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RECORD VALUES FROM HALF LOGISTICS AND INVERSE WEIBULL PROBABILITY DISTRIBUTION FUNCTIONS

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ABSTRACT. Let $X_1, X_2, ..., X_n$ be a sequence of independent and identically distributed random variables with cumulative distribution function $F(x)$. Denote $X_{L(n)} = min\{X_1, X_2, \ldots, X_n\}$, $n = 2, 3, \ldots$. X_j is a lower record of $\{X_n\}$ if and only if $X_{L(j)} < X_{L(j-1)}$, $j = 2, 3, \ldots$ and $X_{L(1)} = X_1$. An analogous definition of records deals with upper record values. By definition, X_1 is an upper as well as lower record value. The subject of the present paper is to introduce the Half logistic and the Inverse weibull probability distributions when applied in studying the performance and behavior with respect to records. In addition to developing the analytical structures, we illustrate the usefulness of the results using environmental and sports data.

Key Words Subject, Classification.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a complete random sample with cumulative probability distribution function (cdf) $F(x)$. To obtain the records as needed for the present study we proceed as follows; first, given a complete random sample $X_1, X_2, \ldots, X_n$, we begin by taking the first observation to be the first record, that is, $X_{L(1)} = X_1$. Secondly, the second record, $X_{L(2)}$, is obtained by observing the X_i's sequentially from $X_2, \ldots, X_n$. The next observation that is less than $X_{L(1)}$ is the second record, $X_{L(2)}$, and the number of trials to get $X_{L(2)}$ is K_1, the inter record time. For example, let the next observation that is less than X_1 be X_5, then $X_{L(2)} = X_5$ and $K_1 = 4$. Now, X_5 is a standard for getting subsequent records. Finally, we will then proceed to use i) the sequence $\{X_{L(i)}, 1 \leq i \leq r\}$ of record values and ii) the sequence of record values and inter record time $X_{L(1)}, K_1, X_{L(2)}, K_2, \ldots, X_{L(r)}, K_r$, with $K_r = 1$. Ahsanullah, [1], has given a very good account of the basic and advanced aspects of records.

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For the record-breaking samples and interrecord times, $X_{L(1)}, K_1, X_{L(2)}, K_2, \ldots, X_{L(r)}, K_r$, we can write the likelihood function as

$$L = \prod_{i=1}^{r} f(x_i)(1 - F(x_i))^{k_i - 1}, \tag{1.1}$$

where $f(x_i)$ is the probability density function (pdf) of the records.

The likelihood function for r, lower record values $X_{L(1)}, X_{L(2)}, \ldots, X_{L(r)}$ from continuous cumulative probability distribution function $F(x)$, is

$$f_{1,2,\ldots,r}(x_1, x_2, \ldots, x_r) = f(x_r) \prod_{i=1}^{r-1} \frac{f(x_i)}{F(x_i)}, \tag{1.2}$$

and the probability density function of $X_{L(r)}$ is given by

$$f_r(x) = \frac{1}{\Gamma(r)}(-\ln(F(x)))^{r-1} f(x), \quad -\infty < x < \infty. \tag{1.3}$$

The cumulative probability distribution function of $X_{L(r)}$ is

$$\begin{aligned} F_r(x) &= \frac{1}{\Gamma(r)} \int_{-\infty}^{x} (-\ln(F(x)))^{r-1} f(x) dx \\ &= 1 - \Gamma_{-\ln(F(x))}(r). \end{aligned} \tag{1.4}$$

The joint probability density function of two lower record values $X_{L(r)}$ and $X_{L(s)}$ is given by

$$\begin{aligned} f(x_r, x_s) &= \frac{1}{\Gamma(r)\Gamma(s-r)}(-\ln(F(x_r)))^{r-1} \\ & [\ln(F(x_r)) - \ln(F(x_s))]^{s-r-1} \frac{f(x_r)}{F(x_r)} f(x_s), \end{aligned} \tag{1.5}$$

where $-\infty < x_s < x_r < \infty$.

Record values have been used in a variety of practical applications, such as industrial stress testing, sports and athletic events, oil and mining surveys, among others. Chandler, [2], was the first to introduce the concept of records for analyzing the breaking strength data of certain materials. The problem of parametric inference for record-breaking data was introduced by Samaniego and Whitaker, [3]. They developed and studied the properties of maximum likelihood estimates of the mean of an underlying exponential probability distribution. Gulati and Padgett, [4] extended the work of Samaniego and Whitaker, [3], to the Weibul probability distribution.

In section 2, we investigate the theory of record values and record times from the Half logistic probability distribution. We shall utilize records to develop key estimates of the parameters that are the key entities in the Half logistic probability distribution. In addition to developing the record estimates of the parameters, we shall illustrate the usefulness of our analytical developments by analyzing the failure times of air conditioning equipment in the Boeing 720 airplane and the failure times in minutes for a specific type of electrical insulation material that was subjected to a continuously increasing voltage stress. In section 3, we consider the Inverse Weibull probability distribution and develop the records theory along with the appropriate statistical estimates. Finally, we obtain some coefficients of the best linear unbiased estimates of the location and scale parameters of the power function probability density function. We illustrate the usefulness of these results with a numerical simulation study.

2. Record values from the Half Logistic Probability Distribution

The logistic probability distribution is very useful in many areas of human endeavor. Berkson, [5], [6], used the subject distribution extensively in analyzing bioassay and quantal response data. The works of Ojo, [7], McDonald and Xu, [8], are of interest among many publications on logistic probability distribution. The simplicity of the logistic probability distribution and its importance as a growth curve have made it even more important in statistical analysis and modeling. The shape of the logistic probability distribution which is similar to that of the Gaussian probability distribution makes it simpler and also profitable on suitable occasions to replace the Gaussian probability distribution by the logistic distribution with negligible errors in the respective theories and applications.

Balakrishnan and Chan, [9], have studied the best linear unbiased estimator of the scaled parameter half logistic probability distribution using double type II censored samples. In this Chapter, we shall study the theory of records for the half logistic probability distribution. In addition to developing the record estimates of the parameters, we shall illustrate their usefulness by applying the results to the failure time data of Boeing 720 airplane. This data has been initially analyzed by Balakrishnan and Chan, [9].

