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HYBRID DIFFERENCE SCHEMES FOR A SINGULARLY PERTURBED SECOND ORDER ORDINARY DIFFERENTIAL EQUATION WITH A DISCONTINUOUS CONVECTION COEFFICIENT ARISING IN CHEMICAL REACTOR THEORY

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ABSTRACT: In this paper, two hybrid difference schemes for singularly perturbed second order ordinary differential equations with a small parameter multiplying the highest derivative with a discontinuous convection coefficient are presented. Parameter-uniform error bounds for the numerical solution and numerical derivative are established. Numerical results are provided to illustrate the theoretical results.

Key Words: Singular perturbation problem, Piecewise uniform mesh, Scaled derivative, Scaled discrete derivative, Hybrid scheme.

1. INTRODUCTION

The theory of singular perturbation is not settled in any direction in mathematics and the path of its development is a dramatic one. In the intensive development of science and technology, many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described dy differential equations involving; large or small parameters, become more complex. In some problems, the perturbations are operative over a very narrow region across which the dependent variable undergoes very rapid changes. These narrow regions frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Consequently, they are usually referred to as boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications and shock layers in fluid and solid mechanics.

In particular, boundary-value problems (BVPs) of the form

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad x \in \Omega \equiv (0,1), \tag{1.1}$$

$$-u'(0) = A, \quad u(1) + \varepsilon u'(1) = B, \tag{1.2}$$

and

$$\varepsilon u''(x) + a(x)u'(x) = b(x,y), \quad x \in \Omega, \tag{1.3}$$

$$-u'(0) = A, \quad u(1) + \varepsilon u'(1) = B, \tag{1.4}$$

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arise in the study of adiabatic tubular chemical flow reactors with axial diffusion [1]. In [1], O'Malley obtained asymptotic solutions of the BVPs arising in chemical reactor theory. It may be noted that the asymptotic solution constructed in [1] converge uniformly to the solution of the reduced problem of the given problem throughout the interval $[0, 1]$ while the derivatives generally converge nonuniformly as $\varepsilon \to 0$ either at $x = 0(a(x) \geq \alpha > 0)$ or at $x = 1(a(x) \leq \alpha < 0)$. There is vast literature dealing with numerical solution for the above type problems; see [2, 3, 4] for a survey.

Various methods for finding numerical solutions for problems involving singularly perturbed second order ordinary differential equations with non - smooth data (discontinuous source term/convection coefficient) using special piecewise uniform meshes (Shishkin mesh and Bakhvalov mesh) have been considered widely in the literature (see [6]–[12] and references therein). While many finite difference methods have been proposed to approximate such solutions, there has been much less research into the finite-difference approximation of their derivatives, even though such approximations are desirable in certain applications. It should be noted that for convection-diffusion problems, the attainment of high accuracy in a computed solution does not automatically lead to good approximation of derivatives of the true solution.

In [13], for singularly perturbed convection-diffusion problems with continuous convection coefficient and source term estimates for numerical derivatives have been derived. Here the scaled derivative is taken on whole domain where as Natalia Kopteva and Martin Stynes [14] have obtained approximation of derivatives with scaling in the boundary layer region and without scaling in the outer region. It may be noted that the source term and convection coefficient are smooth for the problem considered in [13, 14]. R. Mythili Priyadharshini and N. Ramanujam [16], have determined estimate for the scaled derivative for a singularly perturbed reaction-convection-diffusion problem with two parameters.

