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Positive Solutions for Systems of Three-Point Nonlinear Discrete Boundary Value Problems

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Abstract: *Existence of eigenvalues yielding positive solutions for systems of a second order three-point discrete boundary value problem is established. The results are obtained by the use of a Guo-Krasnoselskii fixed point theorem in cones.*

Key words: Discrete three-point boundary value problems, system of difference equations, eigenvalue problem, positive solutions

1 Introduction

We are concerned with determining values of λ, μ (eigenvalues) for which there exist positive solutions for the system of three-point discrete boundary value problems,

$$\begin{aligned} \Delta^2 u(n-1) + \lambda a(n) f(v(n)) = 0, \quad n \in \{1, \ldots, N-1\}, \quad N \geq 2, \\ \Delta^2 v(n-1) + \mu b(n) g(u(n)) = 0, \quad n \in \{1, \ldots, N-1\}, \quad N \geq 2, \end{aligned} \tag{1}$$

$$\begin{aligned} u(0) = 0, \quad u(N) = \alpha u(\eta), \\ v(0) = 0, \quad v(N) = \alpha v(\eta), \end{aligned} \tag{2}$$

where $\eta \in \{1, \ldots, N-1\}, 0 < \alpha < N/\eta$, $\lambda > 0, \mu > 0$ and

(A) $f, g \in C([0, \infty), [0, \infty))$,

(B) $a, b : \{0, \ldots, N\} \to (0, \infty)$,

(C) All of

$$f_0 := \lim_{x \to 0^+} \frac{f(x)}{x}, \qquad g_0 := \lim_{x \to 0^+} \frac{g(x)}{x},$$

$$f_\infty := \lim_{x \to \infty} \frac{f(x)}{x} \quad \text{and} \quad g_\infty := \lim_{x \to \infty} \frac{g(x)}{x}$$

exist as positive real numbers.

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The existence of positive solutions for nonlinear second order multi-point boundary value problems is the last decades in the focus of interest of many researchers. The study of multi-point boundary value problems for second order ordinary differential equations was initiated by Il'in and Moiseev [13]. For a comprehensive bibliography on the subject we refer to the survey paper of Ntouyas [16]. Existence results for positive solutions for second or higher order boundary value problems for discrete difference equations with two- or three-point boundary conditions were studied in [1, 2, 7, 8, 11, 12, 14, 15, 17] and the references cited therein.

Recently the interest of the present authors was focused in the existence of positive solutions for systems of second order multi-point boundary value problems. We refer the interested reader to [3, 9, 10] and the references cited therein. We continue this study here to cover the case of discrete systems.

Note that when $\alpha = 0$ equation (2) reduces to

$$\begin{aligned} u(0) = 0, \quad u(N) = 0, \\ v(0) = 0, \quad v(N) = 0. \end{aligned} \tag{3}$$

The main tool in determining values of λ and μ for which positive solutions (positive with respect to a cone) of (1), (2) exist, is the following well-known Guo-Krasnosel'skii [6] fixed point theorem.

Theorem 1.1 *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(i) $\|Tu\| \le \|u\|, u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|, u \in \mathcal{P} \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \ge \|u\|, u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|, u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2 Some Preliminaries

In this section, we state some preliminary lemmas and the well-known Guo-Krasnosel'skii fixed point theorem. The following lemmas are obvious.

Lemma 2.1 *Let $\eta \in \{1, \dots, N-1\}, 0 < \alpha < N/\eta$; then, for any $y : \{1, \dots, N-1\} \to \mathbb{R}$, the boundary value problem*

$$\Delta^2 u(n-1) + y(n) = 0, \quad t \in \{1, \dots, N-1\}, \tag{4}$$

$$u(0) = 0, \quad u(N) = \alpha u(\eta), \tag{5}$$

has the unique solution

$$u(n) = \frac{n}{N-\alpha\eta}\sum_{s=1}^{N-1}(N-s)y(s) - \frac{\alpha n}{N-\alpha\eta}\sum_{s=1}^{\eta-1}(\eta-s)y(s) - \sum_{s=1}^{n-1}(t-s)y(s). \quad (6)$$

