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Improved Monotone Iterative Techniques for Hyperbolic Initial-Boundary Value Problems

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Abstract

The methods of lower-upper solutions and coupled lower-upper solutions together with the monotone iterative technique are employed to obtain natural monotone, or alternating monotone, or intertwining monotone sequences for hyperbolic initial-boundary value problems when the forcing function is the sum of two monotone functions. The sequences converge uniformly to the extremal solutions or the coupled extremal solutions of the nonlinear initial-boundary value problems.

1. INTRODUCTION

It is well known [3] that the method of lower-upper solutions, coupled with the monotone iterative technique (MIT for short) offers theoretical as well as constructive existence results for a variety of nonlinear problems in a closed set generated by lower and upper solutions. A unification of several known results and some new ones can be accomplished if we split the forcing function into the sum of two monotone functions. In this situation, one can employ a variety of coupled lower-upper solutions, and discuss the MIT depending on the iterative scheme used. See [1, 4, 11, 13, 14]. In this paper, we undertake such a unified study for the initial-boundary value problem associated with the nonlinear hyperbolic partial differential equation

$$u_{xy} = f(x, y, \langle u\rangle) + g(x, y, \langle u\rangle), \ \langle u\rangle = (u, u_x, u_y). \tag{1.1}$$

Utilizing various iterative schemes, we obtain natural monotone, alternating monotone, or intertwining monotone sequences which converge uniformly to the extremal or the coupled extremal solutions of (1.1) when f is montonically nondecreasing and g is monotonically nonincreasing in the last three variables. Furthermore, if f and g satisfy appropriate uniqueness (one-sided Lipschitz) conditions, then the sequences in question converge to the unique solution of equation (1.1). Our results include and improve the earlier results obtained in [1, 2, 3, 5, 6, 7, 8, 10, 11, 12]. To avoid monotony, we give a proof and provide an illustrative example for only one of our results.

2. PRELIMINARIES

For a, $b \in \mathbb{R}$, $a > 0$, $b > 0$, let I, J, and R denote the *intervals* $[0, a]$, $[0, b]$ and the *rectangle* $[0, a] \times [0, b]$ respectively. By $z \in C^2[R, \mathbb{R}]$, we mean that z is a continuous function on R, and its partial derivatives z_x, z_y, and z_{xy} exist and are continuous on R. For $z \in C^2[R, \mathbb{R}]$, the triple (z, z_x, z_y) is denoted by $\langle z\rangle$.

The triple (f_1, f_2, f_3) of constants or functions is denoted by $\{f\}$. The expression $f_1z + f_2z_x + f_3z_y$ denotes the usual inner product $\{f\} \cdot \langle z\rangle$. For v, $w \in C^2[R, \mathbb{R}]$, the inequality $\langle v\rangle \le \langle w\rangle$ means that $v(x, y) \le w(x, y)$, $v_x(x, y) \le w_x(x, y)$, and $v_y(x, y) \le w_y(x, y)$ for $(x, y) \in R$. For v^0, $w^0 \in C^2[R, \mathbb{R}]$ such that $\langle v^0\rangle \le \langle w^0\rangle$ on R, the closed set Ω is defined by

$$\Omega = \left\{(x, y, z, p, q) \ : \ (x, y) \in R, \langle v^0\rangle \le (z, p, q) \le \langle w^0\rangle \text{ on } R\right\}.$$

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Under these notations, consider the initial-boundary value problem (IBVP for short)

$$\begin{aligned} u_{xy} &= f(x,y,\langle u\rangle) + g(x,y,\langle u\rangle),\ (x,y)\in R;\\ u(x,0) = \sigma(x) \text{ for } x\in I,\ &u(0,y)=\tau(y) \text{ for } y\in J,\ \sigma(0)=u_0=\tau(0), \end{aligned} \tag{2.1}$$

where $f,\ g \in C^2[R\times\mathbb{R}^3,\mathbb{R}]$, $\sigma\in C^1[I,\mathbb{R}]$, and $\tau\in C^1[J,\mathbb{R}]$.

There are four possible ways to employ the lower and the upper solutions in the development of monotone methods.

Definition 2.1. Functions $v^0,\ w^0 \in C^2[R,\mathbb{R}]$, $\langle v^0\rangle \le \langle w^0\rangle$, are said to be *natural lower-upper solutions* relative to (2.1) if

$$\begin{aligned} &v^0_{xy} \le f(x,y,\langle v^0\rangle) + g(x,y,\langle v^0\rangle),\ (x,y)\in R;\\ &w^0_{xy} \ge f(x,y,\langle w^0\rangle) + g(x,y,\langle w^0\rangle),\ (x,y)\in R;\\ &v^0_x(x,0)\le\sigma'(x),\ x\in I;\ v^0_y(0,y)\le\tau'(y),\ y\in J;\ v^0(0,0)\le u_0;\\ &w^0_x(x,0)\ge\sigma'(x),\ x\in I;\ w^0_y(0,y)\ge\tau'(y),\ y\in J;\ w^0(0,0)\ge u_0. \end{aligned} \tag{2.2}$$

Definition 2.2. Functions $v^0,\ w^0 \in C^2[R,\mathbb{R}]$, $\langle v^0\rangle \le \langle w^0\rangle$, are said to be *coupled lower-upper solutions of Type I* relative to (2.1) if

$$\begin{aligned} &v^0_{xy} \le f(x,y,\langle v^0\rangle) + g(x,y,\langle w^0\rangle),\ (x,y)\in R;\\ &w^0_{xy} \ge f(x,y,\langle w^0\rangle) + g(x,y,\langle v^0\rangle),\ (x,y)\in R;\\ &v^0_x(x,0)\le\sigma'(x),\ x\in I;\ v^0_y(0,y)\le\tau'(y),\ y\in J;\ v^0(0,0)\le u_0;\\ &w^0_x(x,0)\ge\sigma'(x),\ x\in I;\ w^0_y(0,y)\ge\tau'(y),\ y\in J;\ w^0(0,0)\ge u_0. \end{aligned} \tag{2.3}$$

