Neural, Parallel and Scientific Computations 16 (2008) 189–200

Overlapping operator splitting methods and applications in stiff differential equations

Jürgen Geiser[1] and Christos Kravvaritis[2]

[1]Department of Mathematics, Humboldt Universität zu Berlin,
Unter den Linden 6, D-10099 Berlin, Germany
geiser@mathematik.hu-berlin.de

[2]Department of Mathematics, University of Athens,
Panepistimiopolis 15784, Athens, Greece
ckrav@math.uoa.gr

Abstract: *In this article we study the stability of an overlapping operator-splitting methods based on iterative methods. We discuss the overlapping iterative Operator Splitting method in the context of decoupling the stiff and non-stiff operators. In the context of stabilisation the stiff operators, we present the overlapping ideas as extension to the standard iterative operator splitting method. The efficiency of considering the overlapping method instead of the standard method whole domain in the is discussed. We apply our theoretical results on model problems in stiff parabolic partial differential equations.*

Key words: Operator splitting method, Overlapping operator splitting methods, Schwarz alternating methods, Consistency analysis, Parabolic equations
AMS Subject Classification: 35K45, 35K90, 47D60, 65M06, 65M55

1 Introduction

Overlapping Schwarz waveform relaxation is the name for a combination of two standard algorithms, Schwarz alternating method and wave form relaxation algorithm to solve evolution problems in parallel. The method is defined by partitioning the spatial domain into overlapping sub-domains, as in the classical Schwarz method.

The combined time-space iterative operator-splitting method combines the Schwarz-wave formrelaxation and the iterative operator-splitting method.

The outline of the paper is as follows. For our mathematical model we describe the convection-diffusion-reaction equation in section 2. The Fractional Splitting is introduced in section 3. For the overlapping Schwarz waveform-relaxation method we derive the error-analysis for the scalar and systems (coupled or decoupled systems) and presented the results in section 3. In section 7 we present the numerical results from the solution of selective model problems. We end the article in section 6 with conclusion and comments.

Received January 15, 2008

2 Mathematical Model and Methods

2.1 Model-Problem

The motivation for the study presented below is coming from a computational simulation of heat-transfer [12] and convection-diffusion-reaction-equations [10], [17], [18] and [16].

In our paper we concentrate us to a one dimensional convection-diffusion-reaction equation as our model problem and given by

$$u_t \; - \; D \, u_{xx} \; + \; \nu \, u_x \; = \; -\lambda \, u \; , \; \text{in} \; \Omega \times (0, T) \; , \tag{1}$$

$$u(x, 0) = u_0 \; , \; \text{(Initial-Condition)} \; , \tag{2}$$

$$u(x, t) = u_1 \; , \; \text{on} \; \partial\Omega \times (0, T) \; , \; \text{(Dirichlet-Boundary-Condition)} \; . \tag{3}$$

The unknown $u = u(x,t)$ is considered in $\Omega \times (0,T) \subset \mathbb{R} \times \mathbb{R}$, where $\Omega = [0, L]$. The parameters $u_0, u_1 \in \mathbb{R}^+$ are constants and used as initial- and boundary-parameter respectively. The parameter λ is a constant factor, for example a decay-rate of a chemical reaction. D is constant factor, for example the diffusion factor of a transport-process and **v** is a constant factor, for example the velocity-rate of a transport-process.

The aim of this paper is to present a new method based on a mixed discretization method with Fractional-Splitting and Domain decomposition methods for an effective solver-methods of strong coupled parabolic differential equations.

In the next subsection we discuss the decoupling of the time-scale with a first order fractional splitting-method.

3 Overlapping Schwarz wave form relaxation for the solution of convection reaction diffusion equation

In this section we shall present the necessary conditions for the convergence of the overlapping Schwarz wave form relaxation method for the solution of the convection-reaction diffusion equation with constant coefficients. We will utilize the convergence analysis for the solution of the decoupled and coupled system of convection reaction diffusion equation to elaborate the impact of the coupling on the convergence of the overlapping Schwarz wave form relaxation.

