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Computations of Critical Domains for Quenching Problems by Delta-Shaped Basis Functions

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Abstract

Let D be a bounded convex m-dimensional domain with a piecewise smooth boundary ∂D, $\bar{D}$ be the closure of D, $T\,(\leq \bullet\,)$ be an extended real number, $\Omega = D \times (0,T)$, and $S = \partial D \times (0,T)$. This article introduces a computational method for the critical domain of the singular semilinear problem,

$$\begin{gathered} u_t - \Delta u = f(u) \text{ in } \Omega, \\ u(\mathbf{x},0) = u_0(\mathbf{x}) \text{ on } \bar{D}, \\ u(\mathbf{x},t) = 0 \text{ on } S, \end{gathered}$$

where f and u_0 are given functions such that $\lim_{u \to c^-} f(u) = \bullet$ for some positive constant c, $f(0) > 0$, $f'(u) \geq 0$, $f''(u) \geq 0$, $0 \leq u_0 < c$, $\Delta u_0 + f(u_0) \geq 0$ in D and $u_0 = 0$ on ∂D. By using delta-shaped basis functions and an iterative technique, we solve for the steady-state solution corresponding to the above problem on domains of a given shape. The delta-shaped basis functions are easy to construct and are easily extended to higher dimensions. They are very effective in overcoming the difficulty of meshing an irregular domain. An algorithm for determining the critical domain for domains of a given shape is designed and it is used for one dimensional (1-D) and 2-D quenching problems. The numerical results show that our proposed method can be considered as a unified approach for both regular and irregular domains. Furthermore, the technique can be used for higher dimensional problems.

Keywords - critical domain, quenching, delta-shaped basis functions, collocation method

1. INTRODUCTION

Let D be a bounded convex m-dimensional domain with a piecewise smooth boundary ∂D, $\bar{D}$ be the closure of D, $T\,(\leq \bullet\,)$ be an extended real number, $\Omega = D \times (0,T)$, and

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$S = \partial D \times (0,T)$. Let us consider the following singular semilinear problem,

$$\left.\begin{aligned} u_t - \Delta u = f(u) \text{ in } \Omega, \\ u(\mathbf{x},0) = u_0(\mathbf{x}) \text{ on } \bar{D}, \\ u(\mathbf{x},t) = 0 \text{ on } S, \end{aligned}\right\} \tag{1}$$

where f and u_0 are given functions such that $\lim_{u \to c^-} f(u) = \bullet$ for some positive constant c, $f(0) > 0$, $f'(u) \geq 0$, $f''(u) \geq 0$, $0 \leq u_0 < c$, $\Delta u_0 + f(u_0) \geq 0$ in D and $u_0 = 0$ on ∂D. In accordance with Kawarada [10], a solution u is said to quench if there exists a finite time T such that

$$\sup\{|u_t(\mathbf{x},t)| : \mathbf{x} \in \bar{D}\} \to \bullet \quad \text{as } t \to T^-. \tag{2}$$

The time T when (2) occurs is called the quenching time. When u is an increasing function of t, a necessary condition for (2) is

$$\max\{|u(\mathbf{x},t)| : \mathbf{x} \in \bar{D}\} \to c^- \text{ as } t \to T^-. \tag{3}$$

The spatial point where u reaches c is called a quenching point. Chan and Ke [5] showed that (3) implies (2), and hence the two conditions are equivalent.

A domain D^* is said to be a critical domain for the problem (1) if a solution u exists for all time $t > 0$ when $D \subset D^*$, and there exists a finite time T such that (3) (namely quenching) occurs when $D \supset D^*$. With $u_0 \equiv 0$, Acker and Kawohl [1] showed that within the family of nested domains, each of which has a sufficiently smooth boundary, the problem (1), without requiring $f(u)$ to be convex, has a critical domain, determined by the steady-state problem. Since domains of different shapes have different critical domains, we consider domains of the same shape in order to determine D^* uniquely. We note that once the shape is given, a domain is uniquely determined by its size (namely, area for $m = 2$, and volume for $m \geq 3$). Thus, determining the critical domain among domains of the same shape is equivalent to determining the critical size. With $u_0 \equiv 0$, Chan and Ke [5] showed that for domains of the same shape, a unique critical size for the problem (1) exists without requiring $f(u)$ to be convex in u and existence of f''.