Definition 2.3. Functions $v^0,\ w^0 \in C^2[R,\mathbb{R}]$, $\langle v^0\rangle \le \langle w^0\rangle$, are said to be *coupled lower-upper solutions of Type II* relative to (2.1) if

$$\begin{aligned} &v^0_{xy} \le f(x,y,\langle w^0\rangle) + g(x,y,\langle v^0\rangle),\ (x,y)\in R;\\ &w^0_{xy} \ge f(x,y,\langle v^0\rangle) + g(x,y,\langle w^0\rangle),\ (x,y)\in R;\\ &v^0_x(x,0)\le\sigma'(x),\ x\in I;\ v^0_y(0,y)\le\tau'(y),\ y\in J;\ v^0(0,0)\le u_0;\\ &w^0_x(x,0)\ge\sigma'(x),\ x\in I;\ w^0_y(0,y)\ge\tau'(y),\ y\in J;\ w^0(0,0)\ge u_0. \end{aligned} \tag{2.4}$$

Definition 2.4. Functions $v^0,\ w^0 \in C^2[R,\mathbb{R}]$, $\langle v^0\rangle \le \langle w^0\rangle$, are said to be *coupled lower-upper solutions of Type III* relative to (2.1) if

$$\begin{aligned} &v^0_{xy} \le f(x,y,\langle w^0\rangle) + g(x,y,\langle w^0\rangle),\ (x,y)\in R;\\ &w^0_{xy} \ge f(x,y,\langle v^0\rangle) + g(x,y,\langle v^0\rangle),\ (x,y)\in R;\\ &v^0_x(x,0)\le\sigma'(x),\ x\in I;\ v^0_y(0,y)\le\tau'(y),\ y\in J;\ v^0(0,0)\le u_0;\\ &w^0_x(x,0)\ge\sigma'(x),\ x\in I;\ w^0_y(0,y)\ge\tau'(y),\ y\in J;\ w^0(0,0)\ge u_0. \end{aligned} \tag{2.5}$$

Remark 2.1. Whenever $\langle v^0\rangle \le \langle w^0\rangle$, f is *nondecreasing* in $\langle u\rangle$, and g is *nonincreasing* in $\langle u\rangle$, it is easy to see that the inequalities in (2.2) and (2.5) imply the inequalities in (2.4). Furthermore, in this case, coupled lower-upper solutions of Type II can easily be computed. Indeed we have (see [1]):

Lemma 2.1. *Suppose that $f,\ g\in C[R\times\mathbb{R}^3,\mathbb{R}]$, f is nondecreasing and g is nonincreasing in the last three variables. Then there exist coupled lower-upper solutions v^0, w^0 of Type II for* (2.1) *such that $\langle v^0\rangle \le \langle w^0\rangle$ on R.*

The following known results [1, 9] are required in our subsequent discussion.

Theorem 2.1. *Let v, $w \in C^2[R,\mathbb{R}]$ and $H(x,y,z,p,q,\overline{z},\overline{p},\overline{q}) \in C[R \times \mathbb{R}^6,\mathbb{R}]$ be nondecreasing in z, p, q and nonincreasing in $\overline{z}$, $\overline{p}$, $\overline{q}$. Suppose that any one of the following conditions holds.*

(H$_1$) $v_{xy} \le H(x,y,\langle v\rangle,\langle v\rangle)$, $w_{xy} \ge H(x,y,\langle w\rangle,\langle w\rangle)$, *and there exists $L>0$ such that*

$$H(x,y,z_1,p_1,q_1,\overline{z}_1,\overline{p}_1,\overline{q}_1) - H(x,y,z_2,p_2,q_2,\overline{z}_2,\overline{p}_2,\overline{q}_2) \le L[(z_1-z_2)+(p_1-p_2)+(q_1-q_2)+(\overline{z}_1-\overline{z}_2)+(\overline{p}_1-\overline{p}_2)+(\overline{q}_1-\overline{q}_2)]$$

whenever $z_1 \ge z_2$, $p_1 \ge p_2$, $q_1 \ge q_2$, $\overline{z}_1 \ge \overline{z}_2$, $\overline{p}_1 \ge \overline{p}_2$, $\overline{q}_1 \ge \overline{q}_2$;

(H$_2$) $v_{xy} \le H(x,y,\langle v\rangle,\langle w\rangle)$, $w_{xy} \ge H(x,y,\langle w\rangle,\langle v\rangle)$, *and there exists $L>0$ such that*

$$H(x,y,z_1,p_1,q_1,\overline{z},\overline{p},\overline{q}) - H(x,y,z_2,p_2,q_2,\overline{z},\overline{p},\overline{q}) \le L[(z_1-z_2)+(p_1-p_2)+(q_1-q_2)]$$

whenever $z_1 \ge z_2$, $p_1 \ge p_2$, $q_1 \ge q_2$, and

$$H(x,y,z,p,q,\overline{z}_1,\overline{p}_1,\overline{q}_1) - H(x,y,z,p,q,\overline{z}_2,\overline{p}_2,\overline{q}_2) \ge - L[(\overline{z}_1-\overline{z}_2)+(\overline{p}_1-\overline{p}_2)+(\overline{q}_1-\overline{q}_2)]$$

whenever $\overline{z}_1 \ge \overline{z}_2$, $\overline{p}_1 \ge \overline{p}_2$, $\overline{q}_1 \ge \overline{q}_2$;

(H$_3$) $v_{xy} \le H(x,y,\langle w\rangle,\langle v\rangle)$, $w_{xy} \ge H(x,y,\langle v\rangle,\langle w\rangle)$, *and there exists $L>0$ such that*

$$H(x,y,z,p,q,\overline{z}_1,\overline{p}_1,\overline{q}_1) - H(x,y,z,p,q,\overline{z}_2,\overline{p}_2,\overline{q}_2) \le L[(\overline{z}_1-\overline{z}_2)+(\overline{p}_1-\overline{p}_2)+(\overline{q}_1-\overline{q}_2)]$$

whenever $\overline{z}_1 \ge \overline{z}_2$, $\overline{p}_1 \ge \overline{p}_2$, $\overline{q}_1 \ge \overline{q}_2$, and

$$H(x,y,z_1,p_1,q_1,\overline{z},\overline{p},\overline{q}) - H(x,y,z_2,p_2,q_2,\overline{z},\overline{p},\overline{q}) \ge - L[(z_1-z_2)+(p_1-p_2)+(q_1-q_2)],$$

whenever $z_1 \ge z_2$, $p_1 \ge p_2$, $q_1 \ge q_2$;