Given the following model problem

$$u_t + Lu = f \; , \text{in} \; \Omega \times (0, T) \; , \overline{\Omega} \times (0, T) := \overline{\Omega}_1 \times (0, T) \cup \overline{\Omega}_2 \times (0, T) \; , \tag{4}$$

$$u(x, 0) = u_0 \; , \; \text{(Initial-Condition)} \; , \tag{5}$$

$$u = g \; , \text{on} \; \partial\Omega \times (0, T) \; , \tag{6}$$

where L denotes for each time t a second-order partial differential operator $Lu = -\nabla D \nabla u + v \nabla u + cu$ for the given coeffients $D \in \mathbb{R}^+, v \in \mathbb{R}^n, c \in \mathbb{R}^+$, and n is the dimension of the space. Iteration step consists of two half steps, associated with the two subdomains and we solve 2 subproblems

$$u_1t + Lu_1^n = f \text{ , in } \Omega_1 \times (0,T) \text{ ,} \tag{7}$$
$$u_1(x,0) = u_{10} \text{ , (Initial-Condition) ,} \tag{8}$$
$$u_1^n = g \text{ , on } L_0 = \partial\Omega \times (0,T) \cap \partial\Omega_1 \times (0,T) \text{ ,} \tag{9}$$
$$u_1^n = u_2^{n-1} \text{ , on } L_2 = \partial\Omega_1 \times (0,T) \backslash \partial\Omega \times (0,T) \text{ ,} \tag{10}$$

$$u_{2t} + Lu_2^n = f \text{ , in } \Omega_2 \times (0,T) \text{ ,} \tag{11}$$
$$u_2(x,0) = u_{20} \text{ , (Initial-Condition) ,} \tag{12}$$
$$u_2^n = g \text{ , on } L_3 = \partial\Omega \times (0,T) \cap \partial\Omega_2 \times (0,T) \text{ ,} \tag{13}$$
$$u_2^n = u_1^n \text{ , on } L_1 = \partial\Omega_2 \times (0,T) \backslash \partial\Omega \ \times (0,T); \text{ ,} \tag{14}$$

4 The iterative splitting method

Because of improved The following algorithm is based on the iteration with fixed splitting discretization step-size τ. On the time interval $[t^n, t^{n+1}]$ we solve the following subproblems consecutively for $i = 0, 2, \ldots, 2m$.

Initial idea:

$$\frac{\partial c_i(x,t)}{\partial t} = Ac_i(x,t) \ + \ Bc_{i-1}(x,t), \text{ with } c_i(t^n) = c^n \tag{15}$$
$$\text{and } c_0(t^n) = c^n \ , \ c_{-1} = 0.0,$$
$$\text{and } c_i(x,t) = c_{i-1}(x,t) = c_1 \ , \text{ on } \partial\Omega \times (0,T) \ ,$$
$$\frac{\partial c_{i+1}(x,t)}{\partial t} = \ Ac_i(x,t) \ + \ Bc_{i+1}(x,t), \tag{16}$$
$$\text{with } c_{i+1}(t^n) = c^n \ ,$$
$$\text{and } c_i(x,t) = c_{i-1}(x,t) = c_1 \ , \text{ on } \partial\Omega \times (0,T) \ ,$$

where c^n is the known split approximation at the time level $t = t^n$, cf. [8].

5 The overlapping iterative operator splitting method

The idea behind the overlapping iterative operator splitting method is balancing of the eigenvalues of the different operators by weighting.

$$\frac{\partial c_i(x,t)}{\partial t} = (1-\omega_1)Ac_i(x,t) \ + \ \omega_1(A+B)c_i(x,t) + (1-\omega_1)Bc_{i-1}(x,t), \tag{17}$$
$$\text{with } c_i(t^n) = c^n \text{and } c_0(t^n) = c^n \ , \ c_{-1} = c^n, \tag{18}$$
$$\frac{\partial c_{i+1}(x,t)}{\partial t} = (1-\omega_1)Ac_i(x,t) \ + \ \omega_1(A+B)c_i(x,t) + (1-\omega_1)Bc_{i+1}(x,t), \tag{19}$$
$$\text{with } c_{i+1}(t^n) = c^n \ , \tag{20}$$

where c^n is the known split approximation at the time level $t = t^n$, cf. [8].

6 Consistency and stability analysis of the combined method

Theorem 1 *Let us consider the nonlinear operator-equation in a Banach space* $\mathbf{X}$

$$\begin{aligned} \partial_t c(t) &= A_1(c(t)) + A_2(c(t)) + B_1(c(t)) + B_2(c(t)), \quad 0 < t \leq T \ , \\ c(0) &= c_0 \ , \end{aligned} \tag{21}$$

where $A_1, A_2, B_1, B_2, A_1 + A_2 + B_1 + B_2 : \mathbf{X} \to \mathbf{X}$ *are given linear operators being generators of the* C_0*-semigroup and* $c_0 \in \mathbf{X}$ *is a given element. Then the iteration process* (17)–(20) *is convergent and the rate of the convergence is of second order.*

