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Quenching of the Solution for a Degenerate Semilinear Parabolic Equation

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Abstract: *Let $\Lambda \leq \infty$, q and b be nonnegative constants, and a and c be positive constants. The existence and uniqueness of the solution of the following degenerate semilinear parabolic problem are studied:*

$$\xi^q u_\tau = u_{\xi\xi} - \frac{b}{\xi^2} u + f(u) \text{ in } (0,a) \times (0,\Lambda),$$

$$u(\xi,0) = 0 \text{ on } [0,a],\, u(0,\tau) = 0 = u(a,\tau) \text{ for } 0 < \tau < \Lambda,$$

where $f(u)$ is a given function such that $\lim_{u \to c^-} f(u) = \infty$. Furthermore, we prove that u quenches in a finite time. Also, we investigate the critical length of u.

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1 Introduction

Let $\Lambda \leq \infty$, q and b be nonnegative constants, and a and c be positive constants. We study the existence and uniqueness of the solution of the following degenerate semilinear parabolic problem:

$$\xi^q u_\tau = u_{\xi\xi} - \frac{b}{\xi^2} u + f(u) \text{ in } (0,a) \times (0,\Lambda), \tag{1.1}$$

$$u(\xi,0) = 0 \text{ on } [0,a],\, u(0,\tau) = 0 = u(a,\tau) \text{ for } 0 < \tau < \Lambda, \tag{1.2}$$

where $f \in C^2([0,c))$, $f(0) > 0$, $f'(0) > 0$, $f''(s) > 0$ for $s \in [0,c)$, and $\lim_{u \to c^-} f(u) = \infty$. Furthermore, we prove that u quenches in a finite time. Also, we investigate the critical length of u. Let $\xi = ax$, $\tau = a^{q+2}t$, $\Lambda = a^{q+2}T$, $D = (0,1)$, $\Omega = D \times (0,T)$, $\bar{D} = [0,1]$, $\bar{\Omega} = \bar{D} \times [0,T)$, and $Lu = x^q u_t - u_{xx} + bu/x^2$. The problem (1.1)-(1.2) is transformed to

$$Lu = a^2 f(u) \text{ in } \Omega, \tag{1.3}$$

$$u(x,0) = 0 \text{ on } \bar{D},\, u(0,t) = 0 = u(1,t) \text{ for } 0 < t < T. \tag{1.4}$$

When $T < \infty$, a solution u to the problem (1.3)-(1.4) is said to quench at time T if

$$\max\left\{u(x,t) : x \in \bar{D}\right\} \to c^- \text{ when } t \to T^-.$$

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The length a^* is called the critical length if there exists a global solution u for $a < a^*$, and if u quenches for $a > a^*$.

When $b = 0$ and $q = 1$, the operator L is used to describe the temperature u of the channel flow of a fluid with a temperature-dependent viscosity in the boundary layer (cf. Ockendon [10]).

In the n-dimensional case and $q = 0$, Zhang [16] calculated the lower bound of the fundamental solution of the problem $Lu = 0$ for $b > 0$. On the other hand, Baras and Goldstein [2] studied the existence of the solution of the problem for $b \leq 0$.

When $q \geq 0$, $b \geq 0$, and the forcing term is u^p where $p > 1$, Chan and Chan [4] studied the blow-up for the problem (1.3)-(1.4). They showed that $x = 0$ is the only blow-up point if $1 < p \leq 1 + 2q/\left(1 + \sqrt{1+4b}\right)$. If the forcing term is $\int_0^1 F\left(u\left(\zeta, t\right)\right) d\zeta$ where $F\left(s\right) \geq s^p$ with $p > 1$ for $s \geq 0$, Chan [7] showed that u blows up for every $x \in \bar{D}$.

In Section 2, we shall study the existence and uniqueness of the solution u. Under some conditions, we shall prove that u quenches in a finite time. In Section 3, we shall determine an upper bound of the critical length by constructing a lower solution. Also, we shall use a numerical method to determine the approximated value of a^*. An example will be provided when $f\left(u\right) = 1/\left(1 - u\right)$.

2 Existence and Uniqueness of the Solution

To establish the existence and uniqueness of u, we study the steady state solution v of the problem (1.3)-(1.4) first. v satisfies the following boundary value problem:

$$v'' - \frac{b}{x^2} v = -a^2 f\left(v\right) \text{ in } D, v\left(0\right) = 0 = v\left(1\right). \tag{2.1}$$

As $f\left(v\right) > 0$, from (2.1)

$$v'' - \frac{b}{x^2} v < 0 \text{ in } D, v\left(0\right) = 0 = v\left(1\right).$$

According to Theorem 1.3 of Protter and Weinberger [12, p. 6], $v > 0$ in D. Let $Mv = v'' + \left(1 - b/x^2\right) v$. The general solution of $Mv = 0$ is given by (cf. Weisstein [15, p. 197])

$$y\left(x\right) = x^{1/2} \left(AJ_{\sqrt{1+4b}/2}\left(x\right) + BY_{\sqrt{1+4b}/2}\left(x\right)\right),$$

where $J_{\sqrt{1+4b}/2}\left(x\right)$ and $Y_{\sqrt{1+4b}/2}\left(x\right)$ are Bessel functions of the first and second kind with degree $\sqrt{1+4b}/2$, and A and B are arbitrary constants. The solution $y\left(x\right)$ satisfying $y\left(0\right) = 0$ is denoted by

$$y_1\left(x\right) = x^{1/2} J_{\sqrt{1+4b}/2}\left(x\right).$$

The solution $y\left(x\right)$ satisfying $y\left(1\right) = 0$ is given by

$$y_2\left(x\right) = x^{1/2} \left(J_{\sqrt{1+4b}/2}\left(x\right) - \frac{J_{\sqrt{1+4b}/2}\left(1\right)}{Y_{\sqrt{1+4b}/2}\left(1\right)} Y_{\sqrt{1+4b}/2}\left(x\right)\right).$$

The Green's function $G(x,s)$ for the operator M is

$$G(x,s)=\begin{cases} -\tilde{A}x^{1/2}J_{\sqrt{1+4b}/2}(x)\,\hat{A}s^{1/2}\left(J_{\sqrt{1+4b}/2}(s)-\frac{J_{\sqrt{1+4b}/2}(1)}{Y_{\sqrt{1+4b}/2}(1)}Y_{\sqrt{1+4b}/2}(s)\right) \\ \qquad\qquad \text{if } 0\le x\le s, \\ -\tilde{A}s^{1/2}J_{\sqrt{1+4b}/2}(s)\,\hat{A}x^{1/2}\left(J_{\sqrt{1+4b}/2}(x)-\frac{J_{\sqrt{1+4b}/2}(1)}{Y_{\sqrt{1+4b}/2}(1)}Y_{\sqrt{1+4b}/2}(x)\right) \\ \qquad\qquad \text{if } s\le x\le 1, \end{cases}$$

where $\tilde{A}$ and $\hat{A}$ are constants. According to (9.1.16) of Abramowitz and Stegun [1, p. 360],

$$J_{\sqrt{1+4b}/2}(x)\frac{d}{dx}Y_{\sqrt{1+4b}/2}(x)-Y_{\sqrt{1+4b}/2}(x)\frac{d}{dx}J_{\sqrt{1+4b}/2}(x)=2/(\pi x). \tag{2.2}$$