In [8], the authors have obtained bounds on the errors in approximations to the scaled derivative in the whole domain in the case of discontinuous source term. In [9], the authors have obtained bounds on the errors in approximations to the scaled derivative in the whole domain in the case of discontinuous convection coefficient using hybrid difference schemes. R. Mythili Priyadharshini and N. Ramanujam [10], have derived estimates for the scaled derivative in the boundary layer region and non-scaled derivative in the outer region for the boundary value problems with Robin type boundary conditions and discontinuous convection coefficient and source term. In [5], the authors have suggested a modified upwind scheme for singularly perturbed two-point boundary value problems with smooth data and showed that the scheme is superior to the standard upwind scheme. Zhongdi Cen [6] has suggested a hybrid finite difference scheme for singularly perturbed convection - diffusion problem with discontinuous convection coefficient and Dirichlet type boundary conditions. As far as author's knowledge goes, only few works have been reported in the literature for finding approximation to scaled derivatives of the solution for problems having discontinuous convection coefficient for both upwind and hybrid finite difference schemes on Shishkin mesh. In this paper, we present two hybrid difference schemes which are of higher order convergent for the singularly perturbed second order ordinary differential equations with a discontinuous convection coefficient on a Shishkin mesh. The method is shown to be parameter uniform convergent. Since the derivatives are related to flux or drag in physical and chemical applications, we obtain approximations not only to the solution but also to its derivatives.

Note: Through out this paper, C denotes a generic constant (sometimes subscripted) is independent of the singular perturbation parameter ε and the dimension of the discrete problem N. Note that C can take different values at different place, even in the same argument. Let $y : D \longrightarrow \mathbb{R}$, $D \subset \mathbb{R}$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the supremum norm $\| \; y \; \| = \sup_{x \in D} |y(x)|$. We also assume that $\varepsilon \leq CN^{-1}$ as is generally the case for discretization of convection-dominated problems.

2. **CONTINUOUS PROBLEM**

Consider the singularly perturbed second order ordinary differential equation with discontinuous convection coefficient and source term on the unit interval $\Omega = (0, 1)$.

Find $u \in Y \equiv C^1(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ such that

$$Lu(x) \equiv \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad x \in \Omega^- \cup \Omega^+ \tag{2.1}$$

$$B_0 u(0) \equiv -u'(0) = A, \quad B_1 u(1) \equiv u(1) + \varepsilon u'(1) = B, \tag{2.2}$$

$$-\alpha_1^* < a(x) < -\alpha_1 < 0, \quad \text{for} \quad x > d, \quad \alpha_2^* > a(x) > \alpha_2 > 0, \quad \text{for} \quad x < d,$$
$$b(x) > \beta > 0, \quad x \in \Omega, \quad |[a](d)| \leq C, \quad |[f](d)| \leq C,$$

where $0 < \varepsilon \ll 1$ is a small positive parameter, $\overline{\Omega} = [0, 1]$, $d \in \Omega$, $\Omega^- = (0, d)$, $\Omega^+ = (d, 1)$. For the functions $a(x)$ and $f(x)$ we assume they are sufficiently smooth on $\Omega^- \cup \Omega^+$ and have a jump discontinuity at $x = d$. Further it is assumed that $f(x)$ and $a(x)$ have right and left limits at $x = d$. Note the sign pattern of the coefficient $a(x)$ of the first derivative which is negative to the left of the point of discontinuity and positive to the right of this point. In general, there is an interior layer in the vicinity of the point of discontinuity $x = d$. We denote the jump at d in any function with $[w](d) = w(d+) - w(d-)$.

The sharper bounds on the derivatives of the solution are obtained by decomposing the solution $u = v + w$, into regular component v and an interior layer component w [7, Lemma 4]. Thus the function v is defined by

$$Lv(x) = f(x), \, x \in \Omega^- \cup \Omega^+, \tag{2.3}$$

$$B_0v(0) = B_0u(0),\ v(d-) = v_0(d-) + \varepsilon v_1(d-) + \varepsilon^2 v_2(d-) + \varepsilon^3 v_3(d-), \tag{2.4}$$

$$v(d+) = v_0(d+) + \varepsilon v_1(d+) + \varepsilon^2 v_2(d+) + \varepsilon^3 v_3(d+),\ B_1v(1) = B_1u(1). \tag{2.5}$$

Now, we define the function w as

$$Lw(x) = 0, \quad x \in \Omega^- \cup \Omega^+, \tag{2.6}$$

$$B_0w(0) = 0, \quad [w](d) = -[v](d), \quad [w'](d) = -[v'](d), \quad B_1w(1) = 0. \tag{2.7}$$

Hence $w(d-) = u(d-) - v(d-)$ and $w(d+) = u(d+) - v(d+)$.