2.1. Analytical Formulation of the Record Model.

Let Y be a complete random variable having a half logistic probability density function given by

$$f(x) = \frac{1}{b} \frac{2e^{-\frac{x-\alpha}{b}}}{\left(1 + e^{-\frac{x-\alpha}{b}}\right)^2}, \tag{2.1}$$

where $x \geq \alpha \geq 0$, $b > 0$.

The cumulative probability distribution function of equation 2.1 is given by

$$F(x) = \frac{1 - e^{-\frac{x-\alpha}{b}}}{1 + e^{-\frac{x-\alpha}{b}}}. \tag{2.2}$$

For the record-breaking samples, $x_1, k_1, \ldots, x_r, k_r$, using equations (1.1), (2.1) and (2.2), the likelihood function for the half logistic probability distribution function is given by

$$
\begin{aligned}
L &= \prod_{i=1}^{r} \frac{1}{b} \frac{2e^{-\omega_i}}{(1+e^{-\omega_i})^2} \left(\frac{2e^{-\omega_i}}{1+e^{-\omega_i}} \right)^{k_i-1} \\
&= \prod_{i=1}^{r} \frac{1}{b} \frac{1}{1+e^{-\omega_i}} \left(\frac{2e^{-\omega_i}}{1+e^{-\omega_i}} \right)^{k_i} \qquad (2.3)
\end{aligned}
$$

where $\omega_i = (x_i - \alpha)/b$.

For convenience, the negative loglikelihood function of expression (2.3) is given by

$$
-\log L = \sum_{i=1}^{r} \log(b) + k_i\omega_i + (1+k_i)\log(1+e^{-\omega_i}). \qquad (2.4)
$$

Taking the partial derivatives of equation (2.4) with respect to α and b we have

$$
\frac{\partial(-\log L)}{\partial \alpha} = -\frac{1}{b} \sum_{i=1}^{r} \left[k_i - (1+k_i) \left(\frac{e^{-\omega_i}}{1+e^{-\omega_i}} \right) \right] \qquad (2.5)
$$

and

$$
\frac{\partial(-\log L)}{\partial b} = \frac{1}{b} \sum_{i=1}^{r} \left[1 - k_i\omega_i + (1+k_i) \left(\frac{\omega_i e^{-\omega_i}}{1+e^{-\omega_i}} \right) \right]. \qquad (2.6)
$$

The second partial derivative of equations (2.5) and (2.6) with respect to α, b and αb are given by

$$
\frac{\partial^2(-\log L)}{\partial \alpha^2} = \frac{1}{b^2} \sum_{i=1}^{r} \frac{(1+k_i)e^{-\omega_i}}{(1+e^{-\omega_i})^2} \qquad (2.7)
$$

$$
\begin{aligned}
\frac{\partial^2(-\log L)}{\partial b^2} &= \frac{1}{b^2} \sum_{i=1}^{r} \left[1 - k_i\omega_i + (1+k_i) \left(\frac{\omega_i e^{-\omega_i}}{1+e^{-\omega_i}} \right) \right] \\
&+ \frac{1}{b^2} \sum_{i=1}^{r} k_i\omega_i + (1+k_i) \frac{-\omega_i e^{-\omega_i} - \omega_i e^{-2\omega_i} + \omega_i^2 e^{-\omega_i}}{(1+\omega_i e^{-\omega_i})^2} \qquad (2.8)
\end{aligned}
$$

and

$$
\frac{\partial^2(-\log L)}{\partial \alpha \partial b} = \frac{1}{b^2} \sum_{i=1}^{r} k_i - (1+k_i) \frac{e^{-\omega_i}}{1+e^{-\omega_i}} + (1+k_i) \frac{\omega_i e^{-\omega_i}}{(1+e^{-\omega_i})^2}. \qquad (2.9)
$$

Equating (2.5) and (2.6) to zero, we obtain the maximum likelihood estimates $\hat{\alpha}$ and $\hat{b}$ for α and b, respectively, and equations (2.8) and (2.9) become

$$
\frac{\partial^2(-\log L)}{\partial b^2} = \frac{1}{b^2} \sum_{i=1}^{r} 1 + \frac{(1+k_i)\omega_i^2 e^{-\omega_i}}{(1+\omega_i e^{-\omega_i})^2}, \qquad (2.10)
$$

and

$$
\frac{\partial^2(-\log L)}{\partial \alpha \partial b} = \frac{1}{b^2} \sum_{i=1}^{r} \frac{(1+k_i)\omega_i e^{-\omega_i}}{(1+e^{-\omega_i})^2}. \qquad (2.11)
$$

Using equations, (2.7), (2.10), and (2.11) we obtain the observed information matrix $I(\alpha, b)$, at $(\hat{\alpha}, \hat{b})$, for the half logistic pdf model to be

$$I(\hat{\alpha}, \hat{b}) = \frac{1}{\hat{b}^2}\begin{pmatrix} \sum_{i=1}^{r} \frac{(1+k_i)e^{-\omega_i}}{(1+e^{-\omega_i})^2} & \sum_{i=1}^{r} \frac{(1+k_i)\omega_i e^{-\omega_i}}{(1+e^{-\omega_i})^2} \\ \sum_{i=1}^{r} \frac{(1+k_i)\omega_i e^{-\omega_i}}{(1+e^{-\omega_i})^2} & \sum_{i=1}^{r} 1 + \frac{(1+k_i)\omega_i^2 e^{-\omega_i}}{(1+\omega_i e^{-\omega_i})^2} \end{pmatrix}. \tag{2.12}$$

Note that the inverse of $I(\hat{\alpha}, \hat{b})$ from equation 2.12 gives the variance covariance matrix of $\hat{\alpha}, \hat{b}$.

Next, we proceed to obtain the estimates of the parameters that are inherent in (2.1) as follows: For the complete sample $X_1 = y_1, \ldots, X_n = y_n$, from the half logistic probability density function given by (2.1), we can write the negative log-likelihood function as

$$-\log L = -\sum_{i=1}^{n} \log(b) + \zeta + 2\log(1 + \exp(-\zeta), \tag{2.13}$$

where $\zeta_i = (y_i - \mu/b$.