Two spatial m-dimensional domains D and D_1 are said to have the same shape if there exists $\mathbf{x}_0 \in D \cap D_1$ and a positive constant λ such that

$$D_1 = \{\mathbf{y} : \mathbf{y} = \mathbf{x}_0 + \lambda(\mathbf{x} - \mathbf{x}_0) \text{ for } \mathbf{x} \in D\}. \tag{4}$$

Let the shape of a domain D be given. For a problem defined in a different domain D_1 having the same shape, we may transform it to a problem defined in D. Without loss of generality, we may take $\mathbf{x}_0$ to be the origin. Then, the problem (1) on a domain D_1 given by (4) with the initial data $u_0(\mathbf{x}) = 0$ may be transformed into that on D in the following form:

$$\left.\begin{aligned} \frac{\partial u_\lambda}{\partial t} - \Delta u_\lambda = \lambda^2 f(u_\lambda) \text{ in } \Omega, \\ u_\lambda = 0 \text{ on } \bar{D} \cup S, \end{aligned}\right\} \tag{5}$$

where

$$\lambda = \left(\frac{\text{size of } D_1}{\text{size of } D}\right)^{1/m} \tag{6}$$

(cf. Chan and Ke [5]). Let $F(U) = \lim_{t \to \bullet} f(u(x,t))$. The steady state of the problem (1) is given by

$$-\Delta U = F(U) \text{ in } D, U(\mathbf{x}) = 0 \text{ on } \partial D. \tag{7}$$

Hence, D^* is determined by the supremum of all domains D having the same shape such that the steady-state solution U exists. Let $\tilde{u}$ denote the solution of the problem (1) with $u_0 \equiv 0$. Then, $\tilde{u}$ is a lower solution of u. Since $\tilde{u}$ and u tend to the same steady-state problem (7) if U exists, it follows that U can be obtained by taking $u_0 \equiv 0$.

The concept of quenching was introduced by Kawarada [10] in 1975. He considered a special case of the one-dimensional (1-D) version of the problem (1) with $f(u) = (1-u)^{-1}$, and $D = (0,a)$ for some positive constant a. He showed that for the critical domain $(0,a^*)$, $a^* > 2\sqrt{2}$. Walter [14] showed that $a^* \in (1.5303, \pi/2)$. Acker and Walter [2] improved the above results by showing that $a^* = 1.530$ (to 4 significant figures). Chan and Chen [4] obtained $a^* = 1.5303$ (to 5 significant figures). For the multi-dimensional case, Chan [3] studied the critical domains for domains of the same shape. He developed numerical algorithms by using Green's function to convert the given problem to a nonlinear integral equation, and used the monotone method to find the minimal steady-state solution. The computational method is stable. Since the numerical procedures involve domain integrations, the computations are intensive for multi-dimensional cases. To avoid domain integrations, Chan and Chen [4] used the method of particular solutions, the multiquadric basis and the method of fundamental solutions. For illustration, they obtained exactly the same result as Chan's for the critical domain for elliptic plates of a given shape in much less computational time. By using a finite difference scheme, Chan and Ke [5] computed the critical domain for rectangles of the same shape.

Here, we adopt a unified approach to compute the critical domain D^* for domains of a given shape by using the delta-shaped basis functions. The delta-shaped basis functions are easy to construct, and are effective in approximating a source function given as a scattered data. In Section 2, we discuss delta-shaped basis functions and their use in a straight collocation method for solving partial differential equations (PDEs) of the elliptic type. A comparison is made between the solution by this method and that by the finite element method. The results show that our method is more accurate. Since higher dimensional delta-shaped basis functions can be easily obtained as the product of one dimensional ones, the approximation and the solution solving technique for PDEs can be extended to higher dimensions. In Section 3, we give a computational method, based on Section 2, to find the critical domains. For illustrations, we verify the critical domains for Kawarada's problem, for rectangular domains of a given shape, and for elliptic plates having the same shape. The algorithm is also applied to an irregular domain to show its flexibility.