Lemma 2.1. *For each integer k, satisfying $0 \le k \le 4$, the solutions v and w of* (2.3)–(2.5) *and* (2.6)–(2.7) *respectively satisfy the following bounds:*

$$\| v \| \le C, \quad \| v^{(k)} \|_{\Omega^- \cup \Omega^+} \le C(1 + \varepsilon^{3-k}),$$

$$|[v](d)|,\ |[v'](d)|,\ |[v''](d)|,\ |[v'''](d)| \le C$$

$$\textit{and} \qquad |w^{(k)}(x)| \le \begin{cases} C\varepsilon^{1-k} e^{-(d-x)\alpha_1/\varepsilon}, & x \in \Omega^-, \\ C\varepsilon^{1-k} e^{-(x-d)\alpha_2/\varepsilon}, & x \in \Omega^+. \end{cases}$$

Proof. Using the technique adopted in [7] and applying the argument separately on each of the subintervals $[0, d]$ and $[d, 1]$, the present theorem can be proved. □

3. DISCRETE PROBLEM

A fitted mesh method for the problem (2.1), (2.2) is now introduced. On Ω a piecewise uniform mesh of N mesh intervals are constructed as follows. The domain $\overline{\Omega}$ is subdivided into the four subintervals $[0, d-\sigma_1] \cup [d-\sigma_1, d] \cup [d, d+\sigma_2] \cup [d+\sigma_2, 1]$ for some σ_1, σ_2 that satisfy $0 < \sigma_1 \le \frac{d}{2}$, $\quad 0 < \sigma_2 \le \frac{1-d}{2}$. On each subinterval a uniform mesh with $N/4$ mesh intervals are placed. The interior points of the mesh are denoted by $\Omega^N = \{x_i : 1 \le i \le \frac{N}{2} - 1\} \cup \{x_i : \frac{N}{2} + 1 \le i \le N - 1\}$. Clearly $x_{N/2} = d$ and $\overline{\Omega}^N = \{x_i\}_0^N$. It is fitted to the singular perturbation problem (2.1), (2.2) by choosing σ_1 and σ_2 to be the following functions of N and ε : $\sigma_1 = \min\{\frac{d}{2}, \frac{2\varepsilon}{\alpha} \ln N\}$ and $\sigma_2 = \min\{\frac{1-d}{2}, \frac{2\varepsilon}{\alpha} \ln N\}$ where $\alpha = \min\{\alpha_1, \alpha_2\}$. For our analysis we assume that $\sigma_1 = \sigma_2 = \sigma = \frac{2\varepsilon}{\alpha} \ln N$, since otherwise N^{-1} is exponentially small compared with ε. Then the mesh widths are

$$h_i = \begin{cases} H_1 = \dfrac{4(d-\sigma)}{N}, & i = 1, \dots, N/4, \\ h = \dfrac{4\sigma}{N}, & i = N/4 + 1, \dots, 3N/4 - 1, \\ H_2 = \dfrac{4(1-d-\sigma)}{N}, & i = 3N/4, \dots, N. \end{cases}$$

We discretize (2.1) using the central difference approximation

$$L_c^N U_i \equiv \frac{2\varepsilon}{h_i + h_{i+1}} \left(\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i} \right) + a_i \frac{U_{i+1} - U_{i-1}}{h_i + h_{i+1}} - b_i U_i = f_i, \tag{3.1}$$

whenever the local mesh size allows us to do this without loosing stability, but we employ a second order upwind scheme otherwise:

(3.2)

$$L_u^N U_i \equiv \begin{cases} \frac{2\varepsilon}{h_i+h_{i+1}}(\frac{U_{i+1}-U_i}{h_{i+1}} - \frac{U_i-U_{i-1}}{h_i}) + a_{i-1/2}\frac{U_i-U_{i-1}}{h_i} - b_{i-1/2}\bar{U}_i = f_{i-1/2}, \text{ if } a_i < 0, \\ \frac{2\varepsilon}{h_i+h_{i+1}}(\frac{U_{i+1}-U_i}{h_{i+1}} - \frac{U_i-U_{i-1}}{h_i}) + a_{i+1/2}\frac{U_{i+1}-U_i}{h_{i+1}} - b_{i+1/2}\bar{U}_i = f_{i+1/2}, \text{ if } a_i > 0, \end{cases}$$

where $U_i = U(x_i)$, $\bar{U}_i = \frac{U(x_i)+U(x_{i+1})}{2}$, $a_{i-1/2} \equiv a((x_{i-1}+x_i)/2)$ and $a_{i+1/2} \equiv a((x_i + x_{i+1})/2)$; similarly for $b_{i-1/2}$, $b_{i+1/2}$, $f_{i-1/2}$, $f_{i+1/2}$. At the point $x_{N/2} = d$, we shall use the difference operator

$$L_t^N U_{N/2} \equiv \frac{-U_{N/2+2} + 4U_{N/2+1} - 3U_{N/2}}{2h} - \frac{U_{N/2-2} - 4U_{N/2-1} + 3U_{N/2}}{2h} = 0. \tag{3.3}$$

From equation (3.1) we get

$$U_{N/2-2} = \frac{2h^2}{2\varepsilon - ha_{N/2-1}}(f_{N/2-1} + (\frac{2\varepsilon}{h^2} + b_{N/2-1})U_{N/2-1} - (\frac{\varepsilon}{h^2} + \frac{a_{N/2-1}}{2h})U_{N/2}),$$

$$U_{N/2+2} = \frac{2h^2}{2\varepsilon + ha_{N/2+1}}(f_{N/2+1} + (\frac{2\varepsilon}{h^2} + b_{N/2-1})U_{N/2-1} - (\frac{\varepsilon}{h^2} - \frac{a_{N/2-1}}{2h})U_{N/2}).$$

Inserting the expression for $U_{N/2-2}$ and $U_{N/2+2}$ in (3.3), we get

$$\begin{aligned} L_t^N U_{N/2} =& \frac{1}{2h}(4 - (\frac{2h^2}{2\varepsilon - ha_{N/2-1}})(\frac{2\varepsilon}{h^2} + b_{N/2-1}))U_{N/2-1} - \frac{1}{2h}(6 - (\frac{2h^2}{2\varepsilon - ha_{N/2-1}}) \\ & (\frac{\varepsilon}{h^2} + \frac{a_{N/2-1}}{2h}) + (\frac{2h^2}{2\varepsilon + ha_{N/2+1}})(\frac{\varepsilon}{h^2} - \frac{a_{N/2+1}}{2h}))U_{N/2} + \frac{1}{2h}(4 - (\frac{2h^2}{2\varepsilon + ha_{N/2+1}}) \\ & (\frac{2\varepsilon}{h^2} + b_{N/2+1}))U_{N/2+1} = \frac{hf_{N/2-1}}{2\varepsilon - ha_{N/2-1}} + \frac{hf_{N/2+1}}{2\varepsilon + ha_{N/2+1}}. \end{aligned}$$

Thus, we have

$$L^N U_i = f_i, \quad \text{for} \quad i = 1, \ldots, N-1, \tag{3.4}$$

$$\text{where,} \quad L^N U_i = \begin{cases} L_u^N U_i, & \text{for} \quad i = 1, \ldots, N/4, 3N/4, \ldots, N-1, \\ L_c^N U_i, & \text{for} \quad i = N/4+1, \ldots, N/2-1, N/2+1, \ldots, 3N/4-1, \\ L_t^N U_i, & \text{for} \quad i = N/2 \end{cases}$$

$$\text{and} \quad f_i = \begin{cases} f_{i-1/2}, & \text{for} \quad i = 1, \ldots, N/4, \\ f_{i+1/2}, & \text{for} \quad i = 3N/4, \ldots, N-1, \\ f_i, & \text{for} \quad i = N/4+1, \ldots, N/2-1, N/2+1, \ldots, 3N/4-1, \\ \frac{hf_{N/2-1}}{2\varepsilon - ha_{N/2-1}} + \frac{hf_{N/2+1}}{2\varepsilon + ha_{N/2+1}}, & \text{for} \quad i = N/2. \end{cases}$$