From (6) we have that

$$\begin{aligned} u(n) &\leq -\frac{\alpha n}{N-\alpha\eta}\sum_{s=1}^{\eta-1}(\eta-s)y(s) + \frac{n}{N-\alpha\eta}\sum_{s=1}^{N-1}(N-s)y(s) \\ &\leq \frac{N}{N-\alpha\eta}\sum_{s=1}^{N-1}(N-s)y(s), \quad n \in \{1,\dots,N-1\}, \end{aligned} \quad (7)$$

and

$$\begin{aligned} u(\eta) &= -\frac{\alpha\eta}{N-\alpha\eta}\sum_{s=1}^{\eta-1}(\eta-s)y(s) \\ &\quad + \frac{\eta}{N-\alpha\eta}\sum_{s=1}^{N-1}(N-s)y(s) - \sum_{s=1}^{\eta-1}(\eta-s)y(s) \\ &= \frac{-N}{N-\alpha\eta}\sum_{s=1}^{\eta-1}(\eta-s)y(s) + \frac{\eta}{N-\alpha\eta}\sum_{s=1}^{N-1}(N-s)y(s) \\ &= \frac{-N}{N-\alpha\eta}\sum_{s=1}^{\eta-1}(\eta-s)y(s) + \frac{\eta}{N-\alpha\eta}\sum_{s=1}^{\eta-1}(N-s)y(s) \\ &\quad + \frac{\eta}{N-\alpha\eta}\sum_{s=1}^{\eta-1}(N-s)y(s) \\ &= \frac{N-\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1} sy(s) + \frac{\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)y(s) \\ &\geq \frac{\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)y(s), \quad n \in \{1,\dots,N-1\}. \end{aligned} \quad (8)$$

Lemma 2.2 *Let* $\eta \in \{1,\dots,N-1\}, 0 < \alpha < N/\eta$; *then, for any* $y : \{1,\dots,N-1\} \to \mathbb{R}$, *the Green's function for the boundary value problem*

$$\Delta^2 u(n-1) + y(n) = 0, \quad n \in \{1,\dots,N-1\}, \quad (9)$$

$$u(0) = 0, \quad u(N) = \alpha u(\eta), \quad (10)$$

is given by

$$G(n,s)=\begin{cases}\dfrac{n(N-s)}{N-\alpha\eta}-\dfrac{\alpha n(\eta-s)}{N-\alpha\eta}-(n-s), & s\le n,\ s\le\eta,\\ \dfrac{n(N-s)}{N-\alpha\eta}-\dfrac{\alpha n(\eta-s)}{N-\alpha\eta}, & n\le s\le\eta,\\ \dfrac{n(N-s)}{N-\alpha\eta}, & s>n,\ s>\eta,\\ \dfrac{n(N-s)}{N-\alpha\eta}-(n-s), & \eta\le s\le n.\end{cases} \tag{11}$$

It is obvious that

$$y(n)=\sum_{s=1}^{N-1}G(n,s)y(s).$$

Lemma 2.3 ([14]) *Let* $0<\alpha<N/\eta$ *and assume* (A) *and* (B) *hold. Then, the unique solution of* (4)-(5) *satisfies*

$$\min_{n\in\{\eta,\dots,N\}}u(n)\ge\gamma\|u\|,$$

where $\gamma=\min\left\{\dfrac{\alpha\eta}{N},\dfrac{\alpha(N-\eta)}{N-\alpha\eta},\dfrac{\eta}{N}\right\}$. *(As* $\eta\in\{1,\dots,N-1\}$*, it follows that* $\gamma<1$*).*

We note that a pair $(u(n),v(n))$ is a solution of eigenvalue problem (1), (2) if, and only if,

$$u(n)=\lambda\sum_{s=1}^{N-1}G(n,s)a(s)f\left(\mu\sum_{r=1}^{N-1}G(s,r)b(r)g(u(r))\right),\quad n\in\{0,\dots,N\},$$

where

$$v(n)=\lambda\sum_{s=1}^{N-1}G(n,s)b(s)g(u(s)),\quad n\in\{0,\dots,N\}.$$

A solution $(u(n),v(n))$ of (1), (2) is called a positive solution if $u(i)>0, v(i)>0$ for $i\in\{1,\dots N-1\}$.

Values of λ,μ for which there are positive solutions (positive with respect to a cone) of (1), (2) will be determined via applications of the fixed point theorem, Theorem 1.1.

3 Positive Solutions in a Cone

In this section, we apply Theorem 1.1 to obtain solutions in a cone (that is, positive solutions) of (1), (2). For our construction, let $\mathcal{B}$ be a Banach space of real valued functions defined on $\{0,\dots,N\}$ with supremum norm, $\|\cdot\|$, and define a cone $\mathcal{P}\subset\mathcal{B}$ by

$$\mathcal{P}=\left\{x\in\mathcal{B}\mid x(n)\ge 0 \text{ on } \{0,\dots,N\},\ \text{and}\ \min_{n\in\{\eta,\dots,N\}}x(n)\ge\gamma\|x\|\right\}.$$

