Neural, Parallel, and Scientific Computations 16 (2008) 303-308

A HYBRID FINITE DIFFERENCE SCHEME FOR A CLASS OF SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper a class of singularly perturbed delay differential equations are examined. A hybrid difference scheme with an appropriate piecewise uniform Shishkin-type mesh is derived. We prove that the method is almost second-order convergent in the maximum norm, independently of the small parameter. Numerical experiments support these theoretical results and indicate that the estimates are sharp.

Key Words delay differential equation, singular perturbation, Shishkin mesh, hybrid difference scheme.

1. INTRODUCTION

Differential equations with a small parameter ε multiplying the highest order derivative terms are said to be singularly perturbed and normally boundary layers occur in their solutions. Singular perturbation differential equations are ubiquitous in mathematical problems in the science, engineering and industry (see for example [1]). Since there is sometimes a memory or delayed effect, singularly perturbed delay differential equations have arose in many field, such as in the study of an 'optically bistable device' [2] and in a variety of models for physiological processes or diseases [3]. Such a problem has also appeared to describe the so-called human pupil-light reflex [4].

In this paper we consider the following singularly perturbed delay differential equation

$$Ly(t) \equiv \varepsilon y'(t) + a(t)y(t) = b(t)y(t-1) + g(t)\ , \quad \text{for} \ \ 0 \le t \le T, \tag{1.1}$$

$$y(t) = \varphi(t)\ , -1 \le t \le 0. \tag{1.2}$$

Here $0 < \varepsilon \ll 1$, a, b, g and φ are given smooth functions. In [5] the author gave an asymptotic expansion for the solution of singularly perturbed delay differential equations. Some particular phenomena different from singularly perturbed ordinary differential equations are discovered. For this reason there is a great interest in investigating whether uniformly convergent numerical schemes are still effective for singularly perturbed delay differential equations. In [6] the authors have proposed a

Received January 30, 2008

first-order upwind difference scheme for the problem (1.1)–(1.2). In present paper, we propose a second-order finite difference scheme to solve the singularly perturbed delay differential equation (1.1)–(1.2).

Notation. Throughout the paper, C will denote a generic positive constant (possibly subscripted) that is independent of ε and of the mesh. Note that C is not necessarily the same at each occurence.

Assumption 1. Throughout the paper we shall also assume that $\varepsilon \le CN^{-1}$.

2. PROPERTIES OF THE EXACT SOLUTION

For the construction of layer-adapted meshes and the analysis of associated numerical methods, it is necessary to have precise knowledge of the behavior of the exact solution to be approximated and its derivative. This is provided by the following lemma.

Lemma 1. The solution $y(t)$ of equation (1.1)–(1.2) satisfies

$$|y^{(k)}(t)| \le C(1 + \varepsilon^{-k}\exp(-\alpha(t-l)/\varepsilon)) \tag{2.1}$$

for $k = 0, 1, 2, 3,\ t \in [l, l+1],\ l = 0, 1, 2, \ldots$.

Proof. We analyze the properties of the exact solution by method of steps.

On the first interval $[0, 1]$, applying the maximum principle for the differential operator L to equation (1.1)–(1.2) we obtain

$$|y(t)| \le |y(0)| + \alpha^{-1}\|b(t)\varphi(t-1) + g(t)\|_{\infty,[0,1]} \le C, \tag{2.2}$$

where $\|\cdot\|$ denotes the continuous maximum norm. Hence, from the equation (1.1) and (2.2), we can easily get

$$|y^{(k)}(t)| \le C\varepsilon^{-k}, \quad k = 0, 1, 2, 3, \quad t \in [0, 1]. \tag{2.3}$$

From (2.3) the inequality (2.1) holds for $k = 0$ on $[0, 1]$. Differentiating both sides of (1.1) k times and setting $u(t) = y^{(k)}(t)$, we have $\varepsilon u'(t) + a(t)u(t) = h(t)$, where $h(t)$ depends on $y(t), a(t), b(t), g(t), \varphi(t)$ and their derivatives of order up to and including $k-1$. Then

$$u(t) = u(0)\exp(-\frac{1}{\varepsilon}\int_0^t a(s)\mathrm{d}s) + \frac{1}{\varepsilon}\int_0^t h(\tau)\exp(-\frac{1}{\varepsilon}\int_\tau^t a(s)\mathrm{d}s)\mathrm{d}\tau.$$

Using the inequality

$$\exp(-\frac{1}{\varepsilon}\int_\tau^t a(s)\mathrm{d}s) \le \exp(-\frac{\alpha}{\varepsilon}(t-\tau)) \quad \text{for } \ t \ge \tau$$

and (2.3) we have

$$|u(t)| \le C\varepsilon^{-k}\exp(-\alpha t/\varepsilon) + C\varepsilon^{-1}\int_0^t [\exp(-\alpha(t-\tau)/\varepsilon) + \varepsilon^{-k}\exp(-\alpha t/\varepsilon)]\mathrm{d}\tau$$

and the desired inequality follows from this.