We obtain the iterative result :

Proof. Let us consider the iteration (17)–(20) on the sub-interval $[t^n, t^{n+1}]$. For the error function $e_i(t) = c(t) - c_i(t)$ we have the relations

$$\begin{aligned} &\partial_t e_{i,j}(t) = A_1(e_{i,j}(t)) + A_2(e_{i,j-1}(t)) + B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1}(t)), \\ &t \in (t^n, t^{n+1}], \ e_{i,j}(t^n) = 0 \ , \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\partial_t e_{i+1,j}(t) = A_1(e_{i,j}(t)) + A_2(e_{i,j-1}(t)) + B_1(e_{i+1,j}(t)) + B_2(e_{i-1,j-1}(t)), \\ &t \in (t^n, t^{n+1}], \ e_{i+1,j}(t^n) = 0 \ , \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\partial_t e_{i,j+1}(t) = A_1(e_{i,j}(t)) + A_2(e_{i,j+1}(t)) + B_1(e_{i+1,j}(t)) + B_2(e_{i-1,j-1}(t)), \\ &t \in (t^n, t^{n+1}], \ e_{i,j+1}(t^n) = 0 \ , \end{aligned} \tag{24}$$

and

$$\begin{aligned} &\partial_t e_{i,j}(t) = A_1(e_{i,j}(t)) + A_2(e_{i,j+1}(t)) + B_1(e_{i+1,j}(t)) + B_2(e_{i+1,j+1}(t)), \\ &t \in (t^n, t^{n+1}], \ e_{i,j}(t^n) = 0 \ , \end{aligned} \tag{25}$$

for $i, j = 0, 2, 4, \ldots$, with $e_{0,0}(0) = 0$ and $e_{-1,0} = e_{0,-1} = e_{-1,-1}(t) = c(t)$.

In the following we derive the linear system of equations. We use the notations $\mathbf{X}^2$ for the product space $\mathbf{X} \times \mathbf{X}$ enabled with the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$ $(u, v \in \mathbf{X})$. The elements $\mathcal{E}_i(t)$, $\mathcal{F}_i(t) \in \mathbf{X}^2$ and the linear operator $\mathcal{A} : \mathbf{X}^2 \to \mathbf{X}^2$ are defined as follows

$$\mathcal{E}_{i,j}(t) = \begin{bmatrix} e_{i,j}(t) \\ e_{i+1,j}(t) \\ e_{i,j+1}(t) \\ e_{i+1,j+1}(t) \end{bmatrix} ; \quad \mathcal{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ A_1 & A_2 & 0 & 0 \\ A_1 & A_2 & B_1 & 0 \\ A_1 & A_2 & B_1 & B_2 \end{bmatrix} , \tag{26}$$

$$\mathcal{F}_{i,j}(t) = \begin{bmatrix} A_2(e_{i,j-1}(t)) + B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1}) \\ B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1}) \\ B_2(e_{i-1,j-1}) \\ 0 \end{bmatrix} . \tag{27}$$

Then, using the notations (27), the relations (22)–(25) can be written in the form

$$\begin{aligned} \partial_t \mathcal{E}_{i,j}(t) &= \mathcal{A}\mathcal{E}_{i,j}(t) + \mathcal{F}_{i,j}(t), \quad t \in (t^n, t^{n+1}], \\ \mathcal{E}_{i,j}(t^n) &= 0. \end{aligned} \tag{28}$$

Due to our assumptions, $\mathcal{A}$ is a generator of the one-parameter C_0 semigroup $(\mathcal{A}(t))_{t \geq 0}$. We also assume the estimation of our term $\mathcal{F}_i(t)$ with the growth conditions.

We could estimate the right hand side $\mathcal{F}_i(t)$ in the following lemma :

Lemma 1 *Let us consider the the bounded Jacobian of $A(u)$ and $B(u)$*

We could then estimate the $\mathcal{F}_i(t)$ as

$$||\mathcal{F}_{i,j}(t)|| \leq C ||e_{i-1,j-1}|| \ . \tag{29}$$

Proof. We have the following norm

$||\mathcal{F}_{i,j}(t)|| = \max\{\mathcal{F}_{i,j,1}(t), \mathcal{F}_{i,j,2}(t), \mathcal{F}_{i,j,3}(t), \mathcal{F}_{i,j,4}(t)\}$.