We follow the method of Simmons and Krantz [13, pp. 143-144] and set

$$\lim_{s\to x^-}\frac{\partial}{\partial x}G(x,s)-\lim_{s\to x^+}\frac{\partial}{\partial x}G(x,s)=-1,$$

it gives

$$\tilde{A}\hat{A}=\frac{-\pi}{2}\frac{Y_{\sqrt{1+4b}/2}(1)}{J_{\sqrt{1+4b}/2}(1)}.$$

Hence, the Green's function for the operator M and satisfying the boundary conditions of (2.1) is

$$G(x,s)=\begin{cases} -\frac{\pi}{2}x^{1/2}J_{\sqrt{1+4b}/2}(x)\,s^{1/2}\left(Y_{\sqrt{1+4b}/2}(s)-\frac{Y_{\sqrt{1+4b}/2}(1)}{J_{\sqrt{1+4b}/2}(1)}J_{\sqrt{1+4b}/2}(s)\right) \text{ if } 0\le x\le s, \\ -\frac{\pi}{2}s^{1/2}J_{\sqrt{1+4b}/2}(s)\,x^{1/2}\left(Y_{\sqrt{1+4b}/2}(x)-\frac{Y_{\sqrt{1+4b}/2}(1)}{J_{\sqrt{1+4b}/2}(1)}J_{\sqrt{1+4b}/2}(x)\right) \text{ if } s\le x\le 1. \end{cases} \tag{2.3}$$

Follow the proof of Lemma 3 of Chan and Chen [5], $G(x,s)>0$ for x and s in D.

Lemma 1. *If $R(x)$ is a nonpositive function in D and negative over an interval I where $I\subset D$, then the solution to the boundary value problem,*

$$Mv=R(x)\ \text{in } D,\ v(0)=0=v(1), \tag{2.4}$$

is positive in D.

Proof. The solution v of (2.4) satisfies the integral equation

$$v(x)=\int_0^1 G(x,s)(-R(s))\,ds.$$

To each $x\in D$, $G(x,s)>0$ for $s\in D$. By the assumption, $v(x)>0$ in D. □

Lemma 2. *If $a^2f'(0)\ge 1$, then the boundary value problem (2.1) has the minimal solution $V\ (<c)$.*

Proof. To establish the existence of the minimal solution, we construct a sequence $\{v_n\}$ as follows: $v_0 = 0$, and for $n = 1, 2, \ldots,$

$$\frac{d^2 v_n}{dx^2} + \left(1 - \frac{b}{x^2}\right) v_n = v_{n-1} - a^2 f\left(v_{n-1}\right) \text{ in } D,\, v_n\left(0\right) = 0 = v_n\left(1\right). \tag{2.5}$$

When $n = 1$, (2.5) becomes

$$\frac{d^2 v_1}{dx^2} + \left(1 - \frac{b}{x^2}\right) v_1 = -a^2 f\left(0\right). \tag{2.6}$$

It follows from $f\left(0\right) > 0$,

$$\frac{d^2 v_1}{dx^2} + \left(1 - \frac{b}{x^2}\right) v_1 < 0.$$

By Lemma 1, $v_1 > v_0$ in D. Subtracting (2.6) from (2.1) and by the mean value theorem, we obtain

$$\begin{aligned} &\frac{d^2\left(v - v_1\right)}{dx^2} + \left(1 - \frac{b}{x^2}\right)\left(v - v_1\right) \\ &= v - a^2 f\left(v\right) + a^2 f\left(0\right) \\ &= \left(1 - a^2 f'\left(\psi_1\right)\right) v, \end{aligned}$$

where $\psi_1 \in (0, v)$. Since $f''\left(s\right) > 0$, it implies $1 - a^2 f'\left(\psi_1\right) < 1 - a^2 f'\left(0\right)$. Then,

$$\frac{d^2\left(v - v_1\right)}{dx^2} + \left(1 - \frac{b}{x^2}\right)\left(v - v_1\right) < \left(1 - a^2 f'\left(0\right)\right) v \leq 0.$$

At $x = 0$ and $x = 1$, $v_1 = v$. By Lemma 1, $v_1 < v$ in D. Suppose that $v_0 < v_k < v$ in D for some positive integer k. When $n = k + 1$, by the mean value theorem, there exists a function $\psi_2 \in (0, v_k)$ such that

$$\frac{d^2 v_{k+1}}{dx^2} + \left(1 - \frac{b}{x^2}\right) v_{k+1} = \left(1 - a^2 f'\left(\psi_2\right)\right) v_k - a^2 f\left(0\right) < 0.$$

By $v_{k+1}\left(0\right) = 0 = v_{k+1}\left(1\right)$ and Lemma 1, $v_{k+1} > v_0$ in D. We subtract (2.5) from (2.1)

$$\frac{d^2\left(v - v_{k+1}\right)}{dx^2} + \left(1 - \frac{b}{x^2}\right)\left(v - v_{k+1}\right) = \left(1 - a^2 f'\left(\psi_3\right)\right)\left(v - v_k\right) < 0,$$

where $\psi_3 \in (v_k, v)$. At $x = 0$ and $x = 1$, $v_{k+1} = v$. By Lemma 1, $v_{k+1} < v$ in D. Hence, by the mathematical induction, $v_0 < v_n < v$ in D for $n = 1, 2, \ldots$

Now, suppose that $v_{k-1} < v_k$ in D for some positive integer k. Substituting $n = k + 1$ and $n = k$ respectively in (2.5), we obtain

$$\frac{d^2 v_{k+1}}{dx^2} + \left(1 - \frac{b}{x^2}\right) v_{k+1} = v_k - a^2 f\left(v_k\right), \tag{2.7}$$

$$\frac{d^2 v_k}{dx^2} + \left(1 - \frac{b}{x^2}\right) v_k = v_{k-1} - a^2 f\left(v_{k-1}\right). \tag{2.8}$$