2. DELTA-SHAPED BASIS FUNCTIONS AND A COLLOCATION METHOD

In this section, we give a description of delta-shaped basis functions. They are used to approximate the solution of a partial differential equation. For simplicity, let us first dwell on the 1-D case and assume that all the functions are defined in the interval $[-1,+1]$. Let $(\varphi_n(x),\mu_n)$ be a solution of the following Sturm-Liouville problem on the interval $[-1,+1]$,

$$-\varphi'' = \mu\varphi,\ \varphi(-1) = 0,\ \varphi(1) = 0.$$

Then, $\varphi_n(x) = \sin(n\pi(x+1)/2)$ and $\mu_n = (n\pi/2)^2$. The eigenfunctions $\varphi_n(x)$ form an orthogonal system on $[-1,+1]$.

Let the regularizing coefficients r_n be defined by

$$r_n = r_n(M,\chi) = \left[1-\left(\frac{n}{M+1}\right)^2\right]^\chi,$$

and let

$$c_n(\xi) = r_n\varphi_n(\xi).$$

It follows from Reutskiy [11] that the basis function is given by

$$I_{M,\chi}(x,\xi) = \sum_{n=1}^{M}\left[1-\left(\frac{n}{M+1}\right)^2\right]^\chi \varphi_n(\xi)\varphi_n(x) = \sum_{n=1}^{M} c_n(\xi)\varphi_n(x), \tag{8}$$

where χ and M are positive integers with χ playing the role of regularizing and M playing the role of scaling. We note that the basis function $I_{M,\chi}(x,\xi)$ satisfies the same boundary conditions as those of the eigenfunction $\{\varphi_n(x)\}$. For example,

$$I_{M,\chi}(\pm 1,\xi) = I_{M,\chi}(x,\pm 1) = 0.$$

The multi–dimensional delta-shaped basis functions can be obtained as products of the one-dimensional ones. For example, the 2-D delta-shaped basis functions can be defined as

$$I_{M,\chi}(\mathbf{x},\xi) = I_{M,\chi}(x_1,\xi_1)I_{M,\chi}(x_2,\xi_2),$$

where $\mathbf{x} = (x_1,x_2)$ and $\xi = (\xi_1,\xi_2)$. Using the 1-D basis functions given in (8), we get the 2-D basis functions $I_{M,\chi}(\mathbf{x},\xi)$. In Figure 1, we show the graphs of $I_{20,6}(\mathbf{x},\eta)$ and $I_{40,12}(\mathbf{x},\zeta)$ with centers at $\eta = (-0.25,-0.25)$ and $\zeta = (0.25,0.25)$ respectively. From the graphics, the basis functions $I_{M,\chi}(\mathbf{x},\xi)$ essentially differ from zero only inside some neighborhood of the center point ξ. They are infinitely differentiable and are not identically equal to zero on any interval. From this point of view they can be characterized as approximate compactly supported functions (cf. Chen, Golberg and Shaback [6]).

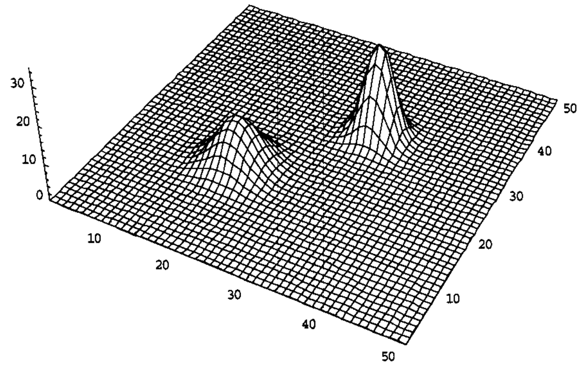

Figure 1: The functions $I_{20,6}(\mathbf{x},\xi)$ and $I_{40,12}(\mathbf{x},\xi)$ centered at $(-0.25,-0.25)$ and $(0.25,0.25)$ respectively

We remark that the regularizing technique described above can be applied to eigenfunctions of other Sturm-Liouville problems to construct basis functions.