Taking the partial derivative of (2.13) with respect to μ and b, we have

$$\frac{\partial(-\log L)}{\partial \mu} = \frac{1}{b}\sum_{i=1}^{n}\left[1 - \frac{2e^{-\zeta_i}}{1+e^{-\zeta_i}}\right] \tag{2.14}$$

and

$$\frac{\partial(-\log L)}{\partial b} = -\frac{1}{b}\sum_{i=1}^{n}\left[1 - \zeta_i + \frac{2e^{-\zeta_i}}{1+e^{-\zeta_i}}\right]. \tag{2.15}$$

Let expressions (2.14) and (2.15) equal to zero and taking the second partial derivative with respect to μ, b and μb, we have

$$\frac{\partial^2(-\log L)}{\partial \mu^2} = \frac{1}{b^2}\sum_{i=1}^{n} \frac{2e^{-\zeta_i}}{(1+e^{-\zeta_i})^2}, \tag{2.16}$$

$$\frac{\partial^2(-\log L)}{\partial b^2} = \frac{1}{b^2}\sum_{i=1}^{n}\left[1 + \frac{2\hat{\zeta}_i^2 e^{-\zeta_i}}{(1+e^{-\hat{\zeta}_i})^2}\right], \tag{2.17}$$

and

$$\frac{\partial^2(-\log L)}{\partial \mu \partial b} = \frac{1}{b^2}\sum_{i=1}^{n} \frac{2\zeta_i e^{-\zeta_i}}{(1+e^{-\zeta_i})^2}. \tag{2.18}$$

Using equations (2.16), (2.17), and (2.18) we obtain the observed information matrix at $(\hat{\mu}, \hat{b})$, that is, $I(\mu, b)$, for the half logistic pdf model to be

$$I(\hat{\mu}, \hat{b}) = \frac{1}{\hat{b}^2}\begin{pmatrix} \sum_{i=1}^{n} \frac{2e^{-\hat{\zeta}_i}}{(1+e^{-\hat{\zeta}_i})^2} & \sum_{i=1}^{n}\left[1 + \frac{2\hat{\zeta}_i^2 e^{-\hat{\zeta}_i}}{(1+e^{-\hat{\zeta}_i})^2}\right] \\ \sum_{i=1}^{n}\left[1 + \frac{2\hat{\zeta}_i^2 e^{-\hat{\zeta}_i}}{(1+e^{-\hat{\zeta}_i})^2}\right] & \sum_{i=1}^{n} \frac{2\hat{\zeta}_i e^{-\hat{\zeta}_i}}{(1+e^{-\hat{\zeta}_i})^2} \end{pmatrix}.$$

Thus, we can use the estimates in equations (2.5) and (2.6) to predict future observations of the phenomenon of interest. We can accomplish this by using the return levels, that is,

$$F(x_s) = 1/s, \quad s > r$$

which gives

$$x_s = \hat{\alpha} - \hat{b}\log\left\{\frac{s-1}{s+1}\right\} \tag{2.19}$$

2.2. **Application.** We shall take α to be zero for comparison purpose consistent with published results,Balakrishnan and Chan, [9]. We consider two applications in this section and compare the results of our analysis with results of other statistical analysis.

2.2.1. *Application 1: Boeing 720 Airplane Data.* The following data which has been initially analyzed by Balakrishnan and Chan, [9], are the failure times of air conditioning equipment in a Boeing 720 airplane.

$$74, 57, 48, 29, 502, 12, 70, 21, 29, 386, 59, 27, 153, 26, 326.$$

Balakrishnan and Chan, [9], have shown that the above data fits the half logistic probability distribution function quite well. Using the scale half logistic probability distribution, that is, letting $\alpha = 0$ in equation (2.1), they obtained the best linear unbiased estimates for b to be 90.92 with a standard error of 19.66.

The following record values and record times can be obtained from the above data, that is,

$$x_i = 74, 57, 47, 29, 12$$

with

$$k_i = 1, 1, 1, 2, 1.$$

Using equations (2.5) and (2.6) and letting $\mu = 0$ in (2.1) we have the maximum likelihood estimate and their standard error of b is $\hat{b}_{mle} = 88.23$ with a standard error of 20.22.

Table 1 presents a comparison of the parameter estimate of our model with that of Balakrishnan and Chan, [9]. As can be observed from the table below, even though we have a reduced sample, our model performs equally well with the maximum likelihood estimate (MLE).

The results of the analysis are displayed in Table 1. As can be observed, the results are not much different from each other despite the fact that we have reduced data.

Method	Estimate(Standard Error)
MLE using Records	88.23(20.22)
BLUE using Complete data	90.92(19.66)
MLE	93.58

TABLE 1. Estimate of b for the Half Logistic pdf for the Boeing 720 Airplane Data

2.2.2. *Application 2: Electrical Insulation Data.* The next application represent the failure times, in minutes, for a specific type of electrical insulation material that was subjected to a continuously increasing voltage stress (Lawless, [10], p.138):

$$12.3, 21.8, 24.4, 28.6, 43.2, 46.9, 70.7, 75.3, 95.5, 98.1, 138.6, 151.9.$$

A random sample of this data was obtained to be

$$138.6, 75.3, 95.5, 151.9, 46.9, 70.7, 24.4, 21.8, 28.6, 43.2, 12.3, 98.1.$$

We shall proceed to fit the scare-parameter half logistic distribution with pdf (2.1), that is, when $\alpha = 0$ to the data.

The best linear unbiased estimate for b has been obtained using the complete data, Balakrishnan and Chan, [9]. Balakrishnan and Chan, [9] obtained the blue (standard error) for b to be 50.50(12.68). Using records, we will obtain the blue for b and compare our result with that of Balakrishnan and Chan, [9].

Using equation (2.15), we obtain the maximum likelihood estimate for the scale parameter of the half logistic pdf to be $\hat{b} = 47.42$.

The appropriateness of the assumption of the half logistic distribution for the above data is checked using the Q-Q plot given by Figure 1. In Figure 1, we plotted the quantile from the half logistic probability distribution versus the empirical quantiles. As can be observed from Figure 1, the half logistic distribution fits the data extremely well. The value of the correlation coefficient in the Q-Q plot is 0.98.

Form the random data above, the following record values and record times have been obtained to be

$$x_i = 138.6, 75.3, 46.9, 24.4, 21.8, 12.3$$

and

$$k_i = 1, 3, 2, 1, 3, 1$$

Using equation (2.6) we calculated the maximum likelihood estimates and their standard errors of b to be $\hat{b}_{mle} = 52.22$ with a standard error of 14.22. The results are summarized in Table 2 below.