Subtract (2.8) from (2.7)

$$\begin{aligned}&\frac{d^2\left(v_{k+1}-v_k\right)}{dx^2}+\left(1-\frac{b}{x^2}\right)\left(v_{k+1}-v_k\right)\\&=\left(v_k-v_{k-1}\right)-a^2f\left(v_k\right)+a^2f\left(v_{k-1}\right),\\&=\left(1-a^2f'\left(\psi_4\right)\right)\left(v_k-v_{k-1}\right)<0,\end{aligned}$$

where $\psi_4\in\left(v_{k-1},v_k\right)$. At $x=0$ and $x=1$, $v_{k+1}=v_k$. By Lemma 1, $v_k<v_{k+1}$ in D. Hence, $v_0<v_k<v_{k+1}<v<c$ in D. By the mathematical induction, the sequence $\{v_n\}$ is increasing and converges strictly monotonically. For $n=0,1,2,\ldots$, the sequence $\{v_n\}$ satisfies the following integral equation

$$v_{n+1}\left(x\right)=\int_0^1 G\left(x,s\right)\left(a^2f\left(v_n\left(s\right)\right)-v_n\left(s\right)\right)ds. \tag{2.9}$$

Let $\lim_{n\to\infty}v_{n+1}=V$. By the construction, $V\ (<c)$ is the minimal solution to the problem (2.1). As the integrand of the above expression is increasing with respect to v_n, by the Monotone Convergence Theorem,

$$V\left(x\right)=\int_0^1 G\left(x,s\right)\left(a^2f\left(V\left(s\right)\right)-V\left(s\right)\right)ds. \tag{2.10}$$

□

In the sequel, let k_i denote appropriate positive constants for $i=1,2,\ldots,10$. It is noted that the term $a^2f\left(v\right)-bv/x^2$ in (2.1) is not a bounded function in x for $x\in D$, this term does not satisfy the one-side Lipschitz condition (cf. Pao [11, p. 99]).

Lemma 3. *$V\in C\left(\bar{D}\right)\cap C^2\left((0,1]\right)$, and V is the unique solution to (2.1).*

Proof. From (2.10) and (2.3), we obtain

$$\begin{aligned}&V\left(x\right)\\&=-\frac{\pi}{2}x^{1/2}\left(Y_{\sqrt{1+4b}/2}\left(x\right)-\frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)}J_{\sqrt{1+4b}/2}\left(x\right)\right)\int_0^x s^{1/2}J_{\sqrt{1+4b}/2}\left(s\right)\\&\qquad\times\left(a^2f\left(V\right)-V\right)ds\\&-\frac{\pi}{2}x^{1/2}J_{\sqrt{1+4b}/2}\left(x\right)\int_x^1 s^{1/2}\left(Y_{\sqrt{1+4b}/2}\left(s\right)-\frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)}J_{\sqrt{1+4b}/2}\left(s\right)\right)\\&\qquad\times\left(a^2f\left(V\right)-V\right)ds.\end{aligned} \tag{2.11}$$

Obviously, $V\left(1\right)=0$. Since $V<c$ and $f\in C^2\left([0,c)\right)$, there exists a positive constant k_1 such that

$$\left|a^2f\left(V\right)-V\right|\le k_1 \tag{2.12}$$

for $x\in\bar{D}$.

For each fixed $x \in (0,1]$,

$$s^{1/2} J_{\sqrt{1+4b}/2}\left(s\right)$$

is an integrable function over the interval $[0,x]$, and

$$s^{1/2}\left(Y_{\sqrt{1+4b}/2}\left(s\right) - Y_{\sqrt{1+4b}/2}\left(1\right) J_{\sqrt{1+4b}/2}\left(s\right) / J_{\sqrt{1+4b}/2}\left(1\right)\right)$$

is integrable over $[x,1]$. By the fundamental theorem of calculus,

$$\int_0^x s^{1/2} J_{\sqrt{1+4b}/2}\left(s\right)\left(a^2 f\left(V\right) - V\right) ds,$$

$$\int_x^1 s^{1/2}\left(Y_{\sqrt{1+4b}/2}\left(s\right) - \frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)} J_{\sqrt{1+4b}/2}\left(s\right)\right)\left(a^2 f\left(V\right) - V\right) ds,$$

are continuous at x. Also, $J_{\sqrt{1+4b}/2}\left(x\right)$ is continuous on $\bar{D}$ and $Y_{\sqrt{1+4b}/2}\left(x\right)$ is continuous in $(0,1]$. Thus, $V\left(x\right)$ is continuous in $(0,1]$. To show that $V\left(x\right)$ is continuous at $x=0$, it necessary to prove that $\lim_{x\to 0} V\left(x\right) = 0$. Let ρ be a positive constant such that $\rho \ll 1$. From (2.11) and (2.12),

$$\begin{aligned}
&\left|\lim_{x\to 0} V\left(x\right)\right| \\
&\leq \lim_{x\to 0} \frac{\pi}{2} k_1 x^{1/2}\left(\left|Y_{\sqrt{1+4b}/2}\left(x\right)\right| + \left|\frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)}\right|\left|J_{\sqrt{1+4b}/2}\left(x\right)\right|\right)\int_0^x s^{1/2}\left|J_{\sqrt{1+4b}/2}\left(s\right)\right| ds \\
&+ \lim_{x\to 0} \frac{\pi}{2} k_1 x^{1/2}\left|J_{\sqrt{1+4b}/2}\left(x\right)\right|\int_x^1 s^{1/2}\left(\left|Y_{\sqrt{1+4b}/2}\left(s\right)\right| + \left|\frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)}\right|\left|J_{\sqrt{1+4b}/2}\left(s\right)\right|\right) ds.
\end{aligned}$$

When $x \ll 1$, by (9.1.7) and (9.1.9) of Abramowitz and Stegun [1, p. 360], $\left|J_{\sqrt{1+4b}/2}\left(x\right)\right| \leq k_2 x^{\sqrt{1+4b}/2}$ and $\left|Y_{\sqrt{1+4b}/2}\left(x\right)\right| \leq k_3 x^{-\sqrt{1+4b}/2}$. For $x < \rho$, we have

$$\begin{aligned}
&\left|\lim_{x\to 0} V\left(x\right)\right| \\
&\leq \lim_{x\to 0} \frac{\pi}{2} k_1 x^{1/2}\left(k_3 x^{-\sqrt{1+4b}/2} + \left|\frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)}\right| k_2 x^{\sqrt{1+4b}/2}\right) k_2 \frac{2x^{\left(3+\sqrt{1+4b}\right)/2}}{3+\sqrt{1+4b}} \\
&+ \lim_{x\to 0} \frac{\pi}{2} k_1 k_2 x^{\left(1+\sqrt{1+4b}\right)/2}\int_x^\rho s^{1/2}\left(k_3 s^{-\sqrt{1+4b}/2} + \left|\frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)}\right| k_2 s^{\sqrt{1+4b}/2}\right) ds \\
&+ \lim_{x\to 0} \frac{\pi}{2} k_1 k_2 x^{\left(1+\sqrt{1+4b}\right)/2}\int_\rho^1 s^{1/2}\left(\left|Y_{\sqrt{1+4b}/2}\left(s\right)\right| + \left|\frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)}\right|\left|J_{\sqrt{1+4b}/2}\left(s\right)\right|\right) ds.
\end{aligned}$$