Since the critical size of the problem (5) is determined by the value $\gamma = \lambda^2$, and the critical domain is determined by the supremum of all domains having the same shape such that the steady-state solution exists, we give a method for solving the steady-state problem (7). By the Rayleigh quotient (cf. Haberman [9, p. 290]), the fundamental eigenvalue σ of the Sturm-Liouville problem,

$$-\Delta U = \sigma U \text{ in } D, U(\mathbf{x}) = 0 \text{ on } \partial D,$$

is positive. Let us construct the sequence $\{U_n\}$ by letting $U_0 \equiv 0$, and for $n = 0,1,2,\ldots,$

$$-\Delta U_{n+1} = F(U_n) \text{ in } D, U_{n+1}(\mathbf{x}) = 0 \text{ on } \partial D. \tag{9}$$

It follows from Stakgold [12, pp. 581-582, and 612-617] that if

$$\limsup_{z\to\bullet} F(z)/z < \sigma, \tag{10}$$

and Green's function corresponding to the steady-state problem (7) exists, then $0 < U_1 < U_2 < \cdots < U_i < U_{i+1}$ for $i = 1,2,\ldots,$ and $\lim_{n\to\bullet} U_n$ is the minimal solution of the steady-state problem (7).

Assuming the above conditions are satisfied, we solve the sequence of linear PDEs defined by (9) until the successive solutions U_n and U_{n+1} are sufficiently close. At each iteration step of (9), the basis functions $I_{M,\chi}(\mathbf{x},\xi)$ can be used for solving PDEs (cf. Tian, Reutskiy and Chen [13]).

At the center points $\{\xi_j\}_{j=1}^{N}$, we use the basis functions $\{I_{M,\chi}(\mathbf{x},\xi_j)\}_{j=1}^{N}$. Let $\{\mathbf{x}_i\}_{i=1}^{N}$ be a set of collocation points for D of which $\{\mathbf{x}_i\}_{i=1}^{N_1}$ are interior points and $\{\mathbf{x}_i\}_{i=N_1+1}^{N}$ are boundary points. For each iteration step n of (9), we find an approximate solution $\widetilde{U}(\mathbf{x})$ in the form:

$$\widetilde{U}(\mathbf{x}) = \sum_{j=1}^{N} p_j I_{M,\chi}(\mathbf{x},\xi_j).$$

Using the governed equation and the boundary condition of (9) at the collocation points, we obtain the collocation system:

$$-\sum_{j=1}^{N} p_j \Delta_{(\mathbf{x})} I_{M,\chi}(\mathbf{x}_i,\xi_j) = F\left(U_n(\mathbf{x}_i)\right),\ 1 \le i \le N_1, \tag{11}$$

$$\sum_{j=1}^{N} p_j I_{M,\chi}(\mathbf{x}_i,\xi_j) = 0,\ N_1+1 \le i \le N, \tag{12}$$

where $\Delta_{(\mathbf{x})}$ denotes the Laplace operator acting on $I_{M,\chi}$ viewed as a function of the first argument. We also use the variant of the collocation when the number of centers K is less than that of collocation points N. In this case, we get an overdetermined system and the least squares method is used to solve it. Thus, the basis functions are just $\{-\Delta_{(\mathbf{x})} I_{M,\chi}(\mathbf{x},\xi_j)\}_{j=1}^{K}$ for interior points and $\{I_{M,\chi}(\mathbf{x},\xi_j)\}_{j=1}^{K}$ for boundary points according to (11) and (12).

To demonstrate the effectiveness of the proposed algorithm, we compare in Example 1 our method and the Poisson solver using the finite element method (FEM) in Matlab version 7.2.0.232. The results show that our method has a higher accuracy.