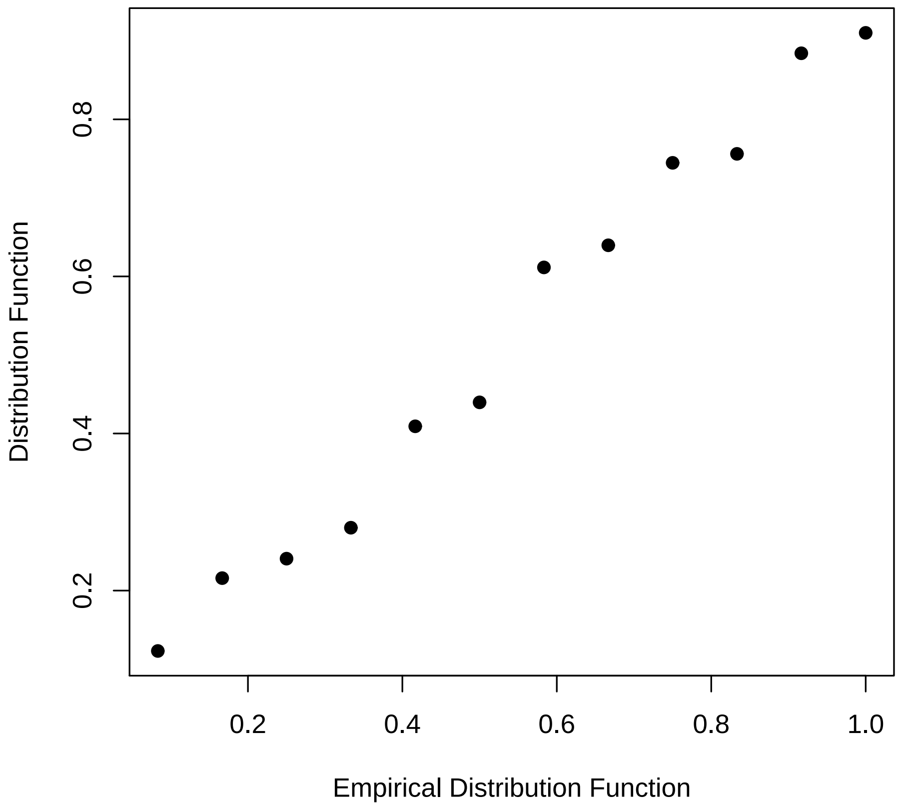

FIGURE 1. QQ plot of the data to verify goodness of fit for the Electrical Insulation Data

Method	Estimate(Standard Error)
MLE using Records	52.23(14.22)
BLUE using Complete data	50.50(12.68)
MLE	47.42

TABLE 2. Estimate of b for the Half Logistic pdf for the Electrical Insulation Data using records

Table 2 presents estimates of the data using our model, with those of Balakrishnan and Chan, [9]. As can be observed, we have successfully presented another method to estimate the parameter of the scale half logistic probability distribution. The subject method, records and inter records, give good result with easier computational process and smaller samples.

3. **Record Values from the Inverse Weibull Probability Distribution**

The Weibull Probability distribution plays a very significant role in statistical analysis and modeling of real world problems. See, Weibull, [11], Lawless, [10], among others. This probability distribution is often used in the field of life data analysis due to its flexibility. It can mimic the behavior of other statistical distributions such as the normal and the exponential. The failure rate is either increasing, decreasing, or remains constant depending on the value of the shape parameter.

Keller and Kamath, [12], introduced the use of the Inverse Weibull probability distribution as a suitable model to describe the degradation phenomena of mechanical components such as the dynamic components (pistons, crankshaft, etc.) of diesel engines. The Inverse Weibull probability distribution distribution also provides a good fit to several data such as the times to breakdown of an insulating fluid, subject to the action of a constant tension, see Nelson, [13]. The Inverse Weibull probability distribution has initiated a large volume of research. For example, Carriere, [14], has used this distribution to model the mortality curve of a population, Mohamed et al., [15], have considered the single and product moments of order statistics from inverse Weibull probability distribution and doubly truncated inverse Weibull probability distributions, Calabria and Pulcini, [16], have discussed the maximum likelihood and least squares estimations of its parameters, and Calabria and Pulcini, [17], have considered Bayes 2-sample prediction of the distribution.

In this chapter we shall use the theory of **records** to obtain some distributional properties of the Inverse Weibull probability distribution. We shall obtain parameter of this distribution and we shall present coefficients of the BLUEs of the location and scale parameters of the Inverse Weibull Probability Distribution.

3.1. **Distributional Properties of Inverse Weibull Probability Distribution using Record Values.** Let X be a complete random variable (rv) from the Inverse Weibull Probability Distribution with pdf given by

$$(3.1)\quad f(y) = \frac{k}{x-\alpha}\left(\frac{b}{x-\alpha}\right)^{k} \exp\left\{-\left(\frac{x-\alpha}{b}\right)^{-k}\right\}, x>0, b>0, \alpha\geq 0, k>0$$
$$= 0, \quad \text{otherwise.}$$

For the record sample $X_{L(1)}, X_{L(2)}, \ldots, X_{L(r)}$ from the inverse Weibull probability distribution, using equation (1.3) and letting $\alpha = 0$ and $b = 1$ in equation (3.1), the

n^{th} moment of $X_{L(r)}$ from the power function probability distribution is given by

$$\begin{aligned} E(X_{L(r)})^n &= \frac{1}{\Gamma(r)} \int_0^\infty x^n f(x_r) dx \\ &= \frac{k}{\Gamma(r)} \int_0^\infty x^{n-kr-1} e^{-x^{-k}} dx \\ &= \frac{\Gamma\left(r - \frac{n}{k}\right)}{\Gamma(r)}, k > n \end{aligned} \tag{3.2}$$

The first moment of $X_{L(r)}$ is obtained from equation 3.4 to be

$$E(X_{L(r)}) = \frac{\Gamma\left(r - \frac{1}{k}\right)}{\Gamma(r)}, k > 1. \tag{3.3}$$

Observe also that

$$\begin{aligned} E(X_{L(1)}) &= \frac{\Gamma(1 - 1/k)}{\Gamma(1)} \\ E(X_{L(2)}) &= (1 - 1/k) E(X_{L(1)}), k > 1. \end{aligned}$$

Recursively we have that

$$E(X_{L(r)}) = E(X_{L(1)}) \prod_{i=1}^{r-2} \left[\frac{i - 1/k}{i} \right], k > 1$$

The moments for the inverse Weibull probability distribution have been computed and are presented in Table 3 for $k = 1.5, 2, 2.5, \ldots, 5$ and $r = 1, 2, \ldots, 10$.

