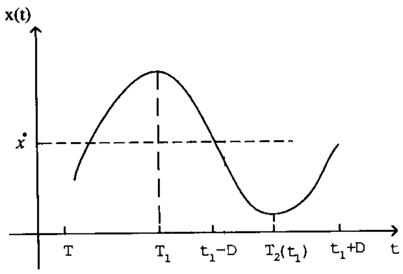

FIGURE 2. The case when $\dot{x}(t_1 - D) < 0$.

Now, we prove that $x(t_1)$ is upper bounded by a positive number denoted by M. Since

$$\begin{aligned} x(t_1) &= x(t_1 - D) + \int_{t_1-D}^{t_1} \kappa(w - x(s-D)p(x(s-D)))ds \\ &= x^* + \kappa w D - \int_{t_1-D}^{t_1} \kappa x(s-D)p(x(s-D)))ds, \end{aligned}$$

where

$$\int_{t_1-D}^{t_1} \kappa x(s-D)p(x(s-D)))ds \geq 0,$$

we get

$$x(t_1) \leq x^* + \kappa w D.$$

Let

$$M = x^* + \kappa w D. \tag{3.1}$$

By carrying out this procedure we can prove that any maximum achieved by $x(t)$ for $t > T$, satisfies

$$x(t) \leq M.$$

(2) $\dot{x}(t_1 - D) < 0$. Without loss of generality we assume that $t_1 - D \in (T_1, T_2)$. Using a procedure similar to that used in case (1) we can show that $x(t)$ achieves a minimum at t_1 in this case (see Figure 2).

Since

$$\begin{aligned} x(t_1) &= x(t_1 - D) + \int_{t_1-D}^{t_1} \kappa(w - x(s-D)p(x(s-D)))ds \\ &= x^* + \kappa w D - \int_{t_1-D}^{t_1} \kappa x(s-D)p(x(s-D)))ds, \end{aligned}$$

where $x(t) \le M$, $p(x) \le 1$, it follows that

$$\begin{aligned} x(t_1) &\ge x^* + \kappa w D - \kappa M D \\ &= x^*(1-\kappa D) + \kappa w D(1 - \kappa D). \end{aligned}$$

When $\kappa D < \frac{1}{2}$, we have

$$x(t_1) > 0.$$

(3) $\dot{x}(t_1 - D) = 0$. In this case, by integrating Eq. (1.1), we get

$$x(t_1) = x(t_1 - D) = x^* \in (0, M].$$

Summarizing the above three cases, we know the extreme points of $x(t)$ are greater than zero and upper bounded by M. So the lemma 3.3 is proved. □

Theorem 3.4. *When $\kappa D < \frac{1}{2}$, the solution of the system (2.3) is global attractable.*

Proof. . We will split the proof of this theorem into four parts.

1). For all $t > T + D$, the function $\hat{x}^2(t)P(\hat{x}(t-D))$ is uniformly continuous.

Based on the Lemma 3.3, for all $t > T + D$, there are $0 < x(t) \le M$. Since $\hat{x}(t) = x(t) - x^*$, we obtain

$$-x^* \le \hat{x}(t) \le \kappa w D,$$

i.e. $|\hat{x}(t)| \le M$. Eq (2.2) implies

$$|\dot{x}(t)| \le 2\kappa M.$$

For any positive number $\varepsilon > 0$, let $\delta = \varepsilon/(2\kappa M)$, when $|t_1 - t_2| < \delta$, we have

$$|x(t_1) - x(t_2)| < \varepsilon.$$

So $x(t)$ is uniformly continuous, i.e. $\hat{x}(t)$ is uniformly continuous.

For all $t \in (T + D, +\infty)$, we derive $x(t) > 0$ from the Lemma 3.3. Since $p'(x) \ge 0$ and $p"(x) \le 0$, it follows that

$$0 \le p'(x(t)) \le p'(0), \ \forall\, t \in (T + D, +\infty).$$

Because $\hat{x}(t)$ is uniformly continuous, this implies that $\hat{x}(t)p(\hat{x}(t-D))$ is uniformly continuous, for all $t \in (T + D, +\infty)$. Therefore, the function $\hat{x}^2(t)p(\hat{x}(t-D))$ is uniformly continuous since $\hat{x}(t)$ and $\hat{x}(t)p(\hat{x}(t-D))$ are bound and uniformly continuous, for all $t \in (T + D, +\infty)$.