Example 1 *We use our basis function to solve the problem,*

$$\left.\begin{aligned} \Delta u = 1,\ (x,y) \in D = [0,1]\times[0,2], \\ u = 0,\ (x,y) \in \partial D. \end{aligned}\right\} \tag{13}$$

Its (exact) solution is given by

$$u_{ex}(x,y) = \sum_{n,m=1}^{\infty} U_{n,m} \sin(n\pi x) \sin\left(\frac{m}{2}\pi y\right). \tag{14}$$

Using the orthogonality conditions,

$$\int_0^1 \sin(n\pi x)\sin(k\pi x)\,dx = \begin{cases} \frac{1}{2}, & \textit{if } n = k, \\ 0, & \textit{if } n \neq k, \end{cases}$$

$$\int_0^2 \sin\left(\frac{m}{2}\pi y\right)\sin\left(\frac{k}{2}\pi y\right)dy = \begin{cases} 1, & \textit{if } m = k, \\ 0, & \textit{if } m \neq k, \end{cases}$$

we represent

$$1 = \sum_{n,m=1}^{\infty} e_{n,m} \sin(n\pi x)\sin\left(\frac{m}{2}\pi y\right),$$

where

$$e_{n,m} = \frac{4}{nm\pi^2}(\cos n\pi - 1)(\cos m\pi - 1).$$

We have

$$\Delta u = \sum_{n,m=1}^{\infty} e_{n,m} \sin(n\pi x) \sin\left(\frac{m}{2}\pi y\right), \ (x,y) \in D.$$

We look for the solution in the form (14). We get

$$U_{n,m} = -\frac{e_{n,m}}{\pi^2 [n^2 + m^2/4]}.$$

In order to use the collocation method, we use the 2-D basis functions defined by

$$I_{M,\chi}((x,y),(\xi,\eta)) = I^{(x)}_{M,\chi}(x,\xi) I^{(y)}_{M,\chi}(y,\eta), \tag{15}$$

where

$$\left.\begin{aligned} I^{(x)}_{M,\chi}(x,\xi) &= \sum_{n=1}^{M} \left[1 - \left(\frac{n}{M+1}\right)^2\right]^{\chi} \varphi_n(\xi)\varphi_n(x), \\ \varphi_n(x) &= \sin\left(\frac{n}{2}\pi(x+1)\right), \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} I^{(y)}_{M,\chi}(y,\eta) &= \sum_{n=1}^{M} \left[1 - \left(\frac{n}{M+1}\right)^2\right]^{\chi} \psi_n(\eta)\psi_n(y), \\ \psi_n(y) &= \sin\left(\frac{n}{4}\pi(y+2)\right). \end{aligned}\right\} \tag{17}$$

We remark that the basis functions $I^{(x)}_{M,\chi}(x,\xi)$ and $I^{(y)}_{M,\chi}(y,\eta)$ satisfy the zero boundary conditions $I^{(x)}_{M,\chi}(\pm 1,\xi) = 0$ and $I^{(y)}_{M,\chi}(\pm 2,\eta) = 0$, respectively. Therefore, the 2-D basis functions defined by (15)-(17) vanish on the boundary of the rectangle $D_2 = [-1,1] \times [-2,2]$. When using the basis to approximate any function, we should avoid placing the data points $\mathbf{x}_i$ and the centers ξ_j close to the boundary ∂D_2. In order to use the basis defined by (15)-(17) to approximate the solution of the problem (13), we transform the problem so that it is defined on $D_s = [-0.5,0.5] \times [-1,1]$. We choose 900 center points (ξ,η) randomly distributed inside the domain D_s and 150 center points (ξ,η) evenly distributed on ∂D_s. We would like to compare the accuracy of our method with that of the Poisson solver by FEM in Matlab. We use each method to obtain the numerical solution at the same set of nodes in D_s, that is, at the FEM nodes. To do this, we calculate the truncated series of (14),

$$u_{M_1,M_2} = \sum_{n,m=1}^{M_1,M_2} U_{n,m} \sin(n\pi x) \sin\left(\frac{m}{2}\pi y\right),$$

for some positive integers M_1 and M_2. Let $\widetilde{u}(x_i,y_i)$ denote the approximate solution at the node (x_i,y_i) in D_s by either FEM or our method. The absolute error is defined as

$$E_a = \max_{(x_i,y_i)} |\widetilde{u}(x_i,y_i) - u_{M_1,M_2}(x_i,y_i)|,$$

and the square root error is defined as

$$E_{sq} = \sqrt{\frac{\sum_{i=1}^{N_1} \left(\widetilde{u}(x_i,y_i) - u_{M_1,M_2}(x_i,y_i)\right)^2}{N_1}}.$$