Simplify the right-hand side,

$$\begin{aligned}
&\left|\lim_{x\to 0} V\left(x\right)\right| \\
&\leq \frac{\pi k_1 k_2}{3+\sqrt{1+4b}} \lim_{x\to 0} x^{1/2}\left(k_3 x^{3/2} + \left|\frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)}\right| k_2 x^{\left(3+2\sqrt{1+4b}\right)/2}\right) \\
&+ \frac{\pi}{2} k_1 k_2 \lim_{x\to 0} x^{\left(1+\sqrt{1+4b}\right)/2}\left(k_3 \rho^{1/2} x^{-\sqrt{1+4b}/2} + \left|\frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)}\right| k_2 \rho^{\left(1+\sqrt{1+4b}\right)/2}\right)\left(\rho - x\right) \\
&+ \frac{\pi}{2} k_1 k_2 \lim_{x\to 0} x^{\left(1+\sqrt{1+4b}\right)/2} \int_\rho^1 s^{1/2}\left(\left|Y_{\sqrt{1+4b}/2}\left(s\right)\right| + \left|\frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)}\right| \left|J_{\sqrt{1+4b}/2}\left(s\right)\right|\right) ds.
\end{aligned}$$

Then, the right-hand side tends to zero when $x \to 0$. Thus, $\lim_{x\to 0} V\left(x\right) = 0$. Hence, $V\left(x\right)$ is continuous on $\bar{D}$.

From (2.11), the derivative of $V\left(x\right)$ is

$$\begin{aligned}
&V'\left(x\right) \\
&= -\frac{\pi}{2}\frac{d}{dx}\left[x^{1/2}\left(Y_{\sqrt{1+4b}/2}\left(x\right) - \frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)} J_{\sqrt{1+4b}/2}\left(x\right)\right)\right] \int_0^x s^{1/2} J_{\sqrt{1+4b}/2}\left(s\right) \\
&\qquad \times \left(a^2 f\left(V\right) - V\right) ds \\
&- \frac{\pi}{2}\frac{d}{dx}\left(x^{1/2} J_{\sqrt{1+4b}/2}\left(x\right)\right) \int_x^1 s^{1/2}\left(Y_{\sqrt{1+4b}/2}\left(s\right) - \frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)} J_{\sqrt{1+4b}/2}\left(s\right)\right) \\
&\qquad \times \left(a^2 f\left(V\right) - V\right) ds.
\end{aligned}$$

Then, the second derivative of $V\left(x\right)$ is given by

$$\begin{aligned}
&V''\left(x\right) \\
&= -\frac{\pi}{2}\frac{d^2}{dx^2}\left[x^{1/2}\left(Y_{\sqrt{1+4b}/2}\left(x\right) - \frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)} J_{\sqrt{1+4b}/2}\left(x\right)\right)\right] \int_0^x s^{1/2} J_{\sqrt{1+4b}/2}\left(s\right) \\
&\qquad \times \left(a^2 f\left(V\right) - V\right) ds \\
&- \frac{\pi}{2}\frac{d^2}{dx^2}\left(x^{1/2} J_{\sqrt{1+4b}/2}\left(x\right)\right) \int_x^1 s^{1/2}\left(Y_{\sqrt{1+4b}/2}\left(s\right) - \frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)} J_{\sqrt{1+4b}/2}\left(s\right)\right) \\
&\qquad \times \left(a^2 f\left(V\right) - V\right) ds \\
&- \frac{\pi}{2}\frac{d}{dx}\left(x^{1/2} Y_{\sqrt{1+4b}/2}\left(x\right)\right) x^{1/2} J_{\sqrt{1+4b}/2}\left(x\right)\left(a^2 f\left(V\left(x\right)\right) - V\left(x\right)\right) \\
&+ \frac{\pi}{2}\frac{d}{dx}\left(x^{1/2} J_{\sqrt{1+4b}/2}\left(x\right)\right) x^{1/2} Y_{\sqrt{1+4b}/2}\left(x\right)\left(a^2 f\left(V\left(x\right)\right) - V\left(x\right)\right). \qquad (2.13)
\end{aligned}$$

Since the second derivative of $x^{1/2}$, $J_{\sqrt{1+4b}/2}\left(x\right)$, and $Y_{\sqrt{1+4b}/2}\left(x\right)$ are continuous in $(0, 1]$, the right-hand side of the above equation is continuous in $(0, 1]$. Hence, $V \in C\left(\bar{D}\right) \cap C^2\left((0, 1]\right)$.

From (2.11), (2.13), $My_1 = 0$, $My_2 = 0$, and (2.2), it yields

$$\begin{aligned} MV &= -\frac{\pi}{2} x \left(a^2 f\left(V\left(x\right)\right) - V\left(x\right)\right) \\ &\quad \times \left(J_{\sqrt{1+4b}/2}\left(x\right) \frac{d}{dx} Y_{\sqrt{1+4b}/2}\left(x\right) - Y_{\sqrt{1+4b}/2}\left(x\right) \frac{d}{dx} J_{\sqrt{1+4b}/2}\left(x\right)\right) \\ &= -\frac{\pi}{2} x \left(a^2 f\left(V\left(x\right)\right) - V\left(x\right)\right) \frac{2}{\pi x} \\ &= V\left(x\right) - a^2 f\left(V\left(x\right)\right). \end{aligned}$$

By Lemma 1, V is the unique solution to (2.1). □

Let ε be a positive number less than 1, $D_\varepsilon = (\varepsilon, 1)$, $\bar{D}_\varepsilon = [\varepsilon, 1]$, $\Omega_\varepsilon = D_\varepsilon \times (0, T)$, $\bar{\Omega}_\varepsilon = \bar{D}_\varepsilon \times [0, T)$, and w be the solution of the following semilinear parabolic problem:

$$Lw = a^2 f\left(w\right) \text{ in } \Omega_\varepsilon, \tag{2.14}$$

$$w\left(x, 0\right) = 0 \text{ on } \bar{D}_\varepsilon,\ w\left(\varepsilon, t\right) = 0 = w\left(1, t\right) \text{ for } 0 < t < T. \tag{2.15}$$

Now, we prove the existence of the solution of the problem (1.3)-(1.4).