2). $\lim_{t\to+\infty} \hat{x}^2(t)p(\hat{x}(t-D)) = 0$.

Based on the Lemma 3.3, $x(t) > 0$ for all $t > T + D$ when $\kappa D < \frac{1}{2}$. Since $p(x) \geq 0$, $p'(x) \geq 0$, we derive from Eq.(2.9)

$$\dot{V}(\hat{x}(t)) \leq -\kappa(1 - 2\kappa D)\hat{x}^2(t)p(\hat{x}(t-D)).$$

By integrating, one deduces that for all $t > T + D$

$$\begin{array}{rcl} V(\hat{x}(t)) & \leq & V(\hat{x}(T+D)) \\ & & -\kappa(1-2\kappa D)\int_{T+D}^{t}\hat{x}^2(s)p(\hat{x}(s-D))ds. \end{array}$$

Since $V(\hat{x}(t))$ is bounded with $|\hat{x}(t)| \leq M$, it follows that

$$\lim_{t\to+\infty}\int_{T+D}^{t}\hat{x}^2(s)p(\hat{x}(s-D))ds < +\infty.$$

When $\hat{x}^2(t)p(\hat{x}(t-D))$ is uniformly continuous for all $t > T+D$, Barbalat's Lemma[22] ensures that

$$\lim_{t\to+\infty}\hat{x}^2(t)p(\hat{x}(t-D)) = 0.$$

3). $\lim_{t\to+\infty} p(x(t)) \neq 0$.

Supposing $\lim_{t\to+\infty} p(x(t)) = 0$, then there exists $T_3 > T + D$ for any $\varepsilon_0 > 0$, when $t > T_3 - D$, we have

$$P(x(t)) < \varepsilon_0.$$

i.e. for all $t > T_3$,

$$P(x(t-D)) - p^* < \varepsilon_0 - p^*.$$

Since

$$\begin{array}{rcl} \hat{x}(t) & = & \hat{x}(T_3) - \kappa\int_{T_3}^{t}[\hat{x}(s-D)p(x(s-D)) \\ & & +x^*(p(x(s-D)) - p^*)]ds \\ & > & \hat{x}(T_3) + \kappa(t-T_3)x^*(P^* - \varepsilon_0) \\ & & -\kappa\int_{T_3}^{t}\hat{x}(s-D)p(x(s-D))ds. \end{array}$$

where $|\hat{x}(t)| \leq M$, for all $t > T_3$, it follows that

$$\hat{x}(t) > \hat{x}(T_3) + \kappa(t-T_3)(x^*P^* - x^*\varepsilon_0 - M\varepsilon_0).$$

If

$$\varepsilon_0 < \frac{x*p^*}{x^* + M},$$

we yield

$$\lim_{t\to+\infty}\hat{x}(t) = +\infty.$$

This is in contradiction with $|\hat{x}(t)| \leq M$, i.e. $\lim_{t\to+\infty} p(x(t)) \neq 0$.

4). The system (2.3) is global attractable.

We have known that

$$\lim_{t\to+\infty}\hat{x}^2(t)p(\hat{x}(t-D)) = 0,$$

but

$$\lim_{t\to+\infty} p(\hat{x}(t)) = \lim_{t\to+\infty} p(x(t)) \neq 0.$$

it implies

$$\lim_{t\to+\infty} \hat{x}^2(t) = 0.$$

Since $\hat{x}(t)$ is bounded when $t > T$, it follows that

$$\lim_{t\to+\infty} \hat{x}(t) = 0.$$

Therefore, the system (2.3) is global attractable. Then, we finish the proof of the Theorem 3.4. □

Now we give the main result of this section

Theorem 3.5. *The system (1.1) is GAS, if* $\kappa D < \frac{1}{2}$.

4. AN EXTENDED RESULT FOR GAS

In this section, we will extend our results for the GAS based on the upper bound of the rate $x(t)$. By Lemma 3.3, there are exists a positive number M such that for any $t > T$, $x(t)$ satisfies $0 < x(t) \leq M$. Let

$$Q = 2p(M) - p(0). \tag{4.1}$$

Lemma 4.1. *Suppose* $f(x) = \kappa(w - xp(x))$. *Then* $|f'(x)| \leq \kappa Q$.