The second moment of $X_{L(r)}$ is obtained from equation 1.3 to be

$$E(X_{L(r)})^2 = \frac{\Gamma\left(r - \frac{2}{k}\right)}{\Gamma(r)}, k > 2. \tag{3.4}$$

From equations 3.3 and 3.4, we have the variance of $X_{L(r)}$ to be

$$Var(X_{L(r)}) = \frac{\Gamma\left(r - \frac{1}{k}\right)}{\Gamma(r)} \left[\frac{\Gamma\left(r - \frac{2}{k}\right)}{\Gamma\left(r - \frac{1}{k}\right)} - \frac{\Gamma\left(r - \frac{1}{k}\right)}{\Gamma(r)} \right]. \tag{3.5}$$

Using equation 1.5, we have the m^{th} and n^{th} joint moments of $X_{L(r)}$ and $X_{L(s)}$, $s > r$ to be

$$\begin{aligned} E(X_{L(r)}^m, X_{L(s)}^n) &= \int_0^\infty \int_0^{x_s} x_r^m x_s^n f(x_r, x_s) dx_r dx_s \\ &= \frac{k^2}{\Gamma(r)\Gamma(s-r)} \int_0^\infty x_s^{n-k-1} e^{-x_s^{-k}} I(x_r) dx_s, \end{aligned} \tag{3.6}$$

where

$$I(x_r) = \int_0^{x_s} \frac{1}{x_r^{rk-m+1}} \left(\frac{1}{x_s^k} - \frac{1}{x_r^k} \right)^{s-r-1} dx_r$$

$$= \frac{1}{k x_s^{ks-m-k}} \int_0^{x_s} \frac{1}{x_r^{rk-m+1}} \left(1 - \left(\frac{x_s}{x_r} \right)^k \right)^{s-r-1} dx_r$$

$$= \frac{1}{k x_s^{ks-m-k}} \frac{\Gamma\left(r - \frac{m}{k}\right)\Gamma(s-r)}{\Gamma\left(s - \frac{m}{k}\right)}. \tag{3.7}$$

Substituting equation (3.7) in equation (3.6) gives

$$E(X_{L(r)}^m, X_{L(s)}^n) = \int_0^\infty \frac{1}{k x^{ks-m-n+1}} e^{-x_s^{-k}} dx_s$$

$$= \frac{k\Gamma\left(r - \frac{m}{k}\right)}{\Gamma\left(s - \frac{m}{k}\right)\Gamma(r)} \int_0^\infty \frac{1}{x_s^{ks-m-n+1}} e^{-x_s^{-k}} dx_s$$

$$= \frac{\Gamma\left(r - \frac{m}{k}\right)\Gamma\left(s - \frac{m+n}{k}\right)}{\Gamma\left(s - \frac{m}{k}\right)\Gamma(r)}, k > m+n. \tag{3.8}$$

The joint moment of $X_{L(r)}$ and $X_{L(s)}$ can be obtain from equation (3.8), by taking $m = n = 1$ to be

$$E(X_{L(r)}, X_{L(s)}) = \frac{\Gamma\left(r - \frac{1}{k}\right)\Gamma\left(s - \frac{2}{k}\right)}{\Gamma\left(s - \frac{1}{k}\right)\Gamma(r)}, s > r, k > 2. \tag{3.9}$$

The covariance of $X_{L(r)}$ and $X_{L(s)}$ is obtain from equations (3.3) and (3.8), to be

$$Cov(X_{L(r)}, X_{L(s)}) = \frac{\Gamma\left(r - \frac{1}{k}\right)}{\Gamma(r)} \left[\frac{\Gamma\left(s - \frac{2}{k}\right)}{\Gamma\left(s - \frac{1}{k}\right)} - \frac{\Gamma\left(s - \frac{1}{k}\right)}{\Gamma(s)} \right], s > r, k > 2. \tag{3.10}$$

Table 4 presents computed values for the variance-covariance matrices of the inverse Weibull probability distribution function for $k = 2.5, 3, 3.5, 4, 4.5, 5$, $r = 1, \ldots, 10$, $s = 1, 2, 3, 4, 5, 6, 7, 8, 10$, for $s > r$.

r	$k = 1.5$	$k = 2$	$k = 2.5$	$k = 3$	$k = 3.5$	$k = 4$	$k = 4.5$	$k = 5$
1	2.67894	1.77245	1.48919	1.35412	1.27599	1.22542	1.19015	1.16423
2	0.89298	0.88623	0.89352	0.90275	0.91142	0.91906	0.92567	0.93138
3	0.59532	0.66467	0.71481	0.75229	0.78122	0.80418	0.82282	0.83825
4	0.46303	0.55389	0.6195	0.6687	0.70682	0.73716	0.76187	0.78236
5	0.38586	0.48466	0.55755	0.61298	0.65633	0.69109	0.71954	0.74324
6	0.33441	0.43619	0.51295	0.57211	0.61883	0.65654	0.68756	0.71351
7	0.29725	0.39984	0.47875	0.54033	0.58936	0.62918	0.6621	0.68973
8	0.26894	0.37128	0.4514	0.5146	0.5653	0.60671	0.64108	0.67002
9	0.24653	0.34808	0.42883	0.49315	0.54511	0.58775	0.62327	0.65327
10	0.22827	0.32874	0.40977	0.47489	0.52781	0.57142	0.60788	0.63876