We have to estimate each term :

$$\begin{aligned} ||\mathcal{F}_{i,j,1}(t)|| &\leq ||A_2(e_{i,j-1}(t)) + B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1})|| \leq C_1 ||(e_{i-1,j-1})|| \\ ||\mathcal{F}_{i,j,2}(t)|| &\leq ||B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1})|| \leq C_2 ||(e_{i-1,j-1})|| \\ ||\mathcal{F}_{i,j,3}(t)|| &\leq ||B_2(e_{i-1,j-1})|| \leq C_3 ||(e_{i-1,j-1})|| \end{aligned}$$

So we obtain the estimation :

$||\mathcal{F}_{i,j}(t)|| \leq \tilde{C} ||e_{i-1,j-1}(t)||$

where $\tilde{C}$ is the maximum value of C_1, C_2 and C_3. □

Hence using the variations of constants formula, the solution of the abstract Cauchy problem (28) with homogeneous initial condition can be written as

$$\mathcal{E}_{i,j}(t) = \int_{t^n}^{t} \exp(\mathcal{A}(t-s)) \mathcal{F}_{i,j}(s) ds, \quad t \in [t^n, t^{n+1}]. \tag{30}$$

(See, e.g. [5].) Hence, using the denotation

$$\|\mathcal{E}_{i,j}\|_\infty = \sup_{t \in [t^n, t^{n+1}]} \|\mathcal{E}_{i,j}(t)\| \quad , \tag{31}$$

we have

$$\begin{aligned} \|\mathcal{E}_{i,j}\|(t) &\leq \|\mathcal{F}_{i,j}\|_\infty \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\| ds = \\ &= C \ \|e_{i-1,j-1}\| \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\| ds, \quad t \in [t^n, t^{n+1}]. \end{aligned} \tag{32}$$

We have estimate $||\mathcal{F}_{i,j}|| \leq C ||e_{i-1,j-1}||$, where C is a constant that bounds the nonlinear terms of $\mathcal{F}_{i,j}(t)$.

Since $(\mathcal{A}(t))_{t \geq 0}$ is a semigroup therefore the so called *growth estimation*

$$\| \exp(\mathcal{A}t) \| \leq K \exp(\omega t); \quad t \geq 0 \ , \tag{33}$$

holds with some numbers $K \geq 0$ and $\omega \in \mathbb{R}$, see [5].

- Assume that $(\mathcal{A}(t))_{t\geq 0}$ is a bounded or exponentially stable semigroup, i.e. (33) holds with some $\omega \leq 0$. Then obviously the estimate

$$\| \exp(\mathcal{A}t)\| \leq K; \quad t \geq 0 \ , \tag{34}$$

holds, and, hence on base of (32), we have the relation

$$\|\mathcal{E}_{i,j}\|(t) \leq K\tau_n \|e_{i-1,j-1}\|, \quad t \in (0, \tau_n). \tag{35}$$

- Assume that $(\mathcal{A}(t))_{t\geq 0}$ has an exponential growth with some $\omega > 0$. Using (32) we have

$$\int_{t^n}^{t^{n+1}} \|\exp(\mathcal{A}(t-s))\| ds \leq K_\omega(t), \quad t \in [t^n, t^{n+1}], \tag{36}$$

where

$$K_\omega(t) = \frac{K}{\omega} \left(\exp(\omega(t - t^n)) - 1\right), \quad t \in [t^n, t^{n+1}] \ , \tag{37}$$

and hence

$$K_\omega(t) \leq \frac{K}{\omega} \left(\exp(\omega \tau_n) - 1\right) = K\tau_n + \mathcal{O}(\tau_n^2) \ , \tag{38}$$

so the estimations (35) and (38) result in

$$\|\mathcal{E}_{i,j}\|_\infty = K\tau_n \|e_{i-1,j-1}\| + \mathcal{O}(\tau_n^2). \tag{39}$$

Taking into the account the definition of $\mathcal{E}_i$ and the norm $\| \cdot \|_\infty$, we obtain

$$\|e_{i,j}\| = K\tau_n \|e_{i-1,j-1}\| + \mathcal{O}(\tau_n^2), \tag{40}$$

and hence

$$\|e_{i+1,j+1}\| = K_1 \tau_n^2 \|e_{i-1,j-1}\| + \mathcal{O}(\tau_n^3), \tag{41}$$

which proves our statement. $\square$

7 Numerical Results

In this section we will present the numerical results from the solution of the Convection-diffusion-reaction equation using several variations of the proposed methods in comparison with already known classical methods.