Proof. The proof is similar to that for Lemma 2.1. From the description of the system (1.1), all solutions $x \geq 0$. Since $p(x)$ is an increasing, concave function, satisfying

$$p'(x) \geq 0,$$

we get

$$p(M) \geq p(0).$$

From $p(x)$ is a concave function, we have

$$p''(x) \leq 0,$$

then

$$xp'(x) \leq \int_0^x p'(s)ds = p(x) - p(0),$$

we obtain

$$p(x) + xp'(x) \leq 2p(x) - p(0).$$

satisfying

$$p(x) + xp'(x) \leq Q.$$

Note that

$$f'(x) = -\kappa(p(x) + xp'(x)),$$

therefore

$$|f'(x)| \leq \kappa Q.$$

Then, we finish the proof of Lemma 4.1. □

By the definition of derivative in the section 2, the following result is similar to Eq.(2.2)

$$|\dot{x}(t)| = |f(x(t-D)) - f(x^*)| \leq \kappa Q |\hat{x}(t-D)|. \tag{4.2}$$

Now we can have the following theorem.

Theorem 4.2. *When $\kappa DQ < 1$, the system (1.1) is Lyapunov stable.*

Proof. We can give a proof similarly to the Theorem 2.2. With the notation $x(t) = x^* + \hat{x}(t)$, the system (1.1) can be transformed into

$$\dot{\hat{x}}(t) = \begin{cases} f(\hat{x}(t-D)), & \hat{x}(t) > -x^*, \\ [f(\hat{x}(t-D))]^+, & \hat{x}(t) = -x^*, \end{cases} \tag{4.3}$$

where

$$\begin{aligned} f(\hat{x}(t-D)) &= -\kappa\hat{x}(t-D)p(\hat{x}(t-D)+x^*) \\ &\quad -\kappa x^*(p(\hat{x}(t-D)+x^*)-p^*). \end{aligned} \tag{4.4}$$

It follows that

$$f(\hat{x}(t-D)) = -\kappa\hat{x}(t-D)[p(\hat{x}(t-D)) + x^* p'(\tilde{x})]. \tag{4.5}$$

where $p(\hat{x}(t-D)) = p(\hat{x}(t-D)+x^*)$, $\tilde{x} = \alpha\hat{x}(t-D) + x^*$, $\alpha \in [0,1]$.

Supposing the solution $\hat{x}(t) = \hat{x}(t,\varphi)$ of the system (4.3) with $\varphi \in [-2D, 0]$, and Lyapunov function $V(\hat{x}(t)) = \frac{1}{2}\hat{x}^2(t)$, we know that the function $V(\hat{x}(t))$ is a continuous function in $t \in [-2D, \infty)$. To apply the Lyapunov-Razumikhin theorem [23], we suppose that $V(\hat{x}(t+\theta)) \leq V(\hat{x}(t))$ for any $\theta \in [-2D, 0]$, which implies that

$$|\hat{x}(t+\theta)| \leq |\hat{x}(t)|, \ \forall\, \theta \in [-2D, 0]. \tag{4.6}$$

In the following, we will compute the derivative of the Lyapunov function along the trajectories of (4.3). Firstly, we consider the case when $\hat{x}(t) = -x^*$. By definition of derivative introduced before we know that $\dot{\hat{x}}(t)$ exists at the equilibrium point and

$$\dot{V}(t) = \hat{x}(t)\dot{\hat{x}}(t) = -x^*[f(\hat{x}(t-D))]^+ \leq 0. \tag{4.7}$$

Now, we consider the case when $\hat{x}(t) > -x^*$. It follows from Eq.(4.5) that

$$\begin{aligned} \dot{V}(t) &= \hat{x}(t)\dot{\hat{x}}(t) \\ &= \hat{x}(t)f(\hat{x}(t-D) \\ &= -\kappa\hat{x}(t)\hat{x}(t-D)[p(\hat{x}(t-D)) + x^* p'(\tilde{x})]. \end{aligned}$$

Since

$$\hat{x}(t-D) = \hat{x}(t) - \int_{t-D}^{t} \dot{\hat{x}}(s)ds, \tag{4.8}$$

we have

$$\begin{aligned}\dot{V}(t) &= -\kappa\hat{x}^2(t)[p(\hat{x}(t-D)) + x^*p'(\tilde{x})] \\ &\quad +\kappa\hat{x}(t)[p(\hat{x}(t-D)) + x^*p'(\tilde{x})]\textstyle\int_{t-D}^{t} \dot{\hat{x}}(s)ds.\end{aligned}$$