TABLE 3. Expected values of Standard Inverse Weibull pdf

s	r	k=2.5	k=3	k=3.5	k=4	k=4.5	k=5
1	1	2.37315	0.8453	0.43935	0.27081	0.18426	0.13376
2	1	0.19967	0.11705	0.07754	0.0554	0.04168	0.03255
2	2	0.1198	0.07803	0.05538	0.04155	0.03242	0.02604
3	1	0.08322	0.05289	0.03692	0.02738	0.02117	0.01688
3	2	0.04993	0.03526	0.02637	0.02053	0.01646	0.01351
3	3	0.03994	0.02938	0.02261	0.01797	0.01463	0.01216
4	1	0.04858	0.03213	0.02304	0.01742	0.01367	0.01103
4	2	0.02915	0.02142	0.01646	0.01307	0.01063	0.00883
4	3	0.02332	0.01785	0.01411	0.01143	0.00945	0.00794
4	4	0.02021	0.01587	0.01277	0.01048	0.00875	0.00741
5	1	0.03293	0.02235	0.01632	0.0125	0.0099	0.00805
5	2	0.01976	0.0149	0.01165	0.00937	0.0077	0.00644
5	3	0.01581	0.01242	0.00999	0.0082	0.00685	0.0058
5	4	0.0137	0.01104	0.00904	0.00752	0.00634	0.00541
5	5	0.01233	0.01012	0.00839	0.00705	0.00599	0.00514
6	1	0.02429	0.0168	0.01243	0.00961	0.00767	0.00628
6	2	0.01457	0.0112	0.00888	0.00721	0.00597	0.00502
6	3	0.01166	0.00933	0.00761	0.00631	0.0053	0.00452
6	4	0.01011	0.0083	0.00688	0.00578	0.00491	0.00422
6	5	0.00909	0.0076	0.00639	0.00542	0.00464	0.00401
6	6	0.00837	0.0071	0.00603	0.00515	0.00443	0.00385
10	1	0.01082	0.00784	0.00599	0.00474	0.00386	0.0032
10	2	0.00649	0.00522	0.00428	0.00356	0.00300	0.00256
10	3	0.00519	0.00435	0.00367	0.00311	0.00267	0.00231
10	4	0.0045	0.00387	0.00332	0.00285	0.00247	0.00215
10	5	0.00405	0.00355	0.00308	0.00268	0.00233	0.00204
10	6	0.00373	0.00331	0.0029	0.00254	0.00223	0.00196
10	7	0.00348	0.00313	0.00277	0.00244	0.00215	0.0019
10	8	0.00328	0.00298	0.00265	0.00235	0.00208	0.00184
10	9	0.00311	0.00285	0.00256	0.00228	0.00202	0.0018
10	10	0.00298	0.00275	0.00248	0.00221	0.00197	0.00176

TABLE 4. Variance Covariance of Standard Inverse Weibull pdf

3.2. **Estimation of Parameter.** In this section, we estimate the parameters of the inverse Weibull probability distribution using lower record values.

3.2.1. *Estimating α and b for known k.* Let $x_1, x_2, \ldots, x_r$ be r lower record values from the standard inverse Weibull probability distribution (3.1) with $\alpha = 0$ and

$b = 1$. Further, let

$$\mathbf{h}' = x_1 + \ldots + x_r,$$

then

$$E(\mathbf{h}') = \mu\mathbf{1} + \sigma^2\delta,$$

and

$$Var(\mathbf{h}') = \sigma^2\mathbf{V},$$

where,

$$\mathbf{1}' = (1, 1, 1, \ldots, 1),$$

$$\delta' = (b_1, b_2, b_3, \ldots b_r),$$

and

$$\mathbf{V} = (v_{ij}), v_{ij} = a_i b_j, 1 \leq i, j \leq r.$$

Let

$$\mathbf{V}^{-1} = (V^{ij}), 1 \leq i < j \leq r,$$

then the entries of $\mathbf{V}^{-1}$ are

$$\begin{aligned} V^{ii} &= \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})} \\ &= \frac{k^2\Gamma(i)}{\Gamma\left(i - \frac{2}{k}\right)} \left[\left(1 + \frac{1}{k}\right)^2 - 2i\left(1 + \frac{2}{k}\right)\right], i = 1, \ldots, r-1 \end{aligned}$$

$$\begin{aligned} V^{ij} &= V^{ji} \\ &= \frac{-1}{a_{i+1}b_i - a_i b_{i+1}} \\ &= -\frac{ik^2(i - \frac{1}{k})\Gamma(i)}{\Gamma(i - \frac{2}{k})}, j = i+1, i = 1, \ldots, r-1, \end{aligned}$$

and

$$V^{ij} = 0 \quad \text{for} \quad |i - j| > 1,$$

$$\begin{aligned} V^{rr} &= \frac{b_{r-1}}{b_r(a_r b_{r-1} - a_{r-1}b_r)} \\ &= \frac{k^2 b_{r-1}}{b_r} \frac{(r - 1 - \frac{1}{k})\Gamma(r)}{\Gamma(r - 1 - \frac{2}{k})}. \end{aligned}$$

Using the method of Lloyd, [18], we have that the BLUE, $\hat{\alpha}$ and $\hat{b}$ for α and b based on r lower record values from the inverse Weibull probability distribution are given by

$$\hat{\alpha} = \frac{\delta' V^{-1}(\delta\mathbf{1}' - \mathbf{1}\delta')\mathbf{V}^{-1}\mathbf{h}}{(\delta'\mathbf{V}^{-1}\delta)(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}) - (\delta'\mathbf{V}^{-1}\mathbf{1})^2},$$

and

$$\hat{b} = \frac{\mathbf{1}'\mathbf{V}^{-1}(\mathbf{1}\delta' - \delta\mathbf{1}')\mathbf{V}^{-1}\mathbf{h}}{(\delta'\mathbf{V}^{-1}\delta)(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}) - (\delta'\mathbf{V}^{-1}\mathbf{1})^2}.$$

The variance and covariance of the estimators are

$$Var(\hat{\alpha}) = \frac{(\delta\mathbf{V}^{-1}\delta)\sigma^2}{(\delta'\mathbf{V}^{-1}\delta)(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}) - (\delta'\mathbf{V}^{-1}\mathbf{1})^2},$$

$$Var(\hat{b}) = \frac{(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1})\sigma^2}{(\delta'\mathbf{V}^{-1}\delta)(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}) - (\delta'\mathbf{V}^{-1}\mathbf{1})^2}$$

$$Cov(\hat{\alpha}, \hat{b}) = -\frac{(\delta'\mathbf{V}^{-1}\mathbf{1})\sigma^2}{(\delta'\mathbf{V}^{-1}\delta)(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}) - (\delta'\mathbf{V}^{-1}\mathbf{1})^2}.$$

Coefficients of the BLUES for α, b and the variance covariance for α and b are given in Tables 5, 6, and 7.