For our first result, define positive numbers L_1 and L_2 by

$$L_1:=\max\left\{\left[\frac{\gamma^2\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)a(s)f_\infty\right]^{-1},\left[\frac{\gamma^2\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)b(s)g_\infty\right]^{-1}\right\},$$

and

$$L_2 := \min\left\{\left[\frac{N}{N-\alpha\eta}\sum_{s=1}^{N-1}(N-s)a(s)f_0\right]^{-1}, \left[\frac{N}{N-\alpha\eta}\sum_{s=1}^{N-1}(N-s)b(s)g_0\right]^{-1}\right\}.$$

Theorem 3.1 *Assume that conditions* (A), (B) *and* (C) *hold. Then, for each* λ, μ *satisfiyng*

$$L_1 < \lambda, \mu < L_2, \tag{12}$$

there exists a pair (u, v) *satisfying* (1), (2) *such that* $u(n) > 0$ *and* $v(n) > 0$ *on* $\{1, \ldots, N-1\}$.

Proof. Let λ, μ as in (12) and let $\epsilon > 0$ be chosen such that

$$\max\left\{\left[\frac{\gamma^2\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)a(s)(f_\infty-\epsilon)\right]^{-1}, \left[\frac{\gamma^2\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)b(s)(g_\infty-\epsilon)\right]^{-1}\right\} \le \lambda, \mu$$

and

$$\lambda, \mu \le \min\left\{\left[\frac{N}{N-\alpha\eta}\sum_{s=1}^{N-1}(N-s)a(s)(f_0+\epsilon)\right]^{-1}, \left[\frac{N}{N-\alpha\eta}\sum_{s=1}^{N-1}(N-s)b(s)(g_0+\epsilon)\right]^{-1}\right\}.$$

Define an operator $T : \mathcal{P} \to \mathcal{B}$ by

$$Tu(n) := \lambda\sum_{s=1}^{N-1}G(n,s)a(s)f\left(\mu\sum_{r=1}^{N-1}G(s,r)b(r)g(u(r))\right), \quad u \in \mathcal{P}. \tag{13}$$

We seek suitable fixed points of T in the cone $\mathcal{P}$.

By Lemma 2.3, $T\mathcal{P} \subset \mathcal{P}$. In addition, standard arguments show that T is completely continuous.

Now, from the definitions of f_0 and g_0, there exists an $H_1 > 0$ such that

$$f(x) \le (f_0+\epsilon)x \text{ and } g(x) \le (g_0+\epsilon)x, \quad 0 < x \le H_1.$$

Let $u \in \mathcal{P}$ with $\|u\| = H_1$. From (7) and the choice of ϵ, we have for $s \in \{0, \ldots, N\}$

$$\begin{aligned}
\mu \sum_{r=1}^{N-1} G(s,r)b(r)g(u(r)) &\leq \mu \frac{N}{N-\alpha\eta} \sum_{r=1}^{N-1} (N-r)b(r)g(u(r)) \\
&\leq \mu \frac{N}{N-\alpha\eta} \sum_{r=1}^{N-1} (N-r)b(r)(g_0+\epsilon)u(r) \\
&\leq \mu \frac{N}{N-\alpha\eta} \sum_{r=1}^{N-1} (N-r)b(r)(g_0+\epsilon)\|u\| \\
&\leq \|u\| \\
&= H_1.
\end{aligned}$$

Consequently, from (8), and the choice of ϵ, we have for $s \in \{0, \ldots, N\}$

$$\begin{aligned}
Tu(n) &= \lambda \sum_{s=1}^{N-1} G(n,s)a(s)f\left(\mu \sum_{r=1}^{N-1} G(s,r)b(r)g(u(r))\right) \\
&\leq \lambda \frac{N}{N-\alpha\eta} \sum_{s=1}^{N-1} (N-s)a(s)f\left(\mu \sum_{r=1}^{N-1} G(s,r)b(r)g(u(r))\right) \\
&\leq \lambda \frac{N}{N-\alpha\eta} \sum_{s=1}^{N-1} (N-s)a(s)(f_0+\epsilon)\mu \sum_{r=1}^{N-1} G(s,r)b(r)g(u(r)) \\
&\leq \lambda \frac{N}{N-\alpha\eta} \sum_{s=1}^{N-1} (N-s)a(s)(f_0+\epsilon)H_1 ds \\
&\leq H_1 \\
&= \|u\|.
\end{aligned}$$

So, $\|Tu\| \leq \|u\|$. If we set

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_1\},$$

then

$$\|Tu\| \leq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{14}$$

By the fact that f_∞ and g_∞ are positive real numbers, it follows that there exists $\overline{H}_2 > 0$ such that

$$f(x) \geq (f_\infty - \epsilon)x \text{ and } g(x) \geq (g_\infty - \epsilon)x, \quad x \geq \overline{H}_2.$$