Substituting (4.2) and (4.6) into the above equation yields

$$\begin{aligned}\dot{V}(t) &\le [p(\hat{x}(t-D)) + x^*p'(\tilde{x})] \\ &\quad \times[-\kappa\hat{x}^2(t) + \kappa|\hat{x}(t)|\kappa Q \textstyle\int_{t-D}^{t} |\hat{x}(s-D)|ds] \\ &\le [p(\hat{x}(t-D)) + x^*p'(\tilde{x})] \\ &\quad \times[-\kappa\hat{x}^2(t) + \kappa^2 DQ\hat{x}^2(t)].\end{aligned}$$

When $\kappa DQ < 1$, we get

$$\begin{aligned}\dot{V}(t) &\le -\kappa(1-\kappa DQ)\hat{x}^2(t)[p(\hat{x}(t-D)) + x^*p'(\tilde{x})] \\ &\le 0.\end{aligned} \tag{4.9}$$

Therefore, the system (4.3) is Lyapunov stable according to the Lyapunov-Razumikhin theorem. Then, we finish the proof of Theorem 4.2. □

We can obtain the following Theorems by applying the proof process similarly to the section 3.

Theorem 4.3. *If $\kappa DQ < 1$, then the solution of the system (4.3) is global attractable.*

Theorem 4.4. *The system (1.1) is GAS, if $\kappa DQ < 1$.*

5. SIMULATION RESULTS

Simulation 1. Firstly, we simulate the result to validate Theorem 3.5. Since our result is only a sufficient condition to ensure the global stability of the system (1.1), it may be stable for Internet congestion control algorithm with the parameter $kD > \frac{1}{2}$. In this simulation, our example is used to test that the system (1.1) is sable when $kD >= \frac{1}{4}$ but $kD < \frac{1}{2}$. Suppose that the congestion control function of the network is set to next expression,

$$p(x) = 1 - e^{-x},$$

we apply this function to analyze the dynamics of this algorithm when the parameter $k = 0.005$, $w = 0.04$, $D = 80ms$. By Eq. (2.1), the equilibrium point can be obtained

$$x^* = 0.21, \ p^* = 0.1915.$$

Since $kD = 0.4$, we follow from the Theorem 3.5 that the system (1.1) is GAS. Let the initial value $x(0)$ of the system (1.1) be $0.02, 0.04, ..., 10(Mb/ms)$, in return, the computer simulations show that the system is global asymptotically stable. Here, we only present Figure 3 and Figure 4 with $x(0) = 0.02Mb/ms$ and $x(0) = 1Mb/ms$,

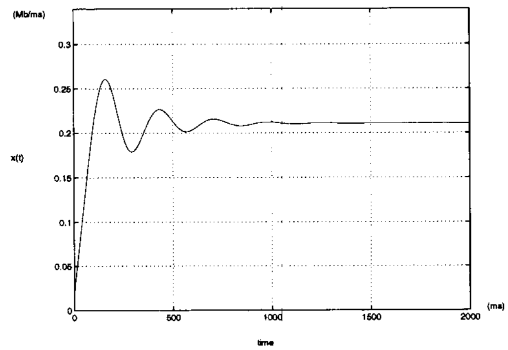

FIGURE 3. State plot with $x(0) = 0.02Mb/ms$.

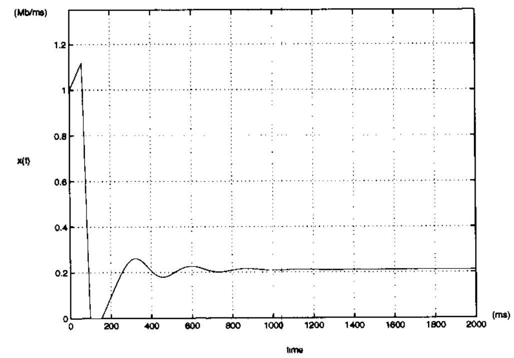

FIGURE 4. State plot with $x(0) = 1Mb/ms$.

respectively, to illuminate the simulation results. Our criterion for GAS is sustained by these computer simulations.