3.3. **Estimation of k when α and b are assumed known.**

3.3.1. *Method of Moments.* For simplicity, we let $\alpha = 0$ and $b = 1$ in (3.1). In the absence of this assumption, the the random variable Y can be replaced by $(Y - \alpha)/b$ if α and b are known. Observe from equation (3.3) that

$$E(X_{L(r)}) = \frac{\Gamma(r - 1/k)}{\Gamma(r)},$$

from which we have that

$$E(X_{L(1)}) = \frac{\Gamma(1 - 1/k)}{\Gamma(1)}.$$

Next,

$$\begin{aligned} E(X_{L(2)}) &= \frac{\Gamma(2 - 1/k)}{\Gamma(2)} \\ &= (1 - 1/k)E(X_{L(1)}), \end{aligned}$$

and

$$\frac{E(X_{L(1)})}{k} = E(X_{L(1)}) - E(X_{L(2)}),$$

similarly,

$$\frac{E(X_{L(2)})}{2k} = E(X_{L(2)}) - E(X_{L(3)}),$$

which gives

$$\frac{E(X_{L(1)})}{k} + \frac{E(X_{L(2)})}{2k} = E(X_{L(1)}) - E(X_{L(3)}) \tag{3.11}$$

The next term results in

$$\frac{E(X_{L(3)})}{3k} = E(X_{L(3)}) - E(X_{L(4)}). \tag{3.12}$$

Adding equations 3.11 and 3.12 together results in

$$\frac{E(X_{L(1)})}{k} + \frac{E(X_{L(2)})}{2k} + \frac{E(X_{L(3)})}{3k} = E(X_{L(1)}) - E(X_{L(4)})$$

Hence, continuing this procedure to the rth lower record value we have

$$\frac{E(X_{L(1)})}{k} + \frac{E(X_{L(2)})}{2k} + \ldots + \frac{E(X_{L(r-1)})}{k(r-1)} = E(X_{L(1)}) - E(X_{L(r)})$$

Dropping the Expectation and solving for k, we have a moment's estimate $\hat{k}_{ME}$ of k to be

$$\hat{k}_{ME} = \frac{1}{X_1 - X_r} \sum_{i=1}^{r-1} \frac{X_i}{i} \tag{3.13}$$

3.3.2. *Method of Maximum Likelihood.* Using equations (1.2) and (3.1), we have that

$$f_{1,2,\ldots,r}(x_1, x_2, \ldots, x_r) = e^{-x_r^{-k}} \prod_{i=1}^{r} \frac{k}{x_i^{k+1}} \tag{3.14}$$

The loglikelihood of equation (3.14) is

$$\log f_{1,2,\ldots,r}(x_1, x_2, \ldots, x_r) = -x_r^{-k} + \sum_{i=1}^{r} \log(k) - (k+1)\log(x_i). \tag{3.15}$$

Differentiating equation (3.15) with respect to k gives

$$\frac{\delta \log f_{1,2,\ldots,r}(x_1, x_2, \ldots, x_r)}{\delta k} = x_r^{-k} \log(x_r) + \sum_{i=1}^{r} \frac{1}{k} - \log(x_i). \tag{3.16}$$

Solving equation (3.16) iteratively gives the maximum likelihood estimate $\hat{k}_{MLE}$ of k.

3.4. **Simulation Study.** To illustrate the performance of the estimators obtained in the previous section, we proceed with a simulation study. We simulated a small random sample of size, $n = 20$ from the power function probability distribution with $k = 3.5$, $\alpha = 3$ and $b = 5$. The simulated values are

$$10.87, 10.37, 10.03, 8.15, 9.14, 10.83, 10.02, 7.60, 10.47, 9.14,$$
$$8.72, 11.09, 8.27, 9.01, 10.40, 7.21, 8.09, 12.92, 10.55, 6.55.$$

From the above sample, we obtained six records, that is,

$$10.87, 10.37, 10.03, 8.15, 7.6, 6.55.$$

Using Table 5 for $k = 3.5$, $r = 6$, we have that the BLUE for α is:

$$\begin{aligned} \hat{\alpha} &= 10.87 \times 3.44 - 10.37 \times 2.02 - 10.03 \times 1.7 - 8.15 \times 2.10 - 7.6 \times 1.55 + 6.55 \times 5.03 \\ &= 3.45, \end{aligned}$$

Using Table 7, we obtained the estimated standard error for $\hat{\alpha}$ to be

$$S.E.(\hat{\mu}) = 3.45\sqrt{0.23929} = 1.69$$

Using Table 6 for $k = 3.5$, $r = 6$, we have that the BLUE for b is

$$\begin{aligned}\hat{b} &= -10.87 \times 4.80 + 10.37 \times 2.04 + 10.03 \times 2.85 + 8.15 \times 3.52 + 7.6 \times 4.11 - 6.55 \times 7.37 \\ &= 6.86.\end{aligned}$$

Using Table 7, we obtained the estimated standard error for $\hat{b}$ to be

$$S.E.(\hat{b}) = 6.86\sqrt{0.69} = 5.70$$

r	$k = 2.5$	$k = 3$	$k = 3.5$	$k = 4$	$k = 4.5$	$k = 5$
3	-0.39628	-0.67253	-0.93153	-1.18546	-1.43740	-1.68844
	-2.28280	-2.30989	-2.46020	-2.65333	-2.86643	-3.09047
	3.67907	3.98241	4.39172	4.83879	5.30383	5.77890
4	0.08390	0.36628	0.98284	2.23483	4.93390	12.28310
	-1.59977	-2.55895	-4.10090	-6.74035	-11.87186	-25.00456
	-2.66628	-3.83842	-5.74126	-8.98713	-15.26382	-31.25570
	5.18215	7.03109	9.85932	14.49265	23.20178	44.97716
5	0.06433	0.25227	0.61255	1.22095	2.20221	3.77892
	-0.83573	-1.31959	-2.00272	-2.96920	-4.35933	-6.42456
	-1.39289	-1.97938	-2.80381	-3.95894	-5.60485	-8.03070
	-1.89939	-2.54491	-3.46353	-4.75072	-6.57961	-9.26619
	5.06368	6.59161	8.65751	11.45791	15.34159	20.94252
6	0.05099	0.19033	0.44115	0.83247	1.40238	2.20284
	-0.51822	-0.81763	-1.21523	-1.73445	-2.40705	-3.27690
	-0.86370	-1.22645	-1.70132	-2.31259	-3.09478	-4.09612
	-1.17777	-1.57686	-2.10163	-2.77511	-3.63300	-4.72629
	-1.47221	-1.89224	-2.45190	-3.17156	-4.08712	-5.25144
	4.98091	6.32285	8.02892	10.16124	12.81957	16.14790
10	-0.00155	-0.00440	-0.00223	-0.01223	-0.01648	-0.02069
	-0.01246	-0.01372	0.00410	-0.01421	-0.01399	-0.01364
	-0.02076	-0.02059	0.00574	-0.01895	-0.01798	-0.01704
	-0.02831	-0.02647	0.00709	-0.02274	-0.02111	-0.01967
	-0.03539	-0.03176	0.00827	-0.02598	-0.02375	-0.02185
	-0.43398	-0.44726	0.83452	-0.46255	-0.46735	-0.47111
	0.95739	0.99739	-2.05534	1.04585	1.06164	1.07420
	-0.67657	-0.67961	1.26150	-0.68186	-0.68231	-0.68260
	-0.06098	-0.04971	0.01211	-0.03618	-0.03183	-0.02841
	1.31261	1.27613	-0.07575	1.22885	1.21316	1.20080