The nodes where we calculate the numerical solution are the FEM nodes $\{(x_i,y_i)\}_{i=1}^{1317}$ *produced by the Matlab Poisson solver. These nodes are used as the collocation points in* D_s *for our method. We also choose 300 collocation points evenly distributed on the boundary* ∂D_s*. We give* E_a *and* E_{sq} *by our method and by the FEM method in Table 1.*

$M_1 = M_2$	E_a	E_{sq}	E_a by FEM	E_{sq} by FEM
150	.83626E-05	.74135E-06	.24179E-03	.42986E-04
200	.81395E-05	.70987E-06	.24176E-03	.42984E-04
300	.75895E-05	.70107E-06	.24173E-03	.42984E-04

Table 1: The absolute error E_a and the square root error E_{sq} by our method and by the FEM Poisson solver in Matlab for the problem (13)

The results show that our approach is more accurate than FEM Poisson solver in Matlab.

In the next example, we demonstrate that the collocation method work well for an irregular domain.

Example 2 *We consider the modified Helmholtz equation with the Dirichlet boundary condition,*

$$\left.\begin{array}{c} \Delta u(\mathbf{x}) - 10u(\mathbf{x}) = p(\mathbf{x}),\ \mathbf{x} \in D, \\ u(\mathbf{x}) = q(\mathbf{x}),\ \mathbf{x} \in \partial D, \end{array}\right\} \tag{18}$$

where D *is a star-shaped region. The parametric equation of the boundary curve* ∂D *is given by*

$$x_1 = 0.25\left(1+\cos^2 4t\right)\cos t,\ x_2 = 0.25\left(1+\cos^2 4t\right)\sin t\ ,\ t \in [0,2\pi).$$

Let us choose the functions $p(\mathbf{x})$ *and* $q(\mathbf{x})$ *such that the (exact) solution of (18) is Franke's function,*

$$F_1(\mathbf{x}) = \frac{3}{4}\exp\left(-\frac{(9x_1+2.5)^2+(9x_2+2.5)^2}{4}\right) + \frac{3}{4}\exp\left(-\frac{(9x_1+5.5)^2+(9x_2+5.5)^2}{49}\right)$$
$$+\frac{1}{2}\exp\left(-\frac{(9x_1-2.5)^2+(9x_2+1.5)^2}{4}\right) - \frac{1}{5}\exp\left(-(9x_1+0.5)^2-(9x_2-2.5)^2\right).$$

The graph of Franke's function is shown in Figure 2.

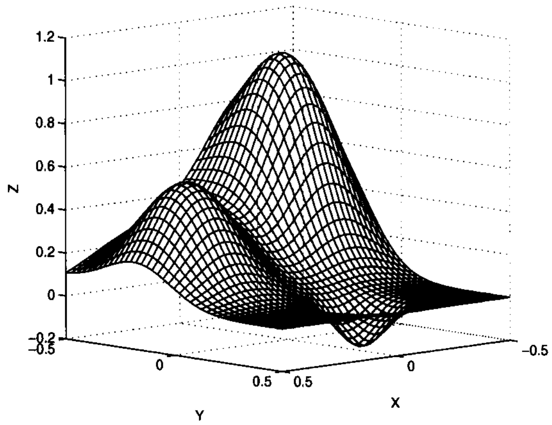

Figure 2: The graph of Franke's function $F_1(\mathbf{x})$

Franke's function is often used as a benchmark problem (cf. Fasshauer [7], and Franke [8]) in approximation. It was initially defined on the unit square $[0,1] \times [0,1]$. However for our purposes, the function is re-scaled to the domain $[-0.5,0.5] \times [-0.5,0.5]$. Since we are looking for the solution of the problem (18) in the star-shaped domain D shown in Figure 3, we only use the data values inside and on the boundary of the domain D.