Simulation 2. Now, we analyze the results of Theorem 3.5 and Theorem 4.4. Suppose that the congestion control function of the network is similar to simulation 1, and the system parameter $w = 0.04$, still. Therefor, the equilibrium point of the system is invariable $x^* = 0.21$. We plot the admissible upper bound of the communication delay on the control gain (Figure 5). From Figure 5, the admissible in Theorem 3.5 is the areas of the dash line and the axes, the admissible in Theorem 4.4 is the areas of the solid line and the axes. We can know that the admissible value in the extended result of Theorem 4.4 is larger than that in Theorem 3.5.

Simulation 3. In this simulation, our example is used to test that the system (1.1) is sable when $kD >= \frac{1}{2}$ but $kDQ < 1$. Suppose the system parameter $k = 0.005$, $w = 0.04$, $D = 200ms$. At this time, the equilibrium point of the system is invariable $x^* = 0.21$, still. Since $kD = 1$, Theorem 3.5 can not been used to judge the GAS of the system. We can calculate $KDQ < 1$, which is satisfied for

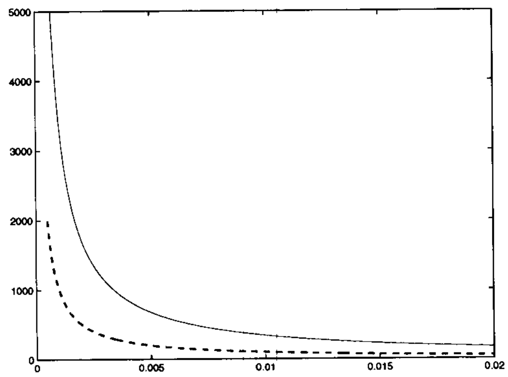

FIGURE 5. Admissible upper bound of the communication delay on the control gain .

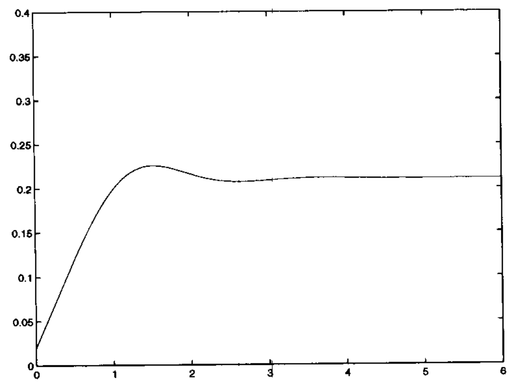

FIGURE 6. State plot with $x(0) = 0.02Mb/ms$.

the requirement of Theorem 4.4. Let the initial value $x(0)$ of the system (1.1) be $0.02, 0.04, ..., 10(Mb/ms)$, in return, the computer simulations show that the system is global asymptotically stable. Here, we only present Figure 6 and Figure 7 with $x(0) = 0.02Mb/ms$ and $x(0) = 1Mb/ms$, respectively, to illuminate the simulation results.

6. CONCLUSIONS

In this paper, we have studied the GAS of Kelly's primal algorithm with communication delay in a single link accessed by a single source. The Lyapunov stability of Kelly's prime algorithm is analyzed by applying Lyapunov-Razumikhin theorem. Based on the global attractability from Barbalat's Lemma, the criteria of the GAS of the algorithm is obtained. An simple ensured upper delay bound guaranteeing

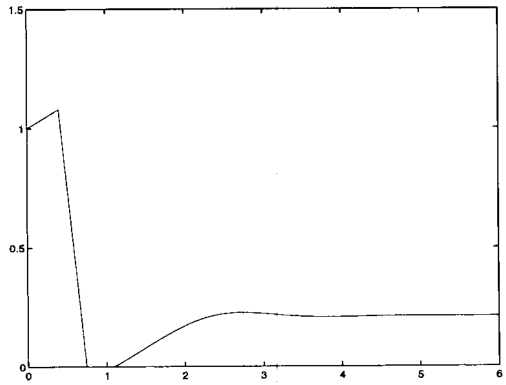

FIGURE 7. State plot with $x(0) = 1Mb/ms$.

the GAS is presented, and it enlarges the admissible upper bound of the communication delay. The switched model of the original Kelly's primal algorithm without any suppositions for the state (the source rate) is investigated, therefore the proof method applied in this paper is more strictly than that in [20, 21]. Finally, simulation examples are given to support the new results.

7. ACNOWLEDGEMENTS

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