We now approximate the boundary conditions (2.2) in two different ways. First we approximate the first derivative by centred finite difference operator:

$$B_0 U_0 \equiv -D^0 U_0 = A \quad \text{and} \quad B_N U_N \equiv U_N + \varepsilon D^0 U_N = B.$$

This is now modified as follows.

$$-\frac{U_1 - U_{-1}}{2H_1} = A \qquad \text{(or)} \qquad U_{-1} = U_1 + 2H_1 A, \tag{3.5}$$

$$U_N + \varepsilon \frac{U_{N+1} - U_{N-1}}{2H_2} = B \qquad \text{(or)} \qquad U_{N+1} = -\frac{2H_2}{\varepsilon} U_N + U_{N-1} + \frac{2H_2}{\varepsilon} B, \tag{3.6}$$

where U_{-1} and U_{N+1} are the functional values at x_{-1} and x_{N+1}. The nodes x_{-1} and x_{N+1} lie outside the interval $[0, 1]$ and are called fictitious nodes. The values U_{-1} and U_{N+1} may be eliminated by assuming that the difference equation (3.2) holds also for $i = 0$ and N, that is at the boundary points x_0 and x_N. Substituting the values U_{-1} and U_{N+1} from (3.5) and (3.6) into the equations (3.2) for $i = 0$ and $i = N$, we get respectively

$$\begin{aligned} B_0^* U_0 &\equiv \left(\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} + \frac{b(x_0)}{2} \right) U_0 - \left(\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} - \frac{b(x_0)}{2} \right) U_1 \\ &= 2H_1 A \left(\frac{\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} - \frac{b(x_0)}{2} \right) - f_{-1/2} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} B_N^* U_N &\equiv \left(\frac{2\varepsilon}{H_2^2} + \frac{a(x_N)}{H_2} + \frac{b(x_N)}{2} + \frac{2}{H_2} + \frac{2a(x_N)}{\varepsilon} - \frac{b(x_N)H_2}{2} \right) U_N \\ &\quad - \left(\frac{2\varepsilon}{H_2^2} + \frac{a(x_N)}{H_2} - \frac{b(x_N)}{2} \right) U_{N-1} \\ &= \frac{2H_2 B}{\varepsilon} \left(\frac{\varepsilon}{H_2^2} + \frac{a(x_N)}{H_2} - \frac{b(x_N)}{2} \right) - f_{N+1/2}. \end{aligned} \tag{3.8}$$

Therefore, the final form of the first difference scheme consists of

$$L^N U_i = f(x_i), \qquad \text{for} \quad i = 1, 2, \ldots, N-1, \tag{3.9}$$

$$B_0^* U_0, \qquad B_N^* U_N. \tag{3.10}$$

Alternatively, we may not use the fictitious points x_{-1} and x_{N+1}. In this case, we may use the following approximation.

$$\Theta_0 U_0 \equiv -[\frac{1}{2H_1}(-3U_0 + 4U_1 - U_2)] = A,$$

$$\text{implies} \qquad 3U_0 - 4U_1 + U_2 = 2H_1 A \tag{3.11}$$

$$\text{and} \qquad \Theta_N U_N \equiv U_N + \varepsilon[\frac{1}{2H_2}(3U_N - 4U_{N-1} + U_{N-2})] = B,$$

$$\text{implies} \qquad (2H_2 + 3\varepsilon)U_N - 4\varepsilon U_{N-1} + \varepsilon U_{N-2} = 2H_2 B. \tag{3.12}$$