Theorem 4. *The problem* (1.3)-(1.4) *has a solution* $u \in C\left(\bar{\Omega}\right) \cap C^{2,1}\left((0, 1] \times [0, T)\right)$.

Proof. Since 0 and V are the lower and upper solutions to the problem (2.14)-(2.15) and $V \in C^2\left(\bar{D}_\varepsilon\right)$, by Theorem 4.2.2 of Ladde, Lakshmikantham, and Vatsala [8, p. 143], there exists a solution $w \in C^{2+\alpha, 1+\alpha/2}\left(\bar{\Omega}_\varepsilon\right)$ of the problem (2.14)-(2.15) such that $0 \le w \le V$ on $\bar{\Omega}_\varepsilon$ for some $\alpha \in (0, 1)$. By the maximum principle (cf. Protter and Weinberger [12, p. 175]), $w > 0$ in Ω_ε and is unique. Let ε_1 and ε_2 be positive real numbers such that $\varepsilon_1 < \varepsilon_2 < 1$. We want to show that $\hat{w} \ge \tilde{w}$ on $\bar{\Omega}_{\varepsilon_2}$, where $\hat{w}$ and $\tilde{w}$ are solutions to the problem (2.14)-(2.15) with $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_2$ respectively. By the mean value theorem,

$$x^q \left(\hat{w} - \tilde{w}\right)_t - \left(\hat{w} - \tilde{w}\right)_{xx} = \left[a^2 f'\left(\psi_5\right) - \frac{b}{x^2}\right] \left(\hat{w} - \tilde{w}\right),$$

where ψ_5 is between $\hat{w}$ and $\tilde{w}$. $\hat{w}\left(1, t\right) = \tilde{w}\left(1, t\right) = 0$ and $\hat{w}\left(\varepsilon_2, t\right) > \tilde{w}\left(\varepsilon_2, t\right) = 0$ for $t \in (0, T)$. Also, $\hat{w}\left(x, 0\right) = \tilde{w}\left(x, 0\right)$ on $\bar{D}_{\varepsilon_2}$. By the maximum principle, $\hat{w} \ge \tilde{w}$ on $\bar{\Omega}_{\varepsilon_2}$. Since $\{w\}$ is a bounded monotone nonincreasing sequence in ε, let $u = \lim_{\varepsilon \to 0} w\left(x, t\right)$. We claim that u is a solution to the problem (1.3)-(1.4). For any $(\underline{x}, \underline{t}) \in \Omega$, there exists a set $E = [b_1, b_2] \times \left[0, \check{t}\right]$ such that $(\underline{x}, \underline{t}) \in E \subset \bar{\Omega}$ (where $b_1 > 0,\ b_2 \le 1$, and $\check{t} < T$). Let $\tilde{q}$ be a positive constant greater than 1.

i. $||w||_{L^{\tilde{q}}(E)} \le ||V||_{L^{\tilde{q}}(E)} \le k_4$,

ii.

$$\begin{aligned} &\left(\int_t^{t+\hat{t}} \left(\int_{b_1}^{b_2} \left|\frac{b}{x^{q+2}}\right|^r dx\right) dt\right)^{1/r} \\ &= \frac{b}{\left[r\left(q+2\right) - 1\right]^{1/r}} \left[b_1^{-r(q+2)+1} - b_2^{-r(q+2)+1}\right]^{1/r} \hat{t}^{1/r}. \end{aligned}$$

The right hand side tends to zero as $\hat{t} \to 0$.

iii. $||x^{-q}a^2 f(w)||_{L^{\tilde{q}}(E)} \leq b_1^{-q}a^2 ||f(V)||_{L^{\tilde{q}}(E)}$.

If we choose $\tilde{q} > 3/(2-\alpha)$, by Theorem 4.9.1 of Ladyženskaja, Solonnikov, and Ural′ceva [9, pp. 341-342] $w \in W_{\tilde{q}}^{2,1}(E)$. By Theorem 2.3.3 there [9, p. 80], $W_{\tilde{q}}^{2,1}(E) \hookrightarrow H^{\alpha,\alpha/2}(E)$. Thus, $||w||_{H^{\alpha,\alpha/2}(E)} \leq k_5$. By the triangular inequality,

$$\begin{aligned}
&\left|\left|bx^{-(q+2)}w\right|\right|_{H^{\alpha,\alpha/2}(E)} \\
&\leq \frac{b}{b_1^{q+2}}||V||_\infty + \frac{b}{b_1^{q+2}} \sup_{\substack{(x,t)\in E \\ (\tilde{x},t)\in E}} \frac{|w(x,t)-w(\tilde{x},t)|}{|x-\tilde{x}|^\alpha} \\
&+ b||V||_\infty \sup_{\substack{(x,t)\in E \\ (\tilde{x},t)\in E}} \frac{\left|x^{-(q+2)}-\tilde{x}^{-(q+2)}\right|}{|x-\tilde{x}|^\alpha} + \frac{b}{b_1^{q+2}} \sup_{\substack{(x,t)\in E \\ (x,\tilde{t})\in E}} \frac{\left|w(x,t)-w\left(x,\tilde{t}\right)\right|}{\left|t-\tilde{t}\right|^{\alpha/2}} \\
&= \frac{b}{b_1^{q+2}}||V||_\infty + \frac{b}{b_1^{q+2}}||w||_{H^{\alpha,\alpha/2}(E)} + b||V||_\infty \left|\left|x^{-(q+2)}\right|\right|_{H^{\alpha,\alpha/2}(E)} \\
&\leq k_6.
\end{aligned}$$