Let

$$H_2 = \max\left\{2H_1, \frac{\overline{H}_2}{\gamma}\right\}.$$

Let $u \in \mathcal{P}$ and $\|u\| = H_2$. Then,

$$u(n) \geq \min_{n \in \{\eta, \ldots, N\}} u(n) \geq \gamma\|u\| \geq \overline{H}_2, \quad n \in \{1, \ldots, N-1\}.$$

Observing that Tu satisfies (4)-(5) for $y(s) = \lambda a(s)f(u(s))$ and in view of (8) and the choice of ϵ, we have for $s \in \{1, \ldots, N-1\}$

$$\begin{aligned}
\mu\sum_{r=1}^{N-1} G(s,r)b(r)g(u(r)) &\geq \mu\frac{\gamma\eta}{N-\alpha\eta}\sum_{r=\eta}^{N-1}(N-r)b(r)g(u(r)) \\
&\geq \mu\frac{\gamma\eta}{N-\alpha\eta}\sum_{r=\eta}^{N-1}(N-r)b(r)g(u(r)) \\
&\geq \mu\frac{\gamma\eta}{N-\alpha\eta}\sum_{r=\eta}^{N-1}(N-r)b(r)(g_\infty-\epsilon)u(r) \\
&\geq \mu\frac{\gamma\eta}{N-\alpha\eta}\sum_{r=\eta}^{N-1}(N-r)b(r)(g_\infty-\epsilon)\gamma\|u\| \\
&\geq \|u\| \\
&= H_2,
\end{aligned}$$

and so, from (12) and the choice of ϵ, we have

$$\begin{aligned}
Tu(\eta) &\geq \lambda\frac{\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)a(s)f\left(\mu\sum_{r=\eta}^{N-1}G(s,r)b(r)g(u(r))\right) \\
&\geq \lambda\frac{\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)a(s)(f_\infty-\epsilon)\mu\sum_{r=\eta}^{N-1}G(s,r)b(r)g(u(r)) \\
&\geq \lambda\frac{\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)a(s)(f_\infty-\epsilon)H_2 \\
&\geq \lambda\frac{\gamma\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)a(s)(f_\infty-\epsilon)H_2 \\
&\geq H_2 \\
&= \|u\|.
\end{aligned}$$

Hence, $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P}$ and $\|u\| = H_2$. So, if we set

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_2\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{15}$$

Applying Theorem 1.1 to (14) and (15), we obtain that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. As such, and with v defined by

$$v(n) = \lambda\sum_{s=1}^{N-1} G(n,s)b(s)g(u(s)),$$

the pair (u, v) is a desired solution of (1), (2) for the given λ. The proof is complete. □

For our next result we define the positive numbers

$$\begin{aligned} L_3 : &= \max\left\{ \left[\frac{\gamma\eta}{N-\alpha\eta} \sum_{s=\eta}^{\eta-1} (N-s)a(s)f_0 \right]^{-1}, \right. \\ &\left. \left[\frac{\gamma\eta}{N-\alpha\eta} \sum_{s=\eta}^{\eta-1} (N-s)b(s)g_0 \right]^{-1} \right\}, \end{aligned}$$

and

$$\begin{aligned} L_4 : &= \min\left\{ \left[\frac{N}{N-\alpha\eta} \sum_{s=1}^{N-1} (N-s)a(s)f_\infty \right]^{-1}, \right. \\ &\left. \left[\frac{N}{N-\alpha\eta} \sum_{s=1}^{N-1} (N-s)b(s)g_\infty \right]^{-1} \right\}. \end{aligned}$$

We are now ready to state and prove our second result.

Theorem 3.2 *Assume that conditions* (A), (B) *and* (C) *hold. Then for each* λ, μ *satisfying*

$$L_3 < \lambda, \mu < L_4, \tag{16}$$

there exists a pair (u, v) *satisfying* (1), (2) *such that* $u(n) > 0$ *and* $v(n) > 0$ *on* $\{1, \dots, N-1\}$.

Proof. Let λ, μ be as in (16) and let $\epsilon > 0$ be chosen such that

$$\begin{aligned} \max & \left\{ \left[\frac{\gamma\eta}{N-\alpha\eta} \sum_{s=\eta}^{\eta-1} (N-s)a(s)(f_0-\epsilon) \right]^{-1}, \right. \\ & \left. \left[\frac{\gamma\eta}{N-\alpha\eta} \sum_{s=\eta}^{\eta-1} (N-s)b(s)(g_0-\epsilon) \right]^{-1} \right\} \leq \lambda, \mu \end{aligned}$$

and

$$\begin{aligned} \lambda, \mu \leq \min & \left\{ \left[\frac{N}{N-\alpha\eta} \sum_{s=1}^{N-1} (N-s)a(s)(f_\infty+\epsilon) \right]^{-1}, \right. \\ & \left. \left[\frac{N}{N-\alpha\eta} \sum_{s=1}^{N-1} (N-s)b(s)(g_\infty+\epsilon) \right]^{-1} \right\}. \end{aligned}$$