(H$_4$) $v_{xy} \le H(x,y,\langle w\rangle,\langle w\rangle)$, $w_{xy} \ge H(x,y,\langle v\rangle,\langle v\rangle)$, *and there exists $L>0$ such that*

$$H(x,y,z_1,p_1,q_1,\overline{z}_1,\overline{p}_1,\overline{q}_1) - H(x,y,z_2,p_2,q_2,\overline{z}_2,\overline{p}_2,\overline{q}_2) \ge -L[(z_1-z_2)+(p_1-p_2)+(q_1-q_2)+(\overline{z}_1-\overline{z}_2)+(\overline{p}_1-\overline{p}_2)+(\overline{q}_1-\overline{q}_2)],$$

whenever $z_1 \ge z_2$, $p_1 \ge p_2$, $q_1 \ge q_2$, $\overline{z}_1 \ge \overline{z}_2$, $\overline{p}_1 \ge \overline{p}_2$, $\overline{q}_1 \ge \overline{q}_2$.

If $v(0,0) \le w(0,0)$, $v_x(x,0) \le w_x(x,0)$, and $v_y(0,y) \le w_y(0,y)$ for $x \in I$ and $y \in J$, then

$$\langle v\rangle \le \langle w\rangle \text{ everywhere in } R. \tag{2.6}$$

Remark 2.2. The one-sided Lipschitz conditions in Theorem 2.1 are required in establishing the nonstrict inequalities only. The following example shows that the conclusion of Theorem 2.1 may be false in the absence of these conditions.

Example 2.1. For $(x,y) \in [0,1] \times [0,1]$, consider the equation

$$u_{xy} = f(x,y,u) = \begin{cases} xy\sqrt[3]{u}, & \text{if } 0 \le u \le 1 \\ 0, & \text{if } u < 0 \\ xy, & \text{if } u > 1 \end{cases} \tag{2.7}$$
$$u(x,0)=0,\ x \in [0,1], \quad u(0,y)=0,\ y \in [0,1].$$

The functions $v(x,y) \equiv 0$ and $w(x,y) = \dfrac{x^3y^3}{27}$ are both solutions of (2.7) which do not satisfy the inequalities in (2.6).

Theorem 2.2. *Suppose that*

$$u_{xy} \le M_1 u + M_2 u_x + M_3 u_y, \quad (x,y) \in R,$$

where M_1, M_2, M_3 are constants and $M_2 \ge 0$, $M_3 \ge 0$. If either

(a) (i) $u(x,0) \le 0$ *for* $x \in I$, $u(0,y) \le 0$ *for* $y \in J$;

(ii) $u_x(x,0) - M_3 u(x,0) \le 0$ *for* $x \in I$,
$u_y(0,y) - M_2 u(0,y) \le 0$ *for* $y \in J$;

(iii) $M_1 + M_2 M_3 \ge 0$;

or

(b) (i)′ $u_x(x,0) \le 0$ *for* $x \in I$, $u_y(0,y) \le 0$ *for* $y \in J$, *and* $u(0,0) = 0$;

(ii)′ $M_1 + M_2 M_3 = 0$;

then $(u, u_x, u_y) \le (0,0,0)$ *everywhere on* R.

For a proof of Theorem 2.2 and several other related results of interest, see [4, 9]. We recall the following result on MIT which is an extension of Theorem 4.1 in [5]. Its proof makes repeated use of Theorem 2.2. See also Remarks 4.3.1 following Theorem 4.3.1 in [4].

Consider the IBVP

$$\begin{gathered} u_{xy} = F(x,y,\langle u \rangle), \quad (x,y) \in R; \\ u(x,0) = \sigma(x) \text{ for } x \in I,\ u(0,y) = \tau(y) \text{ for } y \in J,\ \sigma(0) = u_0 = \tau(0), \end{gathered} \tag{2.8}$$

where $F \in C[R \times \mathbb{R}^3, \mathbb{R}]$, $\sigma \in C^1[I, \mathbb{R}]$, and $\tau \in C^1[J, \mathbb{R}]$.

Theorem 2.3. *Assume that*

(i) v^0, $w^0 \in C^2[R, \mathbb{R}]$, $\langle v^0 \rangle \le \langle w^0 \rangle$ *on* R *and* v^0, w^0 *are respectively the natural lower and upper solutions of* (2.8) *such that* $v^0(0,0) = u_0 = w^0(0,0)$;

(ii) F *satisfies the condition*

$$F(x,y,z,p,q) - F(x,y,\bar{z},\bar{p},\bar{q}) \ge -M_1(z-\bar{z}) - M_2(p-\bar{p}) - M_3(q-\bar{q})$$

whenever $\langle v^0 \rangle \le (\bar{z},\bar{p},\bar{q}) \le (z,p,q) \le \langle w^0 \rangle$ *on* R, *where* M_1, M_2, M_3 *are constants with* $M_2 \le 0$ *and* $M_3 \le 0$.

Then there exist monotone sequences $\{v^n\}$ *and* $\{w^n\}$ *such that* $\lim_n \langle v^n \rangle = \langle \rho \rangle$, $\lim_n \langle w^n \rangle = \langle r \rangle$ *uniformly and monotonically on* R, *where* ρ *and* r *are the minimal and maximal solutions respectively of* (2.8) *and satisfy* $\langle v^0 \rangle \le \langle \rho \rangle \le \langle r \rangle \le \langle w^0 \rangle$ *on* R.