If we eliminate U_2 from (3.11) using the first equation of the set (3.4) and eliminate U_{N-2} from (3.12) using the last equation of the set (3.4), we get respectively

$$\begin{aligned} \Theta_0^* U_0 &\equiv (2 + \frac{a(x_0)H_1}{\varepsilon} + \frac{b(x_0)H_1^2}{2\varepsilon})U_0 - (2 + \frac{a(x_0)H_1}{\varepsilon} - \frac{b(x_0)H_1^2}{2\varepsilon})U_1 \\ &= 2H_1 A - \frac{H_1^2}{\varepsilon} f_{1/2} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \Theta_N^* U_N &\equiv (2H_2 + 2\varepsilon - a(x_N)H_2 + \frac{b(x_N)H_2^2}{2})U_N - (2\varepsilon - a(x_N)H_2 - \frac{b(x_N)H_2^2}{2})U_{N-1} \\ &= 2H_2 B - H_2^2 f_{N-1/2}. \end{aligned} \tag{3.14}$$

Therefore, the final form of the second difference scheme consists of

$$L^N U_i = f(x_i), \qquad \text{for} \quad i = 1, 2, \ldots, N-1, \tag{3.15}$$

$$\Theta_0^* U(x_0), \qquad \Theta_N^* U(x_N). \tag{3.16}$$

Remark 3.1. The truncation error for (3.7) is given by

$$\begin{aligned}
|B_0^*(U-u)(x_0)| &= |(\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} + \frac{b(x_0)}{2})U_0 - (\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} - \frac{b(x_0)}{2})U_1 \\
&- 2H_1 A(\frac{\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} - \frac{b(x_0)}{2}) + f_{-1/2}| \\
&\leq |(\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} + \frac{b(x_0)}{2})U_0 - (\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} - \frac{b(x_0)}{2})(U_0 + H_1 U_0' \\
&+ \frac{H_1^2}{2}U_0'' + \frac{H_1^3}{6}U_0^{(3)} + \cdots) + 2H_1(\frac{\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} - \frac{b(x_0)}{2})u'(x_0) \\
&+ \frac{1}{2}(2\varepsilon u''(x_0) + 2a(x_0)u'(x_0) - 2b(x_0)u(x_0) + b(x_0)H_1 u'(x_0) \\
&- a(x_0)H_1 u''(x_0) - \frac{b(x_0)H_1^2}{2}u''(x_0) - \varepsilon H_1 u^{(3)}(x_0) + \frac{a(x_0)H_1^2}{2}u^{(3)}(x_0) \\
&- \frac{b(x_0)H_1^3}{6}u^{(3)}(x_0) + \cdots)| \leq C\varepsilon H_1|u^{(3)}(x_0)| \quad \leq CH_1^2|u^{(3)}(x_0)|.
\end{aligned}$$

Similarly, the truncation error for (3.8), (3.13) and (3.14) are respectively given as

$$\begin{aligned}
|B_N^*(U-u)(x_N)| &\leq C\varepsilon H_2|u^{(3)}(x_N)| \quad \leq CH_2^2|u^{(3)}(x_N)|, \\
|\Theta_0^*(U-u)(x_0)| &\leq CH_1^3|u^{(3)}(x_0)|, \\
\text{and} \quad |\Theta_N^*(U-u)(x_N)| &\leq CH_2^3|u^{(3)}(x_N)|.
\end{aligned}$$

Further, we have from [15, §4],

$$|L^N(U-u)(x_i)| \leq \begin{cases} \varepsilon H_1 \parallel u^{(3)} \parallel + C_{(\|a\|,\|a'\|)} H_1^2(\parallel u^{(3)} \parallel + \parallel u^{(2)} \parallel), \\ \hfill i = 1, \ldots, N/4, \\ \varepsilon h^2 \parallel u^{(4)} \parallel + \parallel a \parallel h^2 \parallel u^{(3)} \parallel, \quad i = N/4+1, \ldots, 3N/4-1, \\ \varepsilon H_2 \parallel u^{(3)} \parallel + C_{(\|a\|,\|a'\|)} H_2^2(\parallel u^{(3)} \parallel + \parallel u^{(2)} \parallel), \\ \hfill i = 3N/4, \ldots, N-1. \end{cases}$$

Note that $C_{(\|a\|,\|a'\|)}$ is a positive constant that depends on $\parallel a \parallel$ and $\parallel a' \parallel$.