Let T be the cone preserving, completely continuous operator that was defined by (13). From the definitions of f_0 and g_0, there exists $H_3 > 0$ such that

$$f(x) \geq (f_0-\epsilon)x \text{ and } g(x) \geq (g_0-\epsilon)x, \quad 0 < x \leq H_3.$$

Also, from the definition of g_0 it follows that $g(0) = 0$ and so there exists $H_3 \in (0, \overline{H_3})$ such that

$$\mu g(x) \leq \frac{\overline{H_3}}{\frac{N}{N-\alpha\eta}\sum_{r=1}^{N-1}(N-r)b(r)}, \quad 0 \leq x \leq H_3.$$

For $u \in \mathcal{P}$ with $\|u\| = H_3$ we note that for $r \in \{0, \ldots, N\}$, it holds that $0 < u(r) \leq \|u\| = H_3$. As $v(r)$ satisfies (4)-(5) for $y(r) = \mu b(r)g(v(r))$, in view of (7) we have for $s \in \{0, \ldots, N\}$

$$\begin{aligned} \mu\sum_{r=1}^{N-1} G(s,r)b(r)g(u(r)) &\leq \mu\frac{N}{N-\alpha\eta}\sum_{r=1}^{N-1}(N-r)b(r)g(u(r)) \\ &\leq \frac{\frac{N}{N-\alpha\eta}\sum_{r=1}^{N-1}(N-r)b(r)\overline{H_3}}{\frac{N}{N-\alpha\eta}\sum_{r=1}^{N-1}(N-r)b(r)} \\ &\leq \overline{H_3}. \end{aligned}$$

Then, by (8)

$$\begin{aligned} Tu(\eta) &\geq \lambda\frac{\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)a(s)f\left(\mu\sum_{r=1}^{N-1}G(s,r)b(r)g(u(r))\right)ds \\ &\geq \lambda\frac{\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)a(s)(f_0-\epsilon)\mu\frac{\eta}{N-\alpha\eta}\sum_{r=\eta}^{N-1}(N-r)b(r)g(u(r)) \\ &\geq \lambda\frac{\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)a(s)(f_0-\epsilon)\mu\frac{\gamma\eta}{N-\alpha\eta}\sum_{r=\eta}^{N-1}(N-r)b(r)(g_0-\epsilon)\|u\| \\ &\geq \lambda\frac{\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)a(s)(f_0-\epsilon)\|u\| \\ &\geq \lambda\frac{\gamma\eta}{1-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)a(s)(f_0-\epsilon)\|u\| \\ &\geq \|u\|. \end{aligned}$$

So, $\|Tu\| \geq \|u\|$. If we put

$$\Omega_3 = \{x \in \mathcal{B} \mid \|x\| < H_3\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_3. \tag{17}$$

Next, by definitions of f_∞ and g_∞, there exists $\overline{H}_4$ such that

$$f(x) \leq (f_\infty + \epsilon)x \text{ and } g(x) \leq (g_\infty + \epsilon)x, \quad x \geq \overline{H}_4.$$

Clearly, since g_∞ is assumed to be a positive real number, it follows that g is unbounded at ∞, and so, there exists $\widetilde{H_4} > \max\{2H_3, \overline{H}_4\}$ such that $g(x) \leq g(\widetilde{H_4})$, for $0 < x \leq \widetilde{H_4}$.

Set

$$f^*(n) = \sup_{0 \le s \le n} f(s), \quad g^*(n) = \sup_{0 \le s \le n} g(s), \quad \text{for} \quad n \ge 0.$$

Clearly f^* anf g^* are nondecreasing real valued function for which it holds

$$\lim_{x \to \infty} \frac{f^*(x)}{x} = f_\infty, \quad \lim_{x \to \infty} \frac{g^*(x)}{x} = g_\infty.$$

As f^* anf g^* are nondecreasing, for some $H_4 > \overline{H}_4$ we have $f^*(x) \le f^*(H_4)$, $g^*(x) \le g^*(H_4)$ for $0 < x \le H_4$.