The special case where $F(x,y,\langle u \rangle)$ is nondecreasing in $\langle u \rangle$ (in which case we do not require the restriction $v^0(0,0) = u_0 = w^0(0,0)$) is covered by Theorem 2.3. However, the case when $F(x,y,\langle u \rangle)$ is nonincreasing in $\langle u \rangle$ is not included in Theorem 2.3 and is of special interest. In [1] (Theorem 3.4) the authors have employed coupled lower-upper solutions of Type II to develop MIT for the IBVP (2.1) under the additional assumptions "$\langle v^0 \rangle \le \langle v^2 \rangle$ and $\langle w^2 \rangle \le \langle w^0 \rangle$ on R."

A natural question that arises is whether it is possible to obtain monotone sequences $\{v^n\}$, $\{w^n\}$ when F is nonincreasing in $\langle u \rangle$ without the restrictions $\langle v^0 \rangle \le \langle v^2 \rangle$ and $\langle w^2 \rangle \le \langle w^0 \rangle$. The answer is in the affirmative provided we employ coupled lower-upper solutions. In the next section we prove very general results relative to MIT, which include the results of this section as well as several other results of interest.

3. MAIN RESULTS

In the development of MITs relative to the IBVP (2.1), we shall employ the following two iterative schemes:

$$\begin{aligned} v^n_{xy}(x,y) &= f(x,y,\langle v^{n-1}(x,y)\rangle) + g(x,y,\langle w^{n-1}(x,y)\rangle), \quad (x,y)\in R;\\ v^n(x,0) &= \sigma(x),\ x\in I;\ v^n(0,y)=\tau(y),\ y\in J;\ v^n(0,0)=u_0;\\ w^n_{xy}(x,y) &= f(x,y,\langle w^{n-1}(x,y)\rangle) + g(x,y,\langle v^{n-1}(x,y)\rangle), \quad (x,y)\in R;\\ w^n(x,0) &= \sigma(x),\ x\in I;\ w^n(0,y)=\tau(y),\ y\in J;\ w^n(0,0)=u_0, \end{aligned} \tag{S$_1$}$$

and

$$\begin{aligned} v^n_{xy}(x,y) &= f(x,y,\langle w^{n-1}(x,y)\rangle) + g(x,y,\langle v^{n-1}(x,y)\rangle), \quad (x,y)\in R;\\ v^n(x,0) &= \sigma(x),\ x\in I;\ v^n(0,y)=\tau(y),\ y\in J;\ v^n(0,0)=u_0;\\ w^n_{xy}(x,y) &= f(x,y,\langle v^{n-1}(x,y)\rangle) + g(x,y,\langle w^{n-1}(x,y)\rangle), \quad (x,y)\in R;\\ w^n(x,0) &= \sigma(x),\ x\in I;\ w^n(0,y)=\tau(y),\ y\in J;\ w^n(0,0)=u_0, \end{aligned} \tag{S$_2$}$$

for $n = 1, 2, 3, \cdots$.

Our first result, which yields *natural* sequences by utilizing natural upper-lower solutions in conjunction with the scheme (S$_1$), includes and improves several earlier known results in [3, 5, 6, 7, 8, 10, 12].

Theorem 3.1. *Assume that*

(A$_1$) *v^0, $w^0 \in C^2[R,\mathbb{R}]$ are natural upper-lower solutions of (2.1) such that $\langle v^0\rangle \le \langle w^0\rangle$ on R;*

(A$_2$) *f, $g \in C[R\times\mathbb{R}^3,\mathbb{R}]$, f is nondecreasing in $\langle u\rangle$ and g is nonincreasing in $\langle u\rangle$.*

Then there exist sequences $\{v^n\}$, $\{w^n\}$ in Ω such that $\{\langle v^n\rangle\}$ is nonincreasing, $\{\langle w^n\rangle\}$ is nonincreasing and satisfy $\langle v^n\rangle \to \langle v\rangle$, $\langle w^n\rangle \to \langle w\rangle$, where v and w are coupled minimal and maximal solutions respectively of (2.1) on R, that is, v and w satisfy

$$\begin{aligned} v_{xy} &= f(x,y,\langle v\rangle) + g(x,y,\langle w\rangle),\ (x,y)\in R,\\ w_{xy} &= f(x,y,\langle w\rangle) + g(x,y,\langle v\rangle),\ (x,y)\in R, \end{aligned}$$

provided that $\langle v^0\rangle \le \langle v^1\rangle$ and $\langle w^1\rangle \le \langle w^0\rangle$ on R. Also $\langle v^0\rangle \le \langle v\rangle \le \langle w\rangle \le \langle w^0\rangle$ on R.

Proof. For $n = 1, 2, 3, \cdots$, define the iterates as given by scheme (S$_1$). It is easy to see that the solutions of these IBVPs exist and are unique for each $n = 1, 2, 3, \cdots$. We prove that the sequences $\{v^n\}$ and $\{w^n\}$ satisfy the (natural) monotone behavior

$$\langle v^0\rangle \le \langle v^1\rangle \le \cdots \le \langle v^n\rangle \le \langle w^n\rangle \le \cdots \le \langle w^1\rangle \le \langle w^0\rangle \text{ on } R. \tag{3.1}$$

By assumption, we already have $\langle v^0\rangle \le \langle v^1\rangle$ and $\langle w^1\rangle \le \langle w^0\rangle$ on R. We assert that

$$\langle v^1\rangle \le \langle w^1\rangle \text{ on } R. \tag{3.2}$$

To this end, setting $p = v^1 - w^1$ we note that $p_x(x,0) = 0$ if $x\in I$, $p_y(0,y) = 0$ if $y\in J$, $p(0,0) = 0$ and $p_{xy} = f(\langle v^0\rangle) + g(\langle w^0\rangle) - f(\langle w^0\rangle) - g(\langle v^0\rangle) \le 0$ in view of the monotone character of $f(\langle u\rangle)$, $g(\langle u\rangle)$ and the assumption $\langle v^0\rangle \le \langle w^0\rangle$. Hence Theorem 2.2 yields $\langle p\rangle \le \langle 0\rangle$, which in turn establishes (3.2). Thus we have $\langle v^0\rangle \le \langle v^1\rangle \le \langle w^1\rangle \le \langle w^0\rangle$ on R. Assume that, for some $n > 1$,

$$\langle v^{n-1}\rangle \le \langle v^n\rangle \le \langle w^n\rangle \le \langle w^{n-1}\rangle \text{ on } R. \tag{3.3}$$