In this paper, we present the theoretical results and error estimate for the difference scheme (3.9), (3.10) and adopting the same technique one can obtain the same theoretical results and error estimate for the difference scheme (3.15), (3.16).

To guarantee the monotonicity property of the difference operator L^N, we impose the following mild assumption on the minimum number of mesh points

$$\frac{N}{\ln N} \geq 4\max\{\frac{\alpha^*}{\alpha}, \frac{\beta}{\alpha}\}, \quad \text{where} \quad \alpha^* = \min\{\alpha_1^*, \alpha_2^*\}, \; \alpha = \min\{\alpha_1, \alpha_2\}. \tag{3.17}$$

Lemma 3.2. *Assume* (3.17). *Then the operator* L^N *defined by* (3.9) *satisfies a discrete minimum principle, that is, if* $Z(x_i)$, $i = 0, 1, \ldots, N$ *is a mesh function that satisfies* $B_0^* Z(x_0) \geq 0$, $B_N^* Z(x_N) \geq 0$ *and* $L^N Z(x_i) \leq 0$, *for* $1, \ldots, N-1$, *then* $Z(x_i) \geq 0$, *for all* $i = 0, \ldots, N$.

Proof. Define

$$S(x_i) = \begin{cases} 1 - x_i, & i = 0, \\ \dfrac{1}{2} + \dfrac{x_i}{8} - \dfrac{d}{8}, & i = 1, \ldots, N/2, \\ \dfrac{1}{2} - \dfrac{x_i}{4} + \dfrac{d}{4}, & i = N/2, \ldots, N-1, \\ x_i & i = N. \end{cases} \tag{3.18}$$

Then $S(x_i) > 0$, $x_i \in \overline{\Omega}^N$,

$$\begin{aligned} B_0^* S(x_0) &= \left(\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} + \frac{b(x_0)}{2}\right) - \left(\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} - \frac{b(x_0)}{2}\right)(1 - x_1) > 0, \\ B_N^* S(x_N) &= \left(\frac{2\varepsilon}{H_2^2} + \frac{a(x_N)}{H_2} + \frac{b(x_N)}{2} + \frac{2}{H_2} + \frac{2a(x_N)}{\varepsilon} - \frac{b(x_N)H_2}{2}\right) \\ &- \left(\frac{2\varepsilon}{H_2^2} + \frac{a(x_N)}{H_2} - \frac{b(x_N)}{2}\right) x_{N-1} > 0 \end{aligned}$$

$$\begin{aligned} \text{and} \quad L^N S(x_i) &= \begin{cases} \frac{a(x_i)}{8} - b(x_i)s(x_i), & x_i \in \Omega^N \cap (0, d) \\ \frac{-a(x_i)}{4} - b(x_i)s(x_i), & x_i \in \Omega^N \cap (d, 1) \end{cases} \\ &< \begin{cases} 0, & x_i \in \Omega^N \cap (0, d) \\ 0, & x_i \in \Omega^N \cap (d, 1). \end{cases} \end{aligned}$$

We define $\mu = \max\{\max_{0 \leq i \leq N} (\dfrac{-Z}{S})(x_i)\}$. Assume that the theorem is not true. Then $\mu > 0$ and $(Z + \mu S)(x_i) = 0$. Further there exists a $i^* \in \{0, 1, 2, \ldots, N\}$ such that $(Z + \mu S)(x_{i^*}) = 0$ and we consider the following cases.

Case (i): $(Z + \mu S)(x_{i^*}) = 0$, for $i^* = 0$. Therefore,

$$\begin{aligned} 0 \leq B_0^*(Z + \mu S)(x_{i^*}) &\leq (\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} + \frac{b(x_0)}{2})(Z + \mu S)(x_{i^*}) \\ &- (\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} - \frac{b(x_0)}{2})(Z + \mu S)(x_{i^*+1}) < 0, \end{aligned}$$

which is a contradiction.