For $u \in \mathcal{P}$ with $\|u\| = H_4$, by (8) we find for $n \in \{0, \ldots, N\}$

$$\begin{aligned}
Tu(n) &\le \lambda \frac{N}{N - \alpha\eta} \sum_{s=1}^{N-1} (N-s) a(s) f\left(\mu \sum_{r=1}^{N-1} G(s,r) b(r) g(u(r))\right) \\
&\le \lambda \frac{N}{N - \alpha\eta} \sum_{s=1}^{N-1} (N-s) a(s) f^*\left(\mu \sum_{r=1}^{N-1} G(s,r) b(r) g(u(r))\right) \\
&\le \lambda \frac{N}{N - \alpha\eta} \sum_{s=1}^{N-1} (N-s) a(s) f^*\left(\mu \frac{N}{N-\alpha\eta} \sum_{r=1}^{N-1} (N-r) b(r) g^*(u(r))\right) \\
&\le \lambda \frac{N}{N - \alpha\eta} \sum_{s=1}^{N-1} (N-s) a(s) f^*\left(\mu \frac{N}{N-\alpha\eta} \sum_{r=1}^{N-1} (N-r) b(r) g^*(H_4)\right) \\
&\le \lambda \frac{N}{N - \alpha\eta} \sum_{s=1}^{N-1} (N-s) a(s) f^*\left(\mu \frac{N}{N-\alpha\eta} \sum_{r=1}^{N-1} (N-r) b(r) (g_\infty + \epsilon) H_4\right) \\
&\le \lambda \frac{N}{N - \alpha\eta} \sum_{s=1}^{N-1} (N-s) a(s) f^*(H_4) \\
&\le \lambda \frac{N}{N - \alpha\eta} \sum_{s=1}^{N-1} (N-s) a(s) (f_\infty + \epsilon) H_4 \\
&\le H_4 \\
&= \|u\|,
\end{aligned}$$

and so $\|Tu\| \le \|u\|$ for $u \in \mathcal{P}$ with $\|u\| = H_4$. For this case, if we let

$$\Omega_4 = \{x \in \mathcal{B} \mid \|x\| < H_4\},$$

then

$$\|Tu\| \le \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_4. \tag{18}$$

Application of part (ii) of Theorem 1.1 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_4 \setminus \Omega_3)$, which in turn yields a pair (u, v) satisfying (1), (2) for the chosen value of λ. The proof is complete. $\square$

Remark 3.1 Similar results to those in Theorems 3.1, 3.2 can be obtained for the following systems of second order difference equations

$$\begin{aligned} \nabla\Delta u(n) + \lambda a(n) f(v(n)) = 0, \quad n \in \{1, \dots, N-1\}, \quad N \geq 2, \\ \nabla\Delta v(n) + \mu b(n) g(u(n)) = 0, \quad n \in \{1, \dots, N-1\}, \quad N \geq 2, \end{aligned} \tag{19}$$

subject to any one of the following discrete boundary conditions

$$\begin{aligned} u(0) = 0, \quad u(N) = \alpha u(\eta), \\ v(0) = 0, \quad v(N) = \alpha v(\eta), \end{aligned} \tag{20}$$

$$\begin{aligned} u(0) - \beta\Delta u(0) = 0, \quad u(N) = \alpha u(\eta), \\ v(0) - \beta\Delta v(0) = 0, \quad v(N) = \alpha v(\eta), \end{aligned} \tag{21}$$

or

$$\begin{aligned} \Delta u(0) = 0, \quad u(N) = \alpha u(\eta), \\ \Delta v(0) = 0, \quad v(N) = \alpha v(\eta), \end{aligned} \tag{22}$$

where, as usual, Δ is the forward difference operator with stepsize 1, $\Delta u(n) = u(t+1) - u(n)$ and ∇ is the backward difference operator with stepsize 1, $\nabla u(n) = u(n) - u(n-1)$. The corresponding discrete boundary value problems for difference equations were studied in [4, 5]. For example, in [5] it was proved that if $(N+1-\alpha\eta)+\beta(1-\alpha) \neq 0$ and $\beta \neq -1$ the boundary value problem

$$\nabla\Delta u(n) + y(n) = 0, \quad n \in \{1, \dots, N\}$$

$$u(0) - \beta\Delta u(0) = 0, \quad u(N) = \alpha u(\eta),$$

has a unique solution

$$\begin{aligned} u(n) \quad = \quad & -\sum_{s=1}^{n-1}(n-s)y(s) + \sum_{s=1}^{N} \frac{n+\beta}{(N+1-\alpha\eta)+\beta(1-\alpha)}(N+1-s)y(s) \\ & -\sum_{s=1}^{\eta-1} \frac{\alpha(n+\beta)}{(N+1-\alpha\eta)+\beta(1-\alpha)}(\eta-s)y(s), \quad n \in \{0, \dots, N+1\}, \end{aligned}$$

and the Green's function for this problem is given by

$$G(n,s) = \begin{cases} \dfrac{(s+\beta)[N+1-n-\alpha(\eta-n)]]}{(N+1-\alpha\eta)+\beta(1-\alpha)}, & s < n,\ s \leq \eta, \\ \dfrac{(N+1-n)(s+\beta)+\alpha(\eta+\beta)(n-s)}{(N+1-\alpha\eta)+\beta(1-\alpha)}, & \eta \leq s \leq n, \\ \dfrac{(n+\beta)[N+1-s-\alpha(\eta-s)]}{(N+1-\alpha\eta)+\beta(1-\alpha)}, & n \leq s < \eta, \\ \dfrac{(n+\beta)(N+1-s)}{(N+1-\alpha\eta)+\beta(1-\alpha)}, & s \geq n, s \geq \eta. \end{cases}$$

Using these relations and the necessary modifications we can extend our results to the above boundary value problems. We omit the details.