7.1 First numerical example

We consider the one-dimensional convection-reaction-diffusion equation

$$\partial_t u + v \partial_x u - \partial_x D \partial_x u = -\lambda u \ , \text{ in } \Omega \times (T_0, T_f) \ , \tag{42}$$

$$u(x, 0) = u_{ex}(x, 0) \ , \text{ (Initial-Condition)} \ , \tag{43}$$

$$u(x, t) = u_{ex}(x, t) \ , \text{on } \partial\Omega \times (T_0, T_f) \ , \tag{44}$$

where $\Omega \times [T_0, T_f] = [0, 150] \times [100, 10^5]$.

The exact solution is given as

$$u_{ex}(x,t) = \frac{u_0}{2\sqrt{D\pi t}} \exp(-\frac{(x-vt)^2}{4Dt}) \exp(-\lambda t) \ . \tag{45}$$

The initial condition and the Dirichlet boundary conditions are defined using the exact solution (57) at starting time $T_0 = 100$ and with $u_0 = 1.0$. We have $\lambda = 10^{-5}$, $v = 0.001$ and $D = 0.0001$.

7.2 Solution using classical methods

7.2.1 A-B splitting combined with Schwarz wave form relaxation method In order to solve the model problem using overlapping Schwarz wave form relaxation method, we divide the domain Ω in two overlapping sub-domains $\Omega_1 = [0, L_2]$ and $\Omega_2 = [L_1, L]$, where $L_1 < L_2$, and $\Omega_1 \bigcap \Omega_2 = [L_1, L_2]$ is the overlapping region for Ω_1 and Ω_2.

To start the wave form relaxation algorithm we consider first the solution of the model problem (54) over Ω_1 and Ω_2 as follows

$$\begin{aligned} v_t &= Dv_{xx} - \nu v_x - \lambda v \text{ over } \Omega_1 \ , \ \ t \in [T_0, T_f] \\ v(0,t) &= f_1(t) \ , \ \ t \in [T_0, T_f] \\ v(L_2,t) &= w(L_2,t) \ , \ \ t \in [T_0, T_f] \\ v(x,T_0) &= u_0 \ \ \ x \in \Omega_1, \end{aligned} \tag{46}$$

$$\begin{aligned} w_t &= Dw_{xx} - \nu w_x - \lambda w \text{ over } \Omega_2 \ , \ \ t \in [T_0, T_f] \\ w(L_1,t) &= v(L_1,t) \ , \ \ t \in [T_0, T_f] \\ w(L,t) &= f_2(t) \ , \ \ t \in [T_0, T_f] \\ w(x,T_0) &= u_0 \ \ \ x \in \Omega_2, \end{aligned} \tag{47}$$

where $v(x,t) = u(x,t)|_{\Omega_1}$ and $w(x,t) = u(x,t)|_{\Omega_2}$.

Then the Schwarz wave form relaxation is given by

$$\begin{aligned} v_t^{k+1} &= Dv_{xx}^{k+1} - \nu v_x^{k+1} - \lambda v^{k+1} \text{ over } \Omega_1 \ , \ \ t \in [T_0, T_f] \\ v^{k+1}(0,t) &= f_1(t) \ , \ \ t \in [T_0, T_f] \\ v^{k+1}(L_2,t) &= w^k(L_2,t) \ , \ \ t \in [T_0, T_f] \\ v^{k+1}(x,T_0) &= u_0 \ \ \ x \in \Omega_1, \end{aligned} \tag{48}$$

$$\begin{aligned} w_t^{k+1} &= Dw_{xx}^{k+1} - \nu w_x^{k+1} - \lambda w^{k+1} \text{ over } \Omega_2 \ , \ \ t \in [T_0, T_f] \\ w^{k+1}(L_1,t) &= v^k(L_1,t) \ , \ \ t \in [T_0, T_f] \\ w^{k+1}(L,t) &= f_2(t) \ , \ \ t \in [T_0, T_f] \\ w^{k+1}(x,T_0) &= u_0 \ \ \ x \in \Omega_2. \end{aligned} \tag{49}$$

For the solution of (48) and (49) we will apply the sequential operator splitting method (A-B splitting). For this purpose we divide each of these two equations in terms of the operators $A = D\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial x}$ and $B = -\lambda$. The splitting scheme for each of them is given in the following form:

$$\frac{\partial u^*(x,t)}{\partial t} = D\, u^*_{xx} - \nu\, u^*_x\,, \quad \text{with } u^*(x,t^n) = u_0\,, \tag{50}$$

$$\frac{\partial u^{**}(x,t)}{\partial t} = -\lambda u^{**}(t)\,, \quad \text{with } u^{**}(x,t^n) = u^*(x,t^{n+1})\,, \tag{51}$$

where $u^*(x,t) = u^{**}(x,t) = u_1$, on $\partial\Omega \times (0,T)$, are the Dirichlet-Boundary-Conditions for the equations. The solution is given as $u(x,t^{n+1}) = u^{**}(x,t^{n+1})$. We obtain an exact method because of commuting operators.