Using the similar procedure we can obtain the desired result (2.1) on the interval $[l, l+1]$ for $l = 1, 2, \dots$. □

3. MESH AND SCHEME

For our discretization we use a Shishkin-type mesh. Let N be an even integer. Set

$$\tau = \min\left\{\frac{1}{2}, \frac{2}{\alpha}\varepsilon \ln N\right\}.$$

Divide each of the subintervals $[i, i+\tau]$ and $[i+\tau, i+1]$ into $N/2$ equidistant subintervals for $i = 0, 1, 2, \dots$. The mesh width of each subinterval in $[i, i+\tau]$ is h. We use the notation H for the mesh width in $[i+\tau, i+1]$. These mesh widths satisfy

$$N^{-1} \le H \le 2N^{-1}, \quad h = \frac{4}{\alpha}\varepsilon N^{-1} \ln N.$$

Basing on this mesh, we propose a hybrid difference scheme for problem (1.1)–(1.2):

$$L^N y_j^N = f_j \qquad \text{for} \;\; j = 1, 2, \dots, \tag{3.1}$$

$$y_{j-N} = \varphi(t_j - 1) \qquad \text{for} \;\; j = 0, 1, \dots, N, \tag{3.2}$$

where

$$L^N y_j^N = \varepsilon \frac{y_j^N - y_{j-1}^N}{h_j} + a_{j-1/2}\frac{y_{j-1}^N + y_j^N}{2},$$

$$f_j = b_{j-1/2}\frac{y_{j-N-1}^N + y_{j-N}^N}{2} + g_{j-1/2}$$

for $j = kN+1, \dots, kN+N/2,\; k = 0, 1, \dots,$ and

$$L^N y_j^N = \varepsilon \frac{y_j^N - y_{j-1}^N}{h_j} + a_j y_j^N, \quad f_j = b_j y_{j-N}^N + g_j$$

for $j = kN + N/2 + 1, \dots, kN+N,\; k = 0, 1, \dots$. Here $a_{j-1/2} = a((x_{j-1} + x_j)/2)$; similarly for $b_{i-1/2}, g_{i-1/2}$.

Next we shall derive the error estimate for the hybrid difference scheme (3.1)–(3.2). The matrix associated with L^N is a strictly diagonally dominate L_0-matrix. Hence it is an M-matrix. Therefore the following lemma holds true.

Lemma 2. (Discrete comparison principle) The operator L^N satisfies a discrete comparison principle, i.e., if $\{v_j\}$ and $\{w_j\}$ are mesh functions that satisfy $v_0 \le w_0$ and $L^N v_j \le L^N w_j$ for $j = 1, 2, \dots, N$, then $v_j \le w_j$ for all j.

An immediate consequence is the following result.

Lemma 3. (Stability result) If $\{u_j\}_{j=0}^N$ is any mesh function, then

$$\|u\| \le C \max_j\{|u_0|, |L^N u_j|\}.$$

Now we can get our main result for the delay differential equation.

Theorem 1. Let $y(t)$ be the solution to (1.1)–(1.2) and y^N be the solution to the discrete problem (3.1)–(3.2) on the Shishkin-type mesh. Then, for $0 \le x_j \le T$, we have the estimate

$$|y(t_j) - y_j^N| \le CN^{-2} \ln^2 N.$$

Proof. We would verify the theorem by method of steps.

On the first interval $[0, 1]$, we have

$$\begin{aligned}
&|L^N y_j - L^N y_j^N| = |L^N y_j - L y_{j-1/2}| \\
&= |\varepsilon \frac{y_j - y_{j-1}}{h_j} + a_{j-1/2} \frac{y_{j-1} + y_j}{2} - \varepsilon y'_{j-1/2} - a_{j-1/2} y_{j-1/2}| \\
&\le \frac{3\varepsilon}{2} \int_{t_{j-1}}^{t_j} |y'''(t)|(t - t_{j-1})\mathrm{d}t + \frac{3}{2} a_{j-1/2} \int_{t_{j-1}}^{t_j} |y''(t)|(t - t_{j-1})\mathrm{d}t \\
&\le C h_j \int_{t_{j-1}}^{t_j} [1 + \varepsilon^{-2} \exp(-\alpha t/\varepsilon)]\mathrm{d}t
\end{aligned} \tag{3.3}$$

for $j = 1, 2, \ldots, N/2$, and

$$\begin{aligned}
&|L^N y_j - L^N y_j^N| = |L^N y_j - L y_j| = |\varepsilon \frac{y_j - y_{j-1}}{h_j} - \varepsilon y'_j| \\
&\le \varepsilon \int_{t_{j-1}}^{t_j} |y''(t)|\mathrm{d}t \le C \int_{t_{j-1}}^{t_j} (\varepsilon + \varepsilon^{-1} \exp(-\alpha t/\varepsilon))\mathrm{d}t
\end{aligned} \tag{3.4}$$

for $j = N/2 + 1, \ldots, N$.