Similarly, by the mean value theorem,

$$\begin{aligned}
&\left|\left|a^2x^{-q}f(w)\right|\right|_{H^{\alpha,\alpha/2}(E)} \\
&\leq \frac{a^2}{b_1^q}||f(V)||_\infty + \frac{a^2}{b_1^q} \sup_{\substack{(x,t)\in E \\ (\tilde{x},t)\in E}} \frac{|f'(\psi_6)|\,|w(x,t)-w(\tilde{x},t)|}{|x-\tilde{x}|^\alpha} \\
&+ a^2||f(V)||_\infty \sup_{\substack{(x,t)\in E \\ (\tilde{x},t)\in E}} \frac{|x^{-q}-\tilde{x}^{-q}|}{|x-\tilde{x}|^\alpha} + \frac{a^2}{b_1^q} \sup_{\substack{(x,t)\in E \\ (x,\tilde{t})\in E}} \frac{|f'(\psi_7)|\left|w(x,t)-w\left(x,\tilde{t}\right)\right|}{\left|t-\tilde{t}\right|^{\alpha/2}},
\end{aligned}$$

where ψ_6 is between $w(x,t)$ and $w(\tilde{x},t)$, and ψ_7 is between $w(x,t)$ and $w\left(x,\tilde{t}\right)$. As $w \leq V$ and $f''(s) > 0$ for $s > 0$, the following inequality is obtained

$$\begin{aligned}
\left|\left|a^2x^{-q}f(w)\right|\right|_{H^{\alpha,\alpha/2}(E)} &\leq \frac{a^2}{b_1^q}||f(V)||_\infty + \frac{a^2}{b_1^q}||f'(V)||_\infty ||w||_{H^{\alpha,\alpha/2}(E)} \\
&+ a^2||f(V)||_\infty \left|\left|x^{-q}\right|\right|_{H^{\alpha,\alpha/2}(E)} \\
&\leq k_7
\end{aligned}$$

for some positive constant k_7 which is independent of ε. By Theorem 4.10.1 of Ladyženskaja, Solonnikov, and Ural′ceva [9, pp. 351-352], there exists some positive constant k_8 independent of ε such that

$$||w||_{H^{2+\alpha,1+\alpha/2}(E)} \leq k_8.$$

This implies that w, w_t, w_x, and w_{xx} are equicontinuous in E. By the Ascoli-Arzela theorem, we obtain

$$||u||_{H^{2+\alpha,1+\alpha/2}(E)} \leq k_8,$$

and the partial derivatives of u are the limits of the corresponding derivatives of w. Since 0 and V are equal to 0 at $x = 0$ and $x = 1$, $u(0,t) = 0 = u(1,t)$ for $t \in [0,T)$ by the sandwich theorem. Hence, $u \in C\left(\bar{\Omega}\right) \cap C^{2,1}\left((0,1] \times [0,T)\right)$. □

Theorem 5. *The problem* (1.3)-(1.4) *has at most one solution.*

Proof. Suppose that the problem (1.3)-(1.4) has two different solutions $u(x,t)$ and $z(x,t)$. Without loss of generality, let us assume that $z > u$ somewhere, say, $(\bar{x},\bar{t})$ in Ω. Since $z(x,0) - u(x,0) = 0$ on $\bar{D}$, $z(0,t) - u(0,t) = 0$, and $z(1,t) - u(1,t) = 0$, there exists some nonnegative constants a_1, a_2, a_3, and a_4 such that $\bar{x} \in (a_3,a_4) \subset (a_1,a_2) \subset \bar{D}$, and $z(a_1,t) = u(a_1,t)$ and $z(a_2,t) = u(a_2,t)$ for $0 \le t \le \bar{t}$. Also, $z(x,\bar{t}) > u(x,\bar{t})$ for $x \in (a_3,a_4)$, and $z \ge u$ on $[a_1,a_2] \times [0,\bar{t}]$. Let φ and γ denote respectively the fundamental eigenfunction and eigenvalue of the problem,

$$\varphi'' + \gamma\varphi = 0 \text{ for } a_1 < x < a_2,\ \varphi(a_1) = 0 = \varphi(a_2).$$

Then, $\varphi = \sin\left[\pi(x-a_1)/(a_2-a_1)\right]$ which is positive in (a_1,a_2), and $\gamma = \left[\pi/(a_2-a_1)\right]^2$. We have

$$\begin{aligned} 0 &\le \int_0^{\bar{t}}\int_{a_1}^{a_2}(z-u)\gamma\varphi dxdt = -\int_0^{\bar{t}}\int_{a_1}^{a_2}(z-u)\varphi'' dxdt \\ &= -\int_0^{\bar{t}}\int_{a_1}^{a_2}(z-u)_{xx}\varphi dxdt \end{aligned}$$

From (1.3), the above inequality becomes

$$0 \le -\int_0^{\bar{t}}\int_{a_1}^{a_2}\left[x^q(z-u)_t + \frac{b}{x^2}(z-u) - a^2(f(z)-f(u))\right]\varphi dxdt.$$

Since $z(x,0) = u(x,0)$ on $\bar{D}$,

$$\begin{aligned} 0 \le &-\int_{a_1}^{a_2}x^q\left(z(x,\bar{t}) - u(x,\bar{t})\right)\varphi dx - \int_0^{\bar{t}}\int_{a_1}^{a_2}\frac{b}{x^2}(z-u)\varphi dxdt \\ &+ a^2\int_0^{\bar{t}}\int_{a_1}^{a_2}(f(z)-f(u))\varphi dxdt. \end{aligned}$$

As $z \ge u$ on $[a_1,a_2] \times [0,\bar{t}]$, $\varphi(x) > 0$ in (a_1,a_2), and $b \ge 0$, it gives

$$0 \le -\int_{a_1}^{a_2}x^q\left(z(x,\bar{t}) - u(x,\bar{t})\right)\varphi dx + a^2\int_0^{\bar{t}}\int_{a_1}^{a_2}(f(z)-f(u))\varphi dxdt. \tag{2.16}$$

It follows from the mean value theorem for integrals [3, p. 5] that there exists some $\psi_8 \in (a_1,a_2)$ such that

$$\int_{a_1}^{a_2}x^q\varphi\left(z(x,\bar{t}) - u(x,\bar{t})\right)dx = \psi_8^q\int_{a_1}^{a_2}\varphi\left(z(x,\bar{t}) - u(x,\bar{t})\right)dx.$$

By the mean value theorem, there exists some ψ_9 between z and u such that

$$f(z) - f(u) = f'(\psi_9)(z-u) \leq k_9 (z-u).$$

Then, (2.16) becomes

$$\int_{a_1}^{a_2} \varphi\left(z(x,\bar{t}) - u(x,\bar{t})\right) dx \leq \frac{a^2 k_9}{\psi_8^q} \int_0^{\bar{t}} \int_{a_1}^{a_2} \varphi(z-u)\, dx dt.$$

By the Gronwall inequality [14, pp. 14-15],

$$\int_{a_1}^{a_2} \varphi\left(z(x,\bar{t}) - u(x,\bar{t})\right) dx \leq 0.$$

On the other hand, $\varphi\left(z(x,\bar{t}) - u(x,\bar{t})\right) > 0$ for $x \in (a_3, a_4)$ implies

$$\int_{a_1}^{a_2} \varphi\left(z(x,\bar{t}) - u(x,\bar{t})\right) dx > 0.$$

This contradiction shows that the problem (1.3)-(1.4) has at most one solution. □

Lemma 6. *$u > 0$ in Ω, and $u(x,t)$ is a nondecreasing function of t for each $x \in D$.*