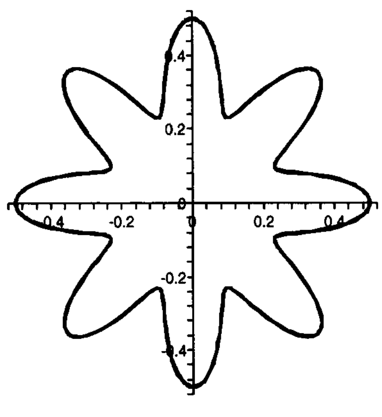

Figure 3: The star-shaped domain

We use K and N_I to denote respectively the number of center and collocation points in D. These points are randomly distributed inside the domain. We use N_b collocation points equally (in terms of angle) distributed on ∂D. The square root error of the computed solution is shown in Table 2. The result is highly accurate for such an oscillating solution over the star-shaped irregular domain.

$I_{M,\chi}$	K	N_1	N_b	E_{sq}
$I_{30,9}$	700	800	200	3.6E-05
$I_{40,12}$	700	800	200	1.2E-06
$I_{40,12}$	1200	1500	300	2.5E-07

Table 2: The square root error of the computed solution on a star-shaped domain

3. ALGORITHM FOR DETERMINING THE CRITICAL DOMAIN

In this section, we design an algorithm to compute the critical domain for the problem (5). As an illustration, we consider a 2-D problem with $f(u) = 1/(1-u)$ and $\lambda^2 = \gamma$:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = \frac{\gamma}{1-u} \text{ in } \Omega, \\ u = 0 \text{ on } \bar{D} \cup S. \end{aligned}\right\} \tag{19}$$

Its steady-state problem is given by

$$-\Delta u = \frac{\gamma}{1-u} \text{ in } D,\ u = 0 \text{ on } \partial D.$$

Since the size of the domain is determined by γ, we develop an algorithm to compute the critical value γ^* for domains having a given shape. The critical domain is then found by (6).

For the function $g(z) = \gamma/(1-z)$, where γ is a positive number, (10) is satisfied. For each of the problems including Kawarada's problem, the problem in a rectangular domain, and the problem in an elliptic plate, Green's function exists.

We use the following procedure to determine the critical size for domains of the same shape:

Step 1. We obtain an upper bound γ of γ^* by solving the problem

$$-\Delta\zeta = 1 \text{ in } D,\ \zeta = 0 \text{ on } \partial D.$$

Then, an initial upper bound for γ^* is $\gamma_{upper} = 1/\max_{\bar{D}} \zeta$. Using $\gamma_{lower} = 0$ as a lower bound, we estimate γ^* with $\gamma^{(1)} = \gamma_{upper}/2$.

Step 2. For each approximation $\gamma^{(k)}$, let us compute the sequence $\left\{U^{(i)}\right\}$ defined by $U^{(0)} = 0$ on $\bar{D}$, and for $i \geq 1$,

$$-\Delta U^{(i)} = \gamma^{(k)}/\left(1 - U^{(i-1)}\right) \text{ in } D, U^{(i)} = 0 \text{ on } \partial D. \tag{20}$$

We calculate $U^{(i)}$ from $U^{(i-1)}$ by the direct collocation method using the delta-shaped basis functions discussed in Section 2. In the iterative process, the linear systems resulted from our collocation method always have the same coefficient matrix. If $\left\{U^{(i)}\right\}$ converges, then we take this value $\gamma^{(k)}$ as a lower bound; otherwise, it is an upper bound.

Step 3. If $\left|\gamma^{(k)}-\gamma^{(k-1)}\right|<\varepsilon$ (a given tolerance), then $\gamma^{(k)}$ is taken to correspond to the critical size.

Step 4. We update $\gamma^{(k)}$ by the following criterion: if the sequence $\left\{U^{(i)}\right\}$ in Step 2 converges, then

$$\gamma^{(k)}=\gamma^{(k-1)}+\frac{1}{2}\left|\gamma^{(k-1)}-\gamma^{(k-2)}\right|;$$

otherwise

$$\gamma^{(k)}=\gamma^{(k-1)}-\frac{1}{2}\left|\gamma^{(k-1)}-\gamma^{(k-2)}\right|.$$

Then, we repeat Steps 2 - 4.