For the discretization of equation (50) we apply the finite-difference method for the spatial discretization and the implicite Euler method for the time discretization. The discretization is given as

$$\begin{aligned}
&\frac{1}{t^{n+1}-t^n}\,(u^*(x_i,t^{n+1}) - u^*(x_i,t^n)) && (52)\\
&= D\,\frac{1}{h_i^2}(-u^*(x_{i+1},t^{n+1}) + 2u^*(x_i,t^{n+1}) - u^*(x_{i-1},t^{n+1}))\\
&- \nu\,\frac{1}{h_i}\,(u^*(x_i,t^{n+1}) - u^*(x_{i-1},t^{n+1}))\,,\\
&\quad \text{with } u^*(x_1,t^n) = u^*(x_2,t^n) = u_0 \text{ and } u^*(x_0,t^n) = u^*(x_m,t^n) = 0\\
&u^{**}(x,t) = \exp(-\lambda(t-t^n)\, u^*(x,t^{n+1})\,, && (53)
\end{aligned}$$

where $h_i = x_{i+1} - x_i$ and we assume a partition with m-nodes.

We are interested in specifying the error between the solution obtained with the above described algorithm and the exact solution. We provide a variety of results for several sizes of space- and time-partition, and also for various overlap sizes. Precisely, we treat the cases $h = 1,\ 0.5,\ 0.25$ as spatial step-size, $\Delta t = 5,\ 10,\ 20$ as time step. The considered subdomains are $\Omega_1 = [0,80]$ and $\Omega_2 = [70,150]$, $\Omega_1 = [0,60]$ and $\Omega_2 = [30,150]$ and $\Omega_1 = [0,100]$ and $\Omega_2 = [30,150]$, with overlap sizes 10, 30 and 70, respectively. Both the approximated and the exact solution are evaluated at the end-time $t = 10^5$. The errors given in Table 1 are the maximum errors that occurred over the whole space domain, i.e. they are calculated using the $\infty-$norm for vectors.

time-step	err	err	err	err	err	err	err	err	err
$\Delta t = 5$	$2.85e-3$	$2.24e-3$	$1.28e-3$	$2.66e-4$	$2.21e-4$	$2.20e-4$	$2.09e-5$	$1.99e-5$	$1.97e-5$
$\Delta t = 10$	$3.94e-3$	$2.61e-3$	$2.56e-3$	$3.03e-4$	$3.02e-4$	$3.01e-4$	$4.55e-5$	$4.34e-5$	$4.29e-5$
$\Delta t = 20$	$5.03e-3$	$2.81e-3$	$2.73e-3$	$8.51e-4$	$5.22e-4$	$5.14e-4$	$8.10e-4$	$5.66e-4$	$4.88e-4$
overlap	10	30	70	10	30	70	10	30	70
space-step	$h = 1$			$h = 0.5$			$h = 0.25$		

Table 1: Error for the scalar convection diffusion reaction-equation using the Schwarz waveform relaxation method for three different sizes of overlapping 10, 30 and 70.

7.3 Solution using the proposed method

For the solution of (54) with the combined time-space iterative splitting method we divide again the equation in terms of the operators $A = D\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial x}$ and $B = -\lambda$. We will utilize the proposed scheme (52)–(53).

The index $k = 0, 1, \ldots, p$ is associated with the subdomains, i.e. for $k = 0, \ldots, p/2$ we are working on Ω_1 and for $k = p/2 + 1, \ldots, p$ on Ω_2. For the first set of values for k we have actually only the effect of the restrictions of the operators A and B on Ω_1. Similarly, the second set of values for k indicates the action of the restrictions of both operators on Ω_2. The outline of the method in Section 3, which is also adopted here, is given without loss of generality for a subdomain-determining value $k = p/2$, just for an overview. This crucial value is determined appropriately according to the three cases of the overlapping subdomains, which we examine in our experiments.

The indices i and j are related to the time- and space-discretization, respectively. For every $k = 0, \ldots, p/2$ and for every interval of the space-discretization we solve the appropriate problems on Ω_1, for every interval of the time-discretization. Similarly for $k = p/2 + 1, \ldots, p$ on Ω_2.