Proof. By Theorem 4, $w > 0$ in Ω_ε. When $\varepsilon \to 0$, this implies $u \geq 0$ in Ω. Suppose that $u(x_0,t_0) = 0$ for some $(x_0,t_0) \in \Omega$. Since $u(x,0) = 0$ on $\bar{D}$, we have $u(x_0,t) = 0$ for $t \in [0,t_0]$. This implies that $u_t(x_0,t_1) = 0$ for some $t_1 \in (0,t_0)$. At t_1, u attains its minimum at x_0, it follows that $u_{xx}(x_0,t_1) \geq 0$. Therefore, at (x_0,t_1)

$$\begin{aligned} &Lu(x_0,t_1) - a^2 f(u(x_0,t_1)) \\ &= x^q u_t(x_0,t_1) - u_{xx}(x_0,t_1) + \frac{b}{x_0^2} u(x_0,t_1) - a^2 f(u(x_0,t_1)) < 0. \end{aligned}$$

This contradicts (1.3). Hence, $u > 0$ in Ω. Let h be a positive number less than T. At $t+h$, (2.14) becomes

$$x^q w_t(x,t+h) - w_{xx}(x,t+h) + \frac{b}{x^2} w(x,t+h) = a^2 f(w(x,t+h)) \text{ in } \Omega_\varepsilon.$$

Subtract (2.14) from the above equation, it yields

$$\begin{aligned} &x^q \left(w(x,t+h) - w(x,t)\right)_t - \left(w(x,t+h) - w(x,t)\right)_{xx} \\ &= \left(a^2 f'(\psi_{10}) - \frac{b}{x^2}\right)\left(w(x,t+h) - w(x,t)\right), \end{aligned}$$

where ψ_{10} is between $w(x,t+h)$ and $w(x,t)$. Also, $w(x,h) > w(x,0)$ in D_ε, and $w(x,t+h) = w(x,t)$ at $x = \varepsilon$ and $x = 1$ for $t \in [0,T)$. By the maximum principle, $w(x,t+h) \geq w(x,t)$ on $\bar{\Omega}_\varepsilon$. Taking $\varepsilon \to 0$, it leads to $u(x,t+h) \geq u(x,t)$ on $\bar{\Omega}$. □

Let ϕ and λ be the fundamental eigenfunction and eigenvalue respectively of the following Sturm-Liouville eigenvalue problem:

$$\phi'' - \frac{b}{x^2}\phi = -\lambda x^q \phi \text{ in } D,\ \phi(0) = 0 = \phi(1). \tag{2.17}$$

From Chan and Chan [4], ϕ is given by

$$\phi(x) = k_{10} x^{1/2} J_{\sqrt{1+4b}/(q+2)} \left(\frac{2\sqrt{\lambda} x^{(q+2)/2}}{q+2} \right),$$

which is positive in D, and $\lambda = \left(j_{\sqrt{1+4b}/(q+2)} (q+2)/2 \right)^2$ where $j_{\sqrt{1+4b}/(q+2)}$ is the first positive zero of $J_{\sqrt{1+4b}/(q+2)}(x)$.

Theorem 7. *If $f(u) \geq 1/(1-u)^{\beta}$ for $u < 1$ where β is a positive constant such that $\beta \in (0,1]$ and $a^2\beta \geq \lambda$, then u quenches in a finite time.*

Proof. Choose k_{10} such that $\int_0^1 x^q \phi(x)\, dx = 1$. Multiply $\phi(x)$ on both sides of (1.3)

$$x^q \phi u_t = \phi u_{xx} - \frac{b}{x^2}\phi u + a^2 \phi f(u).$$

Using integration by parts, (2.17), and $f(u) \geq 1/(1-u)^{\beta}$, we have

$$\left(\int_0^1 x^q \phi u dx \right)_t = \int_0^1 \left(\phi'' u - \frac{b}{x^2}\phi u \right) dx + a^2 \int_0^1 \phi f(u)\, dx$$
$$\geq -\lambda \int_0^1 x^q \phi u dx + a^2 \int_0^1 \frac{\phi}{(1-u)^{\beta}} dx.$$

If follows from $1/(1-u)^{\beta} \geq 1 + \beta u + \beta(\beta+1) u^2/2$ for $u < 1$, the above equation becomes

$$\left(\int_0^1 x^q \phi u dx \right)_t$$
$$\geq -\lambda \int_0^1 x^q \phi u dx + a^2 \int_0^1 \phi \left[1 + \beta u + \frac{\beta(\beta+1)}{2} u^2 \right] dx$$
$$\geq -\lambda \int_0^1 x^q \phi u dx + a^2 \int_0^1 x^q \phi dx + a^2\beta \int_0^1 x^q \phi u dx + a^2 \frac{\beta(\beta+1)}{2} \int_0^1 x^q \phi u^2 dx.$$

By the Jensen's inequality,

$$\left(\int_0^1 x^q \phi u dx \right)_t \geq -\lambda \int_0^1 x^q \phi u dx + a^2 + a^2\beta \int_0^1 x^q \phi u dx + a^2 \frac{\beta(\beta+1)}{2} \left(\int_0^1 x^q \phi u dx \right)^2.$$

Let $U(t) = \int_0^1 x^q \phi u dx$ which is less than 1 before the quenching time, we get

$$U_t \geq a^2 + \left(a^2\beta - \lambda \right) U + a^2 \frac{\beta(\beta+1)}{2} U^2.$$

Since $a^2\beta \geq \lambda$,

$$U_t \geq a^2 + a^2 \frac{\beta(\beta+1)}{2} U^2.$$

Then, integrate the above expression from 0 to t

$$\int_0^U \frac{dU}{1 + \frac{\beta(\beta+1)}{2} U^2} \geq \int_0^t a^2 dt$$

$$\sqrt{\frac{2}{\beta(\beta+1)}} \tan^{-1} \frac{\sqrt{\beta(\beta+1)} U}{\sqrt{2}} \geq a^2 t.$$

As $\beta \in (0,1]$ and $U(t) < 1$, $\sqrt{\beta(\beta+1)}U(t)/\sqrt{2} < 1$. If u exists globally, then t tends to ∞. This implies that $\sqrt{\beta(\beta+1)}U(t)/\sqrt{2}$ approaches $\pi/2\ (>1)$. It leads to a contradiction. Hence, u quenches in a finite time. □