We use the above proposed procedure to calculate the critical size for domains of a given shape for the problem (19). The shapes considered in our numerical examples include a 1-D domain (Kawarada's problem), a 2-D rectangular domain, a 2-D elliptic plate, and an irregular domain. For all of the numerical examples in this section, we use K_1, K_b, to denote the number of center points of the basis functions inside D and on ∂D respectively, and we use N_1, N_b to denote the number of collocation points inside D and on ∂D respectively. So, the total number of center points of the basis functions is $K=K_1+K_b$, and that of the collocation points is $N=N_1+N_b$.

Example 3 *For the 1-D problem, $D=\{x\in R:-a<x<a\}$, where a is a positive number. We choose $a=0.5$. We let $K_1=50$ and $N_1=100$. The type of 1-D basis function used is $I_{30,9}$ defined in (8). The center and collocation points used inside D are quasi-random points. For the 1-D case, we have 2 center points on ∂D with one being -0.5 and the other 0.5. So $K_b=2$. Similarly, $N_b=2$. The total number of center points and collocation points are respectively $K=52$ and $N=102$. An initial upper bound for γ^* is 8.0000. Our computed critical value $\gamma^*=2.3421$. By (6) the critical size of the interval is $\sqrt{\gamma^*}\cdot$length of $D=1.530$ (to three decimal points).*

Now, we consider 2-D problems.

Example 4 *The domain D is a rectangle defined by*

$$D=\{(x,y):-a<x<a,\ -b<y<b\},$$

where a and b are two positive numbers. The ratio a/b determines the shape of the rectangular domain. We let $a=0.5$ and $b=1$. So $a/b=1/2$. The type of basis function used is $I_{30,9}$ defined in (15)-(17). We use $K_1=900$ quasi-random center points inside D and $K_b=150$ center points evenly distributed on ∂D. The total number of center points is $K=1050$. We use the same interior collocation points as in Example 1 plus an additional point $(0,0)$ since at each iterative step (20) (when the iteration converges) the maximum of

$U^{(i)}$ is attained there. So $N_1 = 1318$. We choose $N_b = 300$ collocation points evenly distributed on ∂D. So the total collocation points $N = 1618$. An upper bound for γ^* is 8.7818. The computed critical value $\gamma^* = 2.7993$. By (6), the critical size of the rectangular domain is $\gamma^* \cdot$Area of $D = 5.599$ (to three decimal points) as that given by Chan and Ke [5].

Example 5 *The region D is an elliptic plate defined by*

$$D = \left\{ (x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\},$$

where $a/b = \left(1 + e^{-\pi/2}\right) / \left(1 - e^{-\pi/2}\right)$. Let $a = 0.4575$. We have $b = 0.3000$. The type of basis function used is $I_{30,9}(x,\xi)I_{30,9}(y,\eta)$, where $I_{30,9}(x,\xi)$ and $I_{30,9}(y,\eta)$ are defined by (8). We use $K_1 = 500$ quasi-random center points inside D and $K_b = 150$ center points evenly (in terms of angle) distributed on ∂D. The total number of center points is $K = 650$. We use $N_1 = 1000$ collocation points that are also quasi-random inside D and $N_b = 300$ collocation points evenly (in terms of angle) distributed on ∂D. So $N = 1300$. An initial upper bound for γ^* is 31.7792. Our computed critical value $\gamma^* = 10.3447$. By (6), the critical size of the elliptic plate is $\gamma^* \cdot$Area of $D = 4.460$, which is the same as that by Chan [3].

Our next example involves an irregular domain.

Example 6 *The region D is bounded by the curve $\rho = \left(1 + \cos^2(\theta)\right)/4, 0 \leq \theta \leq 2\pi$. The type of basis function used is the same as in Example 5. We use $K_1 = 500$ center points and $N_1 = 1000$ collocation points that are quasi-random inside D. We distribute $K_b = 150$ center points and $N_b = 300$ collocation points evenly (in terms of angle) on ∂D. So $K = 650$ and $N = 1300$. An initial upper bound for the γ^* is 33.7879. Our computed critical value $\gamma^* = 10.8332$. The critical domain is $\gamma^* \cdot$Area of $D = 10.8332 * .4663 = 5.052$ (to three decimal points).*

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