For $j = 1, \ldots, N/2$ we have

$$\begin{aligned}
&h_j \int_{t_{j-1}}^{t_j} (1 + \varepsilon^{-2} \exp(-\alpha t/\varepsilon))\mathrm{d}t = h_j^2 - \frac{4}{\alpha} N^{-1} \ln N \exp(-\alpha t/\varepsilon)|_{t_{j-1}}^{t_j} \\
&= h_j^2 - \frac{4}{\alpha} N^{-1} \ln N \exp(-\alpha t_j/\varepsilon)(1 - \exp(-\alpha h_j/\varepsilon)) \\
&\le C(h_j^2 + N^{-2} \ln^2 N) \le N^{-2} \ln^2 N,
\end{aligned} \tag{3.5}$$

where we have used mesh widths and Assumption 1.

For $j = N/2 + 1, \ldots, N$ we have

$$\begin{aligned}
&\int_{t_{j-1}}^{t_j} (\varepsilon + \varepsilon^{-1} \exp(-\alpha t/\varepsilon))\mathrm{d}t = \varepsilon h_j - \frac{1}{\alpha} \exp(-\alpha t/\varepsilon)|_{t_{j-1}}^{t_j} \\
&\le \varepsilon h_j + \frac{2}{\alpha} \exp(-\alpha t_{N/2}/\varepsilon) \le C\varepsilon N^{-1} + CN^{-2} \le CN^{-2},
\end{aligned} \tag{3.6}$$

where we also have used mesh widths and Assumption 1.

Combining the inequalities (3.3)–(3.6) to obtain

$$|L^N y_j - L^N y_j^N| \le CN^{-2} \ln^2 N \quad \text{for} \quad j = 1, \ldots, N.$$

Recalling the stability result of lemma 3, we get the error bound

$$|y(t_j) - y_j^N| \le CN^{-2}\ln^2 N \quad \text{for} \quad j = 0, 1, \dots, N.$$

On the second interval $[1, 2]$, we have

$$\begin{aligned}
&|L^N y_j - L^N y_j^N| = |L^N y_j - Ly_j + b_j(y_{j-N} - y_{j-N}^N)| \\
&\le |L^N y_j - Ly_j| + |b_j(y_{j-N} - y_{j-N}^N)| \\
&\le |L^N y_j - Ly_j| + CN^{-2}\ln^2 N
\end{aligned}$$

for $j = N+1, \dots, 2N$. Hence, applying the same techniques as on $[1, 2]$ we have

$$|y_j - y_j^N| \le CN^{-2}\ln^2 N \qquad \text{for} \quad j = N+1, \dots, 2N.$$

This completes the proof on the second interval and we can prove the theorem by induction. □

4. NUMERICAL RESULTS

To illustrate the predicted theory, we solve the following singularly perturbed delay differential equations:

Example Consider the problem

$$\begin{aligned}
&\varepsilon y'(t,\varepsilon) = -y(t,\varepsilon) + \frac{1}{2} y(t-1,\varepsilon)\ ,\ 0 \le t \le 2, \\
&y(t,\varepsilon) = \exp(-t)\ ,\ -1 \le t \le 0.
\end{aligned}$$

Exact solution is given by

$$y(t) = \begin{cases} (1 - \frac{e}{2(1-\varepsilon)})\exp(-t/\varepsilon) + \frac{1}{2(1-\varepsilon)}\exp(1-t), & 0 \le t \le 1, \\ (1 - \frac{e}{2(1-\varepsilon)})(\exp(-t/\varepsilon) - \frac{1}{2\varepsilon}\exp(-(t-1)/\varepsilon)) & \\ -\frac{e}{4(1-\varepsilon)^2}\exp(-(t-1)/\varepsilon) + \frac{1}{4(1-\varepsilon)^2}\exp(2-t) & \\ +\frac{1}{2\varepsilon}(1 - \frac{e}{2(1-\varepsilon)})t\exp(-(t-1)/\varepsilon) + \frac{1}{2(1-\varepsilon)}\exp(-(t-1)/\varepsilon), & 1 < t \le 2. \end{cases}$$

For our test we take $\varepsilon = 10^{-8}$ which is a sufficiently small choice to bring out the singularly perturbed nature of the problem. We measure the accuracy in the discrete maximum norm

$$e^N = \max_i |y_i - y_i^N|,$$

the convergence rate

$$r^N = \log_2(e^N/e^{2N})$$

and the constant in the error estimate

$$C^N = e^N/(N^{-2}\ln^2 N).$$

The numerical results (Table 1) are clear illustrations of the convergence estimate of Theorem 1. They indicate that the theoretical results are fairly sharp.

Table 1. Numerical results for Example

N	*maximal error* $\varepsilon = 10^{-4}$	*maximal error* $\varepsilon = 10^{-8}$
64	8.1554e-3	8.1523e-3
128	4.8558e-3	4.8540e-3
256	2.8122e-3	2.8112e-3
512	1.5942e-3	1.5936e-3
1024	8.8966e-4	8.8932e-4

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Acknowledgement. The work was supported by Zhejiang Province Natural Science Foundation (Grant No. Y607504) and Natural Science Foundation (2007A610048) of Ningbo Municipal of China.

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