3 Critical Length

In this section, we follow the method of Chan and Chen [5] and Chan and Kaper [6] to determine an approximated value of the critical length of u. Firstly, we find an upper bound of the critical length. We look for a lower solution $\hat{u}(x,t)$ which satisfies

$$L\hat{u} \le a^2 f(\hat{u}) \text{ in } \Omega, \tag{3.1}$$

subject to the initial and boundary conditions (1.4). Let us construct $\hat{u}$ in the form of

$$\hat{u}(x,t) = x^{1/2} J_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}x^{(q+2)/2}}{q+2}\right) g(t),$$

where $g(t)$ is a nondecreasing function in t. Clearly, $\hat{u}(0,t) = 0 = \hat{u}(1,t)$. Substitute $\hat{u}$ into (3.1), then by (2.17) and $0 < x < 1$, it gives

$$g'(t) + \lambda g(t) \le \frac{a^2}{x^{1/2} J_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}x^{(q+2)/2}}{q+2}\right)} f\left(x^{1/2} J_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}x^{(q+2)/2}}{q+2}\right) g(t)\right).$$

Let $z = x^{(q+2)/2}$,

$$g'(t) + \lambda g(t) \le \frac{a^2}{z^{1/(q+2)} J_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}}{q+2}z\right)} f\left(z^{1/(q+2)} J_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}}{q+2}z\right) g(t)\right). \tag{3.2}$$

For each t, the minimum value of the right hand side of (3.2) is independent of z. We take the infimum of the expression of the right-hand side with respect to z. Let $K(g(t))$ be a positive function such that

$$K(g(t)) = \inf\left\{\frac{a^2}{z^{1/(q+2)} J_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}}{q+2}z\right)} f\left(z^{1/(q+2)} J_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}}{q+2}z\right) g(t)\right) : z \in \bar{D}\right\}.$$

Then, $g(t)$ can be determined by solving the following initial value problem:

$$g'(t) + \lambda g(t) = K(g(t)) \text{ for } t > 0,\ g(0) = 0. \tag{3.3}$$

Example. Let $f(u) = 1/(1-u)$. The derivative of the right-hand side of (3.2) with respect to z is

$$-\frac{a^2}{q+2}\left[2\sqrt{\lambda}zJ_{\left[\sqrt{1+4b}/(q+2)-1\right]}\left(\frac{2\sqrt{\lambda}}{q+2}z\right)-\left(\sqrt{1+4b}-1\right)J_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}}{q+2}z\right)\right]$$
$$\times\frac{\left[1-2z^{1/(q+2)}J_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}}{q+2}z\right)g(t)\right]}{z^{(q+3)/(q+2)}J^2_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}}{q+2}z\right)\left[1-z^{1/(q+2)}J_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}}{q+2}z\right)g(t)\right]^2}.$$

the right-hand side of (3.3) has an infimum at $z=\varsigma$ where ς is the first positive root of the equation

$$2\sqrt{\lambda}zJ_{\left[\sqrt{1+4b}/(q+2)-1\right]}\left(\frac{2\sqrt{\lambda}}{q+2}z\right)=\left(\sqrt{1+4b}-1\right)J_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}}{q+2}z\right),$$

for $g(t)\in\left(0,(2m)^{-1}\right]$ where

$$m=\varsigma^{1/(q+2)}J_{\sqrt{1+4b}/(q+2)}\left(\frac{2\sqrt{\lambda}}{q+2}\varsigma\right).$$

Since the infimum of the quantity $[Z(1-Z)]^{-1}$ is 4, we have

$$\frac{g'(t)}{g(t)}+\lambda=\begin{cases}a^2/\left[mg(t)\left(1-mg(t)\right)\right] & \text{for } 0<g(t)\leq(2m)^{-1},\\ \\ 4a^2 & \text{for } (2m)^{-1}<g(t)\leq m^{-1},\end{cases}$$

where $g(0)=0$. Let t_2 and t_3 denote the times when $g(t_2)=1/(2m)$ and $g(t_3)=1/m$. Integrate the second equation from t_2 to t_3, it gives

$$\int_{1/(2m)}^{1/m}\frac{1}{g(t)}dg=\int_{t_2}^{t_3}\left(4a^2-\lambda\right)dt.$$

From which we have

$$\frac{\ln 2}{(4a^2-\lambda)}=t_3-t_2.$$

As $t_3>t_2$, $4a^2-\lambda>0$. This implies that u quenches when

$$\frac{j_{\sqrt{1+4b}/(q+2)}(q+2)}{4}<a.$$

Thus, the critical length a^* of u is bounded by

$$a^*\leq\frac{j_{\sqrt{1+4b}/(q+2)}(q+2)}{4}.$$

The procedure of finding the critical length is as follows:

Step 1. Divide the interval $\bar{D}$ into 20 subintervals. Let $x_0=0$, $x_1=0.05,\ldots,x_{20}=1$.

Step 2. Use `Maple`®[1] version 9.03 to compute

$$x_i^{1/2} J_{\sqrt{1+4b}/2}\left(x_i\right),$$

$$x_i^{1/2}\left(Y_{\sqrt{1+4b}/2}\left(x_i\right)-\frac{Y_{\sqrt{1+4b}/2}\left(1\right)}{J_{\sqrt{1+4b}/2}\left(1\right)} J_{\sqrt{1+4b}/2}\left(x_i\right)\right),$$

for $i = 1, 2, ..., 19$. Set $v_{n+1}\left(x_0\right) = 0 = v_{n+1}\left(x_{20}\right)$. Let $a = j_{\sqrt{1+4b}/(q+2)}\left(q+2\right)/4$ and $v_0\left(x\right) = 0$ for $x \in \bar{D}$. From (2.9), we use the numerical integration built in `Maple` to evaluate $v_{n+1}\left(x_i\right)$ for $i = 1, ..., 19$.

Step 3. Use the cubic spline in `Maple` to interpolate $v_{n+1}\left(x\right)$ for $x \in \bar{D}$. Then, calculate

$$\left|\max_{x\in\bar{D}} v_{n+1}\left(x\right) - \max_{x\in\bar{D}} v_n\left(x\right)\right| = \epsilon_n.$$

If ϵ_{n+1} is greater than or equal to ϵ_n, or $\max_{x\in\bar{D}} v_{n+1}\left(x\right) \geq 1$ for some n, then a is not the critical length. If $\epsilon_n < 1 \times 10^{-5}$, we say that u exists globally.

Step 4. If a is not the critical length, decrease the value to obtain a new estimate a for a^*, and repeat Steps 2 and 3 until we find that u exists globally. The method of bisection is used to determine a value of a^{**} such that u exists globally for $a \leq a^{**}$, and u quenches for $a > a^{**}$. a^{**} is an approximation of a^*.

The following table contains the numerical results (in 4 decimal places) of a^* for various b when $q = 0$.

b	Upper bound of a	a^*
0.0000	1.5708	1.5303
0.5000	1.8250	1.7752
1.0000	1.9950	1.9389
2.0000	2.2467	2.1820

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