4 Discussion

A necessary condition for the existence of positive solutions of the BVP (1)-(2) is that the positive numbers L_1 and L_2 defined in Section 3 satisfy $L_1 < L_2$. Setting

$$f(n) = p_2|\sin n| + p_1 n e^{-1/n}, \quad n \in \mathbb{N},$$

$$g(n) = p_2|\sin n| + q_1 e^{-1/n}, \quad n \in \mathbb{N},$$

we immediately observe that

$$\lim_{x\to\infty} \frac{f(x)}{x} = p_1, \quad \lim_{x\to\infty} \frac{f(x)}{x} = q_1,$$

$$\lim_{x\to 0^+} \frac{f(x)}{x} = p_2, \quad \lim_{x\to 0^+} \frac{f(x)}{x} = q_2.$$

Assume that

$$a(n) = \frac{a_1(n)}{N-n}, \quad b(n) = \frac{b_1(n)}{N-n}, \quad n \in \{0, \ldots, N-1\},$$

where $a_1(n) \geq 0, n \in \mathbb{N}$ and $b_1(n) \geq 0, n \in \mathbb{N}$.

Recall that

$$\gamma = \min\left\{\frac{\alpha\eta}{N}, \frac{\alpha(N-\eta)}{N-\alpha\eta}, \frac{\eta}{N}\right\}.$$

Now let us set $r_0 = \frac{\eta}{N} < 1$. Then $\gamma = \min\left\{\alpha r_0, \frac{\alpha(1-r_0)}{1-\alpha r_0}, r_0\right\}$. Assuming that $a \in \left(1, \frac{1}{r_0}\right)$, we have that $\gamma = r_0$ and so

$$\begin{aligned} L_1 &= \max\left\{\left[\frac{\gamma\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)\,a(s)\,f_\infty\right]^{-1}, \left[\frac{\gamma\eta}{N-\alpha\eta}\sum_{s=\eta}^{N-1}(N-s)\,b(s)\,g_\infty\right]^{-1}\right\} \\ &= \max\left\{\left[\frac{r_0^2}{1-\alpha r_0}\sum_{s=\eta}^{N-1}a_1(s)\,p_1\right]^{-1}, \left[\frac{r_0^2}{1-\alpha r_0}\sum_{s=\eta}^{N-1}b_1(s)\,q_1\right]^{-1}\right\} \\ &= \frac{1-\alpha r_0}{r_0^2}\cdot\frac{1}{\min\left\{p_1\sum\limits_{s=\eta}^{N-1}a_1(s), \sum\limits_{s=\eta}^{N-1}b_1(s)\,q_1\right\}}, \end{aligned}$$

i.e.,

$$L_1 = \frac{1-\alpha r_0}{r_0^2\min\left\{p_1\sum\limits_{s=\eta}^{N-1}a_1(s), \sum\limits_{s=\eta}^{N-1}b_1(s)\,q_1\right\}}.$$

In a similar manner

$$\begin{aligned} L_2 &= \min\left\{\left[\frac{N}{N-\alpha\eta}\sum_{s=0}^{N-1}(N-s)\,a(s)\,f_0\right]^{-1}, \left[\frac{N}{N-\alpha\eta}\sum_{s=0}^{N-1}(N-s)\,b(s)\,g_0\right]^{-1}\right\} \\ &= \min\left\{\left[\frac{1}{1-\alpha r_0}\sum_{s=0}^{N-1}a_1(s)\,p_2\right]^{-1}, \left[\frac{1}{1-\alpha r_0}\sum_{s=0}^{N-1}b_1(s)\,q_2\right]^{-1}\right\} \\ &= (1-\alpha r_0)\cdot\frac{1}{\max\left\{\sum_{s=0}^{N-1}a_1(s)\,p_2, \sum_{s=0}^{N-1}b_1(s)\,q_2\right\}}, \end{aligned}$$

i.e.,

$$L_2 = \frac{1-\alpha r_0}{\max\left\{\sum_{s=0}^{N-1}a_1(s)\,p_2, \sum_{s=0}^{N-1}b_1(s)\,q_2\right\}}.$$