From a software development point of view, the above described numerical scheme can be realized with three "for" loops. The first, outer loop is for all values of k. After this loop there exists a control for k, which discriminates two cases for $k < p/2$ and for $k \geq p/2+1$, and sets up the data of the algorithm appropriately for Ω_1 or Ω_2, respectively. The second, middle loop is running for all values of i and the third, inner loop is for all values of j.

By a closer examination of the scheme (52)–(53), taking into account the definitions (50)–(51), we observe that the problems to be solved in the innermost loop are of the form $\partial_t c = Ac + Bc,\ c(x, t^n) = c^n$, where c appears with appropriate indices i and j. These problems are solved with suitable modification and implementation of the iterative operator splitting scheme (15)–(18). The notion of the iterative process takes place in both time- and space-dimensions.

We are interested again in specifying the error between the solution obtained with the above described algorithm and the exact solution. We provide the same variety of results as in the previous subsection, so that a comparison between the proposed and classical methods can be established. Both the approximated and the exact solution are evaluated at the end-time $t = 10^5$. The errors given in the following tables are the maximum errors that occurred over the whole space domain, i.e. they are calculated using the $\infty-$norm for vectors. The results are given in Table 2.

time-step	err	err	err	err	err	err	err	err	err
$\Delta t = 5$	$4.38e-2$	$1.47e-2$	$3.49e-3$	$2.59e-4$	$2.13e-4$	$1.54e-4$	$7.23e-6$	$6.49e-6$	$8.29e-6$
$\Delta t = 10$	$5.12e-2$	$2.26e-2$	$7.46e-3$	$2.45e-4$	$2.22e-4$	$2.15e-4$	$3.49e-5$	$3.47e-5$	$3.37e-5$
$\Delta t = 20$	$6.14e-2$	$4.39e-2$	$1.20e-2$	$7.43e-4$	$5.21e-4$	$4.53e-4$	$5.23e-4$	$5.42e-4$	$3.21e-4$
overlap	10	30	70	10	30	70	10	30	70
space-step	$h = 1$			$h = 0.5$			$h = 0.25$		

Table 2: Error for the scalar convection diffusion reaction-equation using the Schwarz waveform relaxation method for three different sizes of overlapping 10,30 and 70.

7.4 Second numerical example

We consider the one-dimensional convection-reaction-diffusion equation

$$\partial_t u + v\partial_x u - \partial_x D_x \partial_x u - \partial_y D_y \partial_y u = -\lambda u \ , \ \text{in } \Omega \times (T_0, T_f) \ , \tag{54}$$

$$u(x, y, 0) = u_{ex}(x, y, 0) \ , \ \text{(Initial-Condition)} \ , \tag{55}$$

$$u(x, y, t) = u_{ex}(x, y, t) \ , \text{on } \partial\Omega \times (T_0, T_f) \ , \tag{56}$$

where $\Omega \times [T_0, T_f] = [0, 150] \times [0, 150] \times [100, 10^5]$.

The exact solution is given as

$$u_{ex}(x, y, t) = \frac{u_0}{4\sqrt{D_x \pi t}\sqrt{D_y \pi t}} \ \exp(-\frac{(x - vt)^2}{4D_x t}) \ \exp(-\frac{y^2}{4D_y t}) \ \exp(-\lambda t) \ . \tag{57}$$

The initial condition and the Dirichlet boundary conditions are defined using the exact solution (57) at starting time $T_0 = 100$ and with $u_0 = 1.0$. We have $\lambda = 10^{-5}$, $v = 0.001$ and $D_x = 0.0001$, $D_y = 0.0005$.

Please do the computations to the 2 dim problem. The decoupling is the same as in the 1 d case.

8 Conclusions and Discussions

We present decomposition methods for complex equations based on the one hand with to classical methods, overlapping Schwarz wave form relaxation method for the space and A-B splitting for time and on the other hand with a combined space-time iterative operator splitting method. The combined method allows more accurate results and improved convergence results.

References

[1] N. Bubner, O. Klein, P. Philip, J. Sprekels, and K. Wilmanski. *A transient model for the sublimation growth of silicon carbide single crystals*. Journal of Crystal Growth, 205: 294-304, 1999.

[2] X.C. Cai. Additive Schwarz algorithms for parabolic convection-diffusion equations. *Numer. Math.*, 60(1):41–61, 1991.

[3] X.C. Cai. Multiplicative Schwarz methods for parabolic problems. *SIAM J. Sci Comput.*, 15(3):587–603, 1994.