We shall prove that

$$\langle v^n\rangle \le \langle v^{n+1}\rangle \le \langle w^{n+1}\rangle \le \langle w^n\rangle \text{ on } R. \tag{3.4}$$

To do this, let $p = v^n - w^{n+1}$, so that $p_x(x,0) = 0$ if $x\in I$, $p_y(0,y) = 0$ if $y\in J$, and $p(0,0) = 0$. Then $p_{xy} = f(\langle v^{n-1}\rangle) + g(\langle v^{n-1}\rangle) - f(\langle v^n\rangle) - g(\langle v^n\rangle) \le 0$ because of (3.3) and the monotonicity of $f(\langle u\rangle)$, $g(\langle u\rangle)$. Therefore we have $\langle v^n\rangle \le \langle v^{n+1}\rangle$ on R by Theorem 2.2. A similar argument yields $\langle w^{n+1}\rangle \le \langle w^n\rangle$ on R. Now letting $p = v^{n+1} - w^{n+1}$ we find that $p_x(x,0) = 0$ if $x \in I$, $p_y(0,y) = 0$ if $y \in J$, $p(0,0) = 0$, and $p_{xy} = f(\langle v^n\rangle) + g(\langle w^n\rangle) - f(\langle w^n\rangle) - g(\langle v^n\rangle) \le 0$, in view of (3.3) and the monotone nature of $f(\langle u\rangle)$, $g(\langle u\rangle)$. Hence by Theorem 2.2 we obtain $\langle v^{n+1}\rangle \le \langle w^{n+1}\rangle$ on R, proving (3.4). Hence by induction (3.1) is established. Using the monotone character of the sequences $\{v^n\}$, $\{w^n\}$ in Ω, together with the Ascoli-Arzelà theorem, it follows by using standard arguments that $\langle v^n\rangle \to \langle v\rangle$, $\langle w^n\rangle \to \langle w\rangle$, where v, $w \in C^2[R,\mathbb{R}]$ are coupled solutions of the IBVP (2.1). To show that v, w are in fact the coupled extremal solutions of (2.1), let $u \in \Omega$ be any solution of (2.1). Assume for some $n > 0$, that we have

$$\langle v^n\rangle \le \langle u\rangle \le \langle w^n\rangle \text{ on } R. \tag{3.5}$$

Then letting $p = v^{n+1} - u$, we note that $p_x(x,0) = 0$ if $x \in I$, $p_y(0,y) = 0$ if $y \in J$, $p(0,0) = 0$ and $p_{xy} = f(\langle v^n\rangle) + g(\langle w^n\rangle) - f(\langle u\rangle) - g(\langle u\rangle) \le 0$ because of the monotonicity of $f(\langle u\rangle)$, $g(\langle u\rangle)$, and (3.5). Theorem 2.2 now implies that $\langle v^n\rangle \le \langle u\rangle$ on R. Similarly, $\langle u\rangle \le \langle w^n\rangle$ on R. It therefore follows by induction that $\langle v^n\rangle \le \langle u\rangle \le \langle w^n\rangle$ on R for all $n = 1,2,3,\cdots$, and this implies, in turn, that $\langle v\rangle \le \langle u\rangle \le \langle w\rangle$ on R, proving thereby that v and w are coupled minimal and maximal solutions respectively of the IBVP (2.1) on R. This completes the proof. □

Corollary 3.1. *In addition to the assumptions of Theorem 3.1 suppose that f, g satisfy*

$$\begin{aligned} f(\langle u_1\rangle) - f(\langle u_2\rangle) &\le \ [L]\cdot\langle u_1 - u_2\rangle\,, \\ g(\langle u_1\rangle) - g(\langle u_2\rangle) &\ge -[L]\cdot\langle u_1 - u_2\rangle\,,\ L > 0 \text{ a constant}, \end{aligned}$$

whenever $\langle v^0\rangle \le \langle u_1\rangle \le \langle u_2\rangle \le \langle w^0\rangle$ on R, where $[L]\cdot\langle u\rangle = L(u + u_x + u_y)$. Then $v \equiv u \equiv w$ is the unique solution of (2.1) *in Ω.*

Proof. Since $\langle v\rangle \le \langle w\rangle$ it suffices to show that $\langle w\rangle \le \langle v\rangle$ on R. To this end, setting $p = w - v$, we find that $p_x(x,0) \equiv 0$ on I, $p_y(0,y) \equiv 0$ on J, $p(0,0) = 0$, and

$$\begin{aligned} p_{xy} &= f(\langle w\rangle) + g(\langle v\rangle) - f(\langle v\rangle) - g(\langle w\rangle) \\ &\le [2L]\cdot\langle p\rangle\,. \end{aligned}$$

Theorem 2.1 with (H_2) now implies $\langle p\rangle \le \langle 0\rangle$, completing the proof. □

Several special cases of interest can be deduced from Theorem 3.1. First let us make the following observation.