Observe that $L_1 < L_2$ is equivalent to

$$\frac{1-\alpha r_0}{r_0^2\min\left\{p_1\sum_{s=r_0N}^{N-1}a_1(s), \sum_{s=r_0N}^{N-1}b_1(s)\,q_1\right\}} < \frac{1-\alpha r_0}{\max\left\{\sum_{s=0}^{N-1}a_1(s)\,p_2, \sum_{s=0}^{N-1}b_1(s)\,q_2\right\}},$$

i.e., to

$$\max\left\{p_2\sum_{s=0}^{N-1}a_1(s), q_2\sum_{s=0}^{N-1}b_1(s)\right\} < r_0^2\min\left\{p_1\sum_{s=r_0N}^{N-1}a_1(s), q_1\sum_{s=r_0N}^{N-1}b_1(s)\right\},$$

which can be written as

$$\frac{\max\left\{p_2\sum_{s=0}^{N-1}a_1(s), q_2\sum_{s=0}^{N-1}b_1(s)\right\}}{\min\left\{p_1\sum_{s=r_0N}^{N-1}a_1(s), q_1\sum_{s=r_0N}^{N-1}b_1(s)\right\}} < r_0^2. \tag{23}$$

A weaker - but easier to be verified - sufficient condition for (23) to hold is

$$\frac{\max\{p_2,q_2\}\max\left\{\sum_{s=0}^{N-1}a_1(s), \sum_{s=0}^{N-1}b_1(s)\right\}}{\min\{p_1,q_1\}\min\left\{\sum_{s=r_0N}^{N-1}a_1(s), \sum_{s=r_0N}^{N-1}b_1(s)\right\}} < r_0^2, \tag{24}$$

while a sufficient condition for (24) to hold is

$$\frac{\max\{p_2,q_2\}}{\min\{p_1,q_1\}}\cdot\frac{\sum_{s=0}^{N-1}\max\{a_1(s),b_1(s)\}}{\sum_{s=r_0N}^{N-1}\min\{a_1(s),b_1(s)\}} < r_0^2. \tag{25}$$

Example 4.1 Consider the system of three point boundary value problems

$$\Delta^2 u(n-1) + \lambda\frac{a_0}{N-n}\left[p_2\left|\sin(v(n))\right| + p_1 v(n)\,e^{-1/v(n)}\right] = 0,$$

$$\Delta^2 v(n-1) + \mu\frac{b_0}{N-n}\left[q_2\left|\sin(u(n))\right| + q_1 u(n)\,e^{-1/u(n)}\right] = 0,$$

$$
\begin{aligned}
u(0) = 0, \quad & u(N) = \frac{4}{3} u\left(\frac{2}{3} N\right), \\
v(0) = 0, \quad & v(N) = \frac{4}{3} u\left(\frac{2}{3} N\right),
\end{aligned}
$$

where $a_0, b_0, p_2, p_1, p_0, q_2$ are positive real numbers.

We observe that $1 < \alpha = \frac{4}{3} < \frac{3}{2} = \frac{N}{\eta}$, and that for the special case that $a_1(n) = a_0$ and $b_1(n) = b_0$, (23) becomes

$$
\frac{N}{N-\eta} \cdot \frac{\max\{p_2 a_0, q_2 b_0\}}{\min\{p_1 a_0, q_1 b_0\}} < \left(\frac{\eta}{N}\right)^2,
$$

i.e.,

$$
\frac{\max\{p_2 a_0, q_2 b_0\}}{\min\{p_1 a_0, q_1 b_0\}} < \left(\frac{\eta}{N}\right)^2 \frac{N-\eta}{N}.
$$

In view of the above discussion, we have the following result:

If

$$
\frac{\max\{p_2 a_0, q_2 b_0\}}{\min\{p_1 a_0, q_1 b_0\}} < \frac{2}{27},
$$

then there exist some positive numbers λ and μ for which the above system has positive solutions.

We note that the results obtained in the above discussion may easily be applied to BVPs containing equations like

$$
\begin{aligned}
\Delta^2 u(n-1) + \lambda \frac{a_0}{N-n}\left[\sum_{i=1}^{k} \left|\sin\left(p_{2i} v(n)\right)\right| + \sum_{i=1}^{l} p_{1i} v(n)\, c_i^{-1/d_i v(n)}\right] &= 0, \\
\Delta^2 v(n-1) + \mu \frac{b_0}{N-n}\left[\sum_{i=1}^{m} \left|\sin\left(q_{2i} u(n)\right)\right| + \sum_{i=1}^{\kappa} q_{1i} u(n)\, e^{-1/\beta_i u(n)}\right] &= 0,
\end{aligned}
$$

where the constants involved are positive.

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