[4] D.S. Daoud, M.J. Gander *Overlapping Schwarz Waveform Relaxation for Convection Reaction Diffusion Problems* Procedding of DD13 conference, France, published by CIMNE, Barcelona, Spain, April 2002 (first edition).

[5] K.-J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*. Springer, New York, 2000.

[6] R.E. Ewing. Up-scaling of biological processes and multiphase flow in porous media. *IIMA Volumes in Mathematics and its Applications*, Springer-Verlag, 295 (2002), 195-215.

[7] I. Farago, and Agnes Havasi. *On the convergence and local splitting error of different splitting schemes*. Eötvös Lorand University, Budapest, 2004.

[8] I. Farago, J. Geiser. *Iterative Operator-Splitting methods for Linear Problems.* Preprint No. 1043 of the Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany, June 2005.

[9] M.J. Gander and H. Zhao. *Overlapping Schwarz waveform relaxation for parabolic problems in higher dimension.* In A. Handlovičová, Magda Komorníkova, and Karol Mikula, editors, *Proceedings of Algoritmy 14*, pages 42–51. Slovak Technical University, September 1997.

[10] J. Geiser. *Discretisation Methods with embedded analytical solutions for convection dominated transport in porous media* Proceeding of Numerical Analysis and Applications, Third international conference, Rousse, Bulgaria, 2004, Lect.Notes in Mathematics (Springer), vol.3401, 2005.

[11] J. Geiser, R.E. Ewing, J. Liu. *Operator Splitting Methods for Transport Equations with Nonlinear Reactions.* Proceedings of the Third MIT Conference on Computational Fluid and Solid Mechanic, Cambridge, MA, June 14-17, 2005.

[12] J. Geiser, O. Klein, and P. Philip. *Anisotropic thermal conductivity in apparatus insulation: Numerical study of effects on the temperature field during sublimation growth of silicon carbide single crystals.* Preprint Weierstra-Institut fr Angewandte Analysis und Stochastik, Berlin, 2005.

[13] E. Giladi and H. Keller. Space time domain decomposition for parabolic problems. Technical Report 97-4, Center for research on parallel computation CRPC, Caltech, 1997.

[14] M.S. Glockenbach. *Partial Differential Equation : Analytical and Numerical Methods*. SIAM, Society for Industrial and Applied Mathematics, Philadelphia, OT 79, 2002.

[15] W.H. Hundsdorfer. *Numerical Solution od Advection-Diffusion-Reaction Equations.* Technical Report NM-N9603, CWI, 1996.

[16] W. Hundsdorfer and J.G. Verwer. *Numerical Solution of Time-dependent Advection-Diffusion-Reaction Equations.* Springer Series in Computational Mathematics, 33, Springer Verlag, 2003.

[17] K.H. Karlsen and N.H. Risebro. *Corrected operator splitting for nonlinear parabolic equations*. SIAM J. Numer. Anal., 37(3):980–1003, 2000.

[18] K.H. Karlsen, K.A. Lie, J.R. Natvig, H.F. Nordhaug, and H.K. Dahle. *Operator splitting methods for systems of convection- diffusion equations: nonlinear error mechanisms and correction strategies*. J. Comput. Phys., 173(2):636–663, 2001.

[19] G.I Marchuk. *Some applicatons of splitting-up methods to the solution of problems in mathematical physics.* Aplikace Matematiky, 1 (1968) 103-132.

[20] Gérard A. Meurant. *Numerical experiments with a domain decomposition method for parabolic problems on parallel computers.* In Roland Glowinski, Yuri A. Kuznetsov, Gérard A. Meurant, Jacques Périaux, and Olof Widlund, editors, *Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Philadelphia, PA, 1991. SIAM.

[21] C.V. Pao *Non Linear Parabolic and Elliptic Equation* Plenum Press, New York, 1992.

[22] G. Strang. *On the construction and comparision of difference schemes*. SIAM J. Numer. Anal., 5:506–517, 1968.

[23] J.G. Verwer and B. Sportisse. *A note on operator splitting in a stiff linear case.* MAS-R9830, ISSN 1386-3703, 1998.

[24] Z. Zlatev. *Computer Treatment of Large Air Pollution Models.* Kluwer Academic Publishers, 1995.

[25] H.A. Schwarz. *ber einige Abbildungsaufgaben.* Journal fr Reine und Angewandte Mathematik, 70:105–120, 1869.

[26] C. N. Dawson, Q. Du, and D. F. Dupont. *A finite Difference Domain Decomposition Algorithm for Numerical Solution of the Heat Equation.* Mathematics of Computation, 57:63-71, 1991