Consider the IBVP

$$\begin{gathered} u_{xy} = f(x,y,\langle u\rangle),\ (x,y) \in R; \\ u(x,0) = \sigma(x) \text{ for } x \in I,\ u(0,y) = \tau(y) \text{ for } y \in J,\ \sigma(0) = u_0 = \tau(0), \end{gathered} \tag{3.6}$$

and let $F(x,y,\langle u\rangle) = f(x,y,\langle u\rangle) + \{M\}\cdot\langle u\rangle$, where $\{M\} = (M_1, M_2, M_3)$, M_i, $1 \le i \le 3$, are constants with $M_1 - M_2M_3 = 0$. Then we have

$$e^{M_3x+M_2y}\left(u_{xy} + \{M\}\cdot\langle u\rangle\right) = e^{M_3x+M_2y}\,F(x,y,\langle u\rangle)\,,$$

which gives

$$\left[e^{M_3x+M_2y}\,u\right]_{xy} = e^{M_3x+M_2y}\,F(x,y,\langle u\rangle).$$

Using the transformations

$$\overline{u} = e^{(M_3x+M_2y)}\,u \text{ and } \overline{F}(x,y.\langle u\rangle) = e^{(M_3x+M_2y)}\,F(x,y,\langle u\rangle),$$

we obtain

$$\overline{u}_{xy} = \overline{F}(x,y,\langle e^{-(M_3x+M_2y)}\,\overline{u}\rangle).$$

Now suppose that v^0, w^0 are natural lower-upper solutions of (3.6). Then we have

$$\begin{aligned} v^0_{xy} &\le f(x,y,\langle v^0\rangle)\,, \\ v^0_{xy} + \{M\}\cdot\langle v^0\rangle &\le f(x,y,\langle v^0\rangle) + \{M\}\cdot\langle v^0\rangle\,, \\ \left[e^{(M_3x+M_2y)}\,v^0\right]_{xy} &\le e^{M_3x+M_2y}\,F(x,y,\langle v^0\rangle)\,, \\ \overline{v}^0_{xy} &\le \overline{F}(x,y,\langle e^{-(M_3x+M_2y)}\,\overline{v}^0\rangle)\,. \end{aligned}$$

Similarly, $\overline{w}^0_{xy} \geq \overline{F}(x, y, \langle e^{-(M_3x+M_2y)}\,\overline{w}^0\rangle)$, which shows that $\overline{v}^0$ is a lower solution and $\overline{w}^0$ is an upper solution. Utilizing this fact, it is now possible to deduce several special cases of Theorem 3.1. See, for example, [4, 13, 14].

Example 3.1. For $(x, y) \in R = [0,1] \times [0,1]$, consider

$$u_{xy} = f(x,y,u) = \begin{cases} xy\sqrt{u-1} & \text{if } 1 \leq u \leq 2 \\ 0 & \text{if } u < 1 \\ xy & \text{if } u > 2 \end{cases} \tag{3.7}$$

$$u(x,0) = 1 \text{ for } x \in [0,1]\,,\; u(0,y) = 1 \text{ for } y \in [0,1]\,. \tag{3.8}$$

The forcing function f in (3.7) is nondecreasing in $\langle u\rangle$ for fixed $(x, y) \in R$. Equation (3.7) has *two* solutions satisfying the initial boundary conditions (3.8):

$$v(x,y) \equiv 1\,;\; w(x,y) = \frac{x^4y^4}{256} + 1\,.$$

The functions $v^0(x, y) \equiv 1$ and $w^0(x, y) = xy + 1$ form a pair of natural lower-upper solutions for (3.7). Clearly $\langle v^0\rangle \leq \langle v^1\rangle$ on R. Also $w^1(x, y) = 1 + (\frac{2}{5})^2(xy)^{5/2} \leq w^0(x, y)$ on R, so that all the conditions of Theorem 3.1 are satisfied. It is easy to see that $v^n(x, y) \equiv 1$ for all $n = 1, 2, 3, \cdots$, so that $\langle v^n(x, y)\rangle \to \langle v(x, y)\rangle$ uniformly on J and satisfies the monotone behaviors $\langle v^0\rangle \leq \langle v^1\rangle \leq \langle v^2\rangle \cdots \leq \langle v^n\rangle \leq \cdots$. The graphs of $\{\langle w^n\rangle\}_{0\leq n\leq 6}$ and $\langle w\rangle$ and their numerical values at four preselected points $(x, y) \in R$ are given below.

Table 1: Table of Values for w^0 to w^5 and w

	(x,y)	(x,y)	(x,y)	(x,y)
	$(0.1, 0.2)$	$(0.3, 0.4)$	$(0.6, 0.5)$	$(0.8, 0.7)$
$w^0(x,y)$	1.0200000000	1.1200000000	1.3000000000	1.5600000000
$w^1(x,y)$	1.0000090510	1.0007981290	1.0078872048	1.0375482802
$w^2(x,y)$	1.0000001139	1.0000385152	1.0007567237	1.0057531322
$w^3(x,y)$	1.0000000103	1.0000068008	1.0001884062	1.0018101402
$w^4(x,y)$	1.0000000028	1.0000025836	1.0000849905	1.0009179361
$w^5(x,y)$	1.0000000014	1.0000015169	1.0000543760	1.0006226761
$w(x,y)$	1.0000000006	1.0000008100	1.0000316406	1.0003841600

Table 2: Table of Values for w^0_x to w^5_x and w_x

	(x,y)	(x,y)	(x,y)	(x,y)
	$(0.1, 0.2)$	$(0.3, 0.4)$	$(0.6, 0.5)$	$(0.8, 0.7)$
$w^0_x(x,y)$	0.2000000000	0.4000000000	0.5000000000	0.7000000000
$w^1_x(x,y)$	0.0002262742	0.0066510751	0.0328633535	0.1173383756
$w^2_x(x,y)$	0.0000037027	0.0004179482	0.0040989201	0.0233720997
$w^3_x(x,y)$	0.0000003725	0.0000821768	0.0011382974	0.0082021977
$w^4_x(x,y)$	0.0000001063	0.0000328332	0.0005400438	0.0043745393
$w^5_x(x,y)$	0.0000000541	0.0000197512	0.0003540107	0.0030404108
$w_x(x,y)$	0.0000000250	0.0000108000	0.0002109375	0.0019208000