TABLE 5. Coefficients of the BLUE for α in terms of b.

r	$k = 2.5$	$k = 3$	$k = 3.5$	$k = 4$	$k = 4.5$	$k = 5$
3	0.57493	0.91662	1.21424	1.49452	1.76581	2.03173
	3.10453	2.97993	3.06617	3.22463	3.41620	3.62566
	-3.67946	-3.89654	-4.28042	-4.71915	-5.18201	-5.65739
4	-0.16832	-0.63722	-1.56650	-3.34699	-7.04898	-16.90941
	2.64841	3.99737	6.12633	9.71121	16.59580	34.07033
	4.41401	5.99605	8.57686	12.94827	21.33745	42.58792
	-6.89410	-9.35621	-13.13669	-19.31249	-30.88427	-59.74884
5	-0.13966	-0.47463	-1.04811	-1.94912	-3.33327	-5.48303
	1.52954	2.22987	3.18893	4.51182	6.37709	9.10590
	2.54923	3.34481	4.46450	6.01576	8.19911	11.38238
	3.47623	4.30047	5.51497	7.21891	9.62505	13.13351
	-7.41534	-9.40052	-12.12030	-15.79736	-20.86797	-28.13876
6	-0.11857	-0.38113	-0.79766	-1.39593	-2.21862	-3.32708
	1.02750	1.47211	2.03826	2.75352	3.65634	4.80015
	1.71249	2.20817	2.85357	3.67136	4.70101	6.00018
	2.33522	2.83908	3.52499	4.40564	5.51857	6.92329
	2.91902	3.40689	4.11249	5.03501	6.20839	7.69254
	-7.87566	-9.54513	-11.73165	-14.46961	-17.86569	-22.08908
10	-0.00086	-0.00168	-0.00223	-0.00256	-0.00274	-0.00283
	0.00491	0.00452	0.00410	0.00372	0.00339	0.00310
	0.00818	0.00678	0.00574	0.00496	0.00436	0.00388
	0.01116	0.00871	0.00709	0.00595	0.00511	0.00448
	0.01395	0.01045	0.00827	0.00680	0.00575	0.00497
	1.00968	0.90396	0.83452	0.78554	0.74920	0.72119
	-2.53035	-2.24240	-2.05534	-1.92442	-1.82782	-1.75371
	1.59718	1.39219	1.26150	1.17124	1.10532	1.05514
	0.02404	0.01636	0.01211	0.00947	0.00771	0.00647
	-0.13788	-0.09889	-0.07575	-0.06071	-0.05028	-0.04269

TABLE 6. Coefficients of the BLUE for b in tems of b.

4. **Conclusion**

In the present study, we have introduced the concepts of "Records" for a given phenomenon that is probabilistically characterized by the half logistic and inverse Weibull. We have developed the analytical structure of the estimates of the records. The usefulness of the analytical results of the half logistics pdf were illustrated by analyzing two real applications, namely, the air conditioning system of Boeing 720 and the Electrical Insulation Data that were initially analyzed by Balakrishnan and

r	$k = 2.5$	$k = 3$	$k = 3.5$	$k = 4$	$k = 4.5$	$k = 5$
3	0.5988546	0.5156701	0.4844271	0.4702911	0.46347	0.46023
	1.181577	0.9067008	0.7869827	0.7205605	0.67856	0.64971
	-0.814424	-0.6652533	-0.6037476	-0.5715508	-0.55236	-0.53993
4	0.41967	0.57127	0.80749	1.1947	1.91954	3.72366
	1.20515	1.43263	1.83086	2.50222	3.76888	6.92783
	-0.69477	-0.89239	-1.20631	-1.72127	-2.68335	-5.07373
5	0.21924	0.29459	0.39435	0.52628	0.70485	0.95674
	0.77532	0.8699	1.02113	1.23163	1.52146	1.93268
	-0.40125	-0.49781	-0.62792	-0.7997	-1.0311	-1.35604
6	0.13595	0.18253	0.23929	0.30742	0.38919	0.48799
	0.56707	0.61452	0.69006	0.78784	0.90838	1.05556
	-0.2695	-0.32864	-0.40135	-0.488	-0.59119	-0.71483
10	0.00327	0.00306	0.00279	0.00252	0.00226	0.00203
	0.00326	0.00219	0.00157	0.00118	0.00092	0.00074
	-0.00129	-0.00101	-0.00081	-0.00066	-0.00055	-0.00046

TABLE 7. Variance Covariance of α and b in terms of b.

Chan, [9]. In addition to showing that both data sets fit the half logistic probability distribution, our estimate of the half logistic parameter are as good as the published estimates using classical approach. Thus, the present method offers an advantage in the computational aspect as well as reduced sample sizes. We have also developed the estimates of the location and scale parameters of the inverse Weibull probability distribution given the shape parameter and also the estimates of the shape parameter has been obtained given that the location and scale parameters are known. In addition, we have tabulated the means and variance of lower record values from the inverse Weibull probability distribution function. Coefficients of the best linear unbiased estimates of the location and scale parameters of the inverse Weibull probability distribution function have been obtained. In addition we have shown the importance of our results using simulation study.

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