Figure 1: Graphs of w^0 to w^5 and w

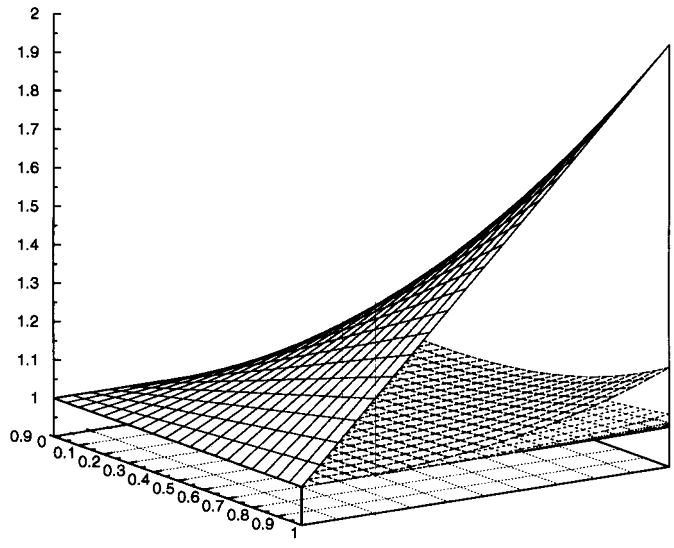


Figure 2: Graphs of w_x^0 to w_x^5 and w_x

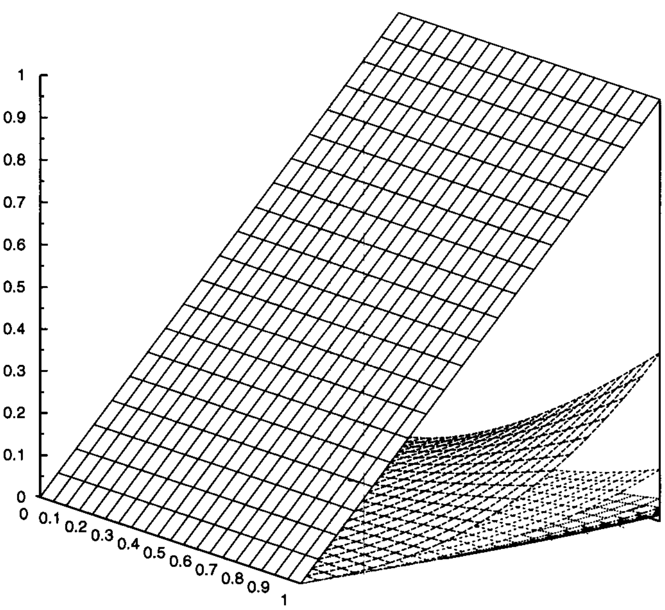
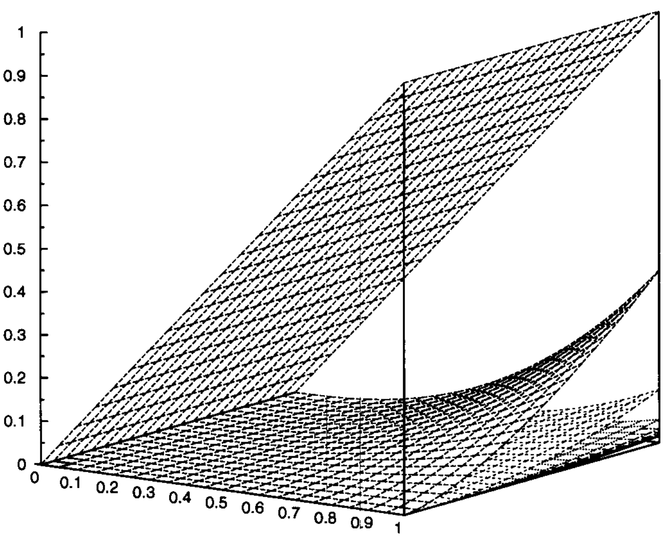


Table 3: Table of Values for w_y^0 to w_y^5 and w_y

	(x,y)	(x,y)	(x,y)	(x,y)
	$(0.1, 0.2)$	$(0.3, 0.4)$	$(0.6, 0.5)$	$(0.8, 0.7)$
$w_y^0(x,y)$	0.1000000000	0.3000000000	0.6000000000	0.8000000000
$w_y^1(x,y)$	0.0001131371	0.0049883063	0.0394360241	0.1341010007
$w_y^2(x,y)$	0.0000018514	0.0003129361	0.0049187042	0.0267109711
$w_y^3(x,y)$	0.0000001862	0.0000616326	0.0013659449	0.0093739403
$w_y^4(x,y)$	0.0000000532	0.0000246249	0.0006480526	0.0049994735
$w_y^5(x,y)$	0.0000000270	0.0000148134	0.0004248129	0.0034747551
$w_y(x,y)$	0.0000000125	0.0000081000	0.0002531250	0.0021952000

Figure 3: Graphs of w_y^0 to w_y^5 and w_y



The foregoing computations were done using the standard differentiation and plotting utilities of the software package Maxima (`http://maxima.sourceforge.net`).

In the next result, we employ scheme (S_2) and obtain sequences which are alternately monotone, under the same hypotheses (A_1) and (A_2) as in Theorem 3.1, but by dropping the assumptions $\langle v^0\rangle \le \langle v^1\rangle$, $\langle w^1\rangle \le \langle w^0\rangle$ and adding the new ones $\langle v^0\rangle \le \langle v^2\rangle$, $\langle w^2\rangle \le \langle w^0\rangle$.

Theorem 3.2. *Let the assumptions* (A_1) *and* (A_2) *of Theorem 3.1 hold. Then for any solution* u *of* (2.1), *the sequences* $\{v^n\}$, $\{w^n\}$, *in the iterative scheme* (S_2) *satisfy the inequalities*

$$\langle v^0\rangle \le \langle v^2\rangle \le \cdots \le \langle v^{2n-2}\rangle \le \langle u\rangle \le \langle v^{2n-1}\rangle \le \cdots \le \langle v^3\rangle \le \langle v^1\rangle,$$
$$\langle w^1\rangle \le \langle w^3\rangle \le \cdots \le \langle w^{2n-1}\rangle \le \langle u\rangle \le \langle w^{2n-2}\rangle \le \cdots \le \langle w^2\rangle \le \langle w^0\rangle,$$

on R, *provided that* $\langle v^0\rangle \le \langle v^2\rangle$ *and* $\langle w^2\rangle \le \langle w^0\rangle$ *on* R. *Moreover, the monotone sequences* $\{v^{2n}\}$, $\{v^{2n-1}\}$, $\{w^{2n}\}$, $\{w^{2n-1}\}$, *converge uniformly to* v, w, v^* *and* w^* *respectively on* R *and satisfy the equations*

$$\begin{aligned}
w_{xy} &= f(x,y,\langle v^*\rangle) + g(x,y,\langle v\rangle),\ (x,y)\in R;\\
w(x,0) = \sigma(x),\ x\in I;\ & w(0,y) = \tau(y),\ y\in J;\ w(0,0) = u_0,\\
v_{xy} &= f(x,y,\langle w^*\rangle) + g(x,y,\langle w\rangle),\ (x,y)\in R;\\
v(x,0) = \sigma(x),\ x\in I;\ & v(0,y) = \tau(y),\ y\in J;\ v(0,0) = u_0,\\
w^*_{xy} &= f(x,y,\langle v\rangle) + g(x,y,\langle v^*\rangle),\ (x,y)\in R;\\
w^*(x,0) = \sigma(x),\ x\in I;\ & w^*(0,y) = \tau(y),\ y\in J;\ w^*(0,0) = u_0,\\
v^*_{xy} &= f(x,y,\langle w\rangle) + g(x,y,\langle w^*\rangle),\ (x,y)\in R;\\
v^*(x,0) = \sigma(x),\ x\in I;\ & v^*(0,y) = \tau(y),\ y\in J;\ v^*(0,0) = u_0.
\end{aligned}$$

Corollary 3.2. *In addition to the assumptions of Theorem 3.2, suppose that* f *and* g *satisfy one-sided Lipschitz conditions as in Corollary 3.1. Then* $v \equiv w \equiv v^* \equiv w^* \equiv u$ *is the unique solution of* (2.1) *in* Ω.

Our third result employs coupled lower-upper solutions of Type I together with the scheme (S_2) and yields intertwined monotone sequences without any additional requirements. Compare with Theorem 4.3.1 in [4], wherein coupled lower-upper solutions of Type I are utilized in conjunction with the scheme (S_1).

Theorem 3.3. *Assume that*

(B_1) v^0, $w^0 \in C^2[R,\mathbb{R}]$ *are coupled lower-upper solutions of Type I of* (2.1) *such that* $\langle v^0\rangle \le \langle w^0\rangle$ *on* R;

(B_2) f, $g \in C^2[R\times\mathbb{R}^3,\mathbb{R}]$, f *is nondecreasing in* $\langle u\rangle$ *and* g *is nonincreasing in* $\langle u\rangle$.

Then the sequences $\{v^n\}$, $\{w^n\}$ *generated by the scheme* (S_2) *satisfy the intertwined property*

$$\langle v^0\rangle \le \langle w^1\rangle \le \cdots \le \langle v^{2n-2}\rangle \le \langle w^{2n-1}\rangle \le \langle u\rangle \le$$
$$\langle v^{2n-1}\rangle \le \langle w^{2n-2}\rangle \le \cdots \le \langle v^1\rangle \le \langle w^0\rangle,$$

on R, *where* u *is any solution of* (2.1) *in* Ω. *The sequences* $\{\langle v^{2n}\rangle, \langle w^{2n-1}\rangle\} \to \langle v\rangle$, $\{\langle w^{2n}\rangle, \langle v^{2n-1}\rangle\} \to \langle w\rangle$, *uniformly where* v *and* w *are coupled lower and upper solutions respectively of* (2.1) *on* R, *that is* v *and* w *satisfy*

$$\begin{aligned}
v_{xy} &= f(x,y,\langle v\rangle) + g(x,y,\langle w\rangle),\ (x,y)\in R,\\
w_{xy} &= f(x,y,\langle w\rangle) + g(x,y,\langle v\rangle),\ (x,y)\in R.
\end{aligned}$$

Also, $\langle v\rangle \le \langle u\rangle \le \langle w\rangle$ *on* R. *Furthermore, if* f *and* g *satisfy the appropriate one-sided Lipschitz conditions, then* $v \equiv w \equiv u$ *is the unique solution of* (2.1) *in* Ω.

Our final two results employ coupled lower-upper solutions of Type II whose existence follows from Lemma 2.1. Theorem 3.4 below shows that we get the same conclusions as in Theorem 3.1, under the same additional conditions, whereas Theorem 3.5 asserts the same conclusions as in Theorem 3.3 under different additional conditions. (Compare with Theorem 3.2.) We skip their proofs.

Theorem 3.4. *Under the conditions of Lemma 2.1, let v^0, w^0 be coupled lower-upper solutions of Type II relative to* (2.1), *with $\langle v^0\rangle \leq \langle w^0\rangle$ on R. Then the sequences $\{v^n\}$, $\{w^n\}$ generated by the scheme* (S_1) *satisfy the natural monotone property*

$$\langle v^0\rangle \leq \langle v^1\rangle \leq \langle v^2\rangle \leq \cdots \leq \langle v^n\rangle \leq \langle w^n\rangle \leq \cdots \leq \langle w^2\rangle \leq \langle w^1\rangle \leq \langle w^0\rangle$$

on R, provided $\langle v^0\rangle \leq \langle v^1\rangle$ and $\langle w^1\rangle \leq \langle w^0\rangle$ on R. Furthermore, $\langle v^n\rangle \to \langle v\rangle$, $\langle w^n\rangle \to \langle w\rangle$, where v and w are coupled minimal and maximal solutions respectively of (2.1) *on R. If in addition f and g satisfy the one-sided Lipschitz conditions in Corollary 3.1, then $v \equiv w \equiv u$ is the unique solution of* (2.1) *in Ω.*

Theorem 3.5. *Under the conditions of Lemma 2.1, let v^0, w^0 be coupled lower-upper solutions of Type II relative to* (2.1) *with $\langle v^0\rangle \leq \langle w^0\rangle$ on R. Then the sequences $\{v^n\}$, $\{w^n\}$ generated by the scheme* (S_2) *satisfy the conclusions of Theorem 3.3, provided $\langle v^0\rangle \leq \langle w^1\rangle$ and $\langle v^1\rangle \leq \langle w^0\rangle$ on R.*

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