Case (ii): $(Z + \mu S)(x_{i^*}) = 0$, for $0 < i^* < N$.

Therefore, $0 \geq L^N(Z + \mu S)(x_{i^*}) > 0$, which is a contradiction.

Case (iii): $(Z + \mu S)(x_{i^*}) = 0$, for $i^* = N$. Therefore,

$$0 \leq B_N(Z + \mu S)(x_{i^*}) \leq (\frac{2\varepsilon}{H_2^2} + \frac{a(x_N)}{H_2} + \frac{b(x_N)}{2} + \frac{2}{H_2} + \frac{2a(x_N)}{\varepsilon} - \frac{b(x_N)H_2}{2})$$

$$(Z+\mu S)(x_{i^*}) - (\frac{2\varepsilon}{H_2^2} + \frac{a(x_N)}{H_2} - \frac{b(x_N)}{2})(Z+\mu S)(x_{i^*-1}) < 0,$$

which is a contradiction.

Hence the proof of the theorem. □

Lemma 3.3. *Let U_i be a solution of* (3.9), (3.10), *then*

$$\| U \| \leq \max\{|B_0^* U_0|, |B_N^* U_N|\} + \frac{1}{\gamma} \| f \|,$$

where $\gamma = \min\{\frac{\alpha_1}{d}, \frac{\alpha_2}{1-d}\}$.

Proof. Let $C_1 = C(\max\{|B_0^* U_0|, |B_N^* U_N|\} + \frac{1}{\gamma} \| f \|)$. Define the mesh functions

$$\Psi^{\pm}(x_i) = C_1 S(x_i) \pm U(x_i),$$

where $S(x_i)$ is given in (3.18). Then we have $B_0^* \Psi^{\pm}(x_0) \geq 0$, $B_N^* \Psi^{\pm}(x_N) \geq 0$ and $L^N \Psi^{\pm}(x_i) \leq 0, \quad x_i \in \Omega^N$. By Lemma 3.2, we get the required result. □

Using the technique adopted in [6, pp 695], we can decompose the discrete solution as $U(x_i) = V(x_i) + W(x_i)$.

Lemma 3.4. *At each mesh point x_i, the regular component of the error satisfies the estimate*

$$|(V-v)(x_i)| \leq \begin{cases} C\dfrac{1+x_i}{1+d} N^{-2}, & \text{for} \quad i = 0, \ldots, N/2 \\ C\dfrac{2-x_i}{2-d} N^{-2}, & \text{for} \quad i = N/2+1, \ldots, N. \end{cases} \tag{3.19}$$

Proof. Let us now consider the truncation error at the mesh points. As given in Remark 3.1, we can prove that

$$\begin{aligned} |B_0^*(V-v)(x_0)| &\leq CN^{-2} |v^{(3)}(x_0)| \leq CN^{-2}, \\ |B_N^*(V-v)(x_N)| &\leq CN^{-2} |v^{(3)}(x_N)| \leq CN^{-2}, \\ |L^N(V-v)(x_i)| &\leq \begin{cases} \varepsilon H_1 \| v^{(3)} \| + C_{(\|a\|, \|a'\|)} H_1^2 (\| v^{(3)} \| + \| v^{(2)} \|), \\ \hfill i = 1, \ldots, N/4, \\ \varepsilon h^2 \| v^{(4)} \| + \| a \| h^2 \| v^{(3)} \|, \quad i = N/4+1, \ldots, 3N/4-1, \\ \varepsilon H_2 \| v^{(3)} \| + C_{(\|a\|, \|a'\|)} H_2^2 (\| v^{(3)} \| + \| v^{(2)} \|), \\ \hfill i = 3N/4, \ldots, N-1, \end{cases} \\ &\leq CN^{-2}, \quad i = 1, \ldots, N-1. \end{aligned}$$

Consider the two mesh functions $\Psi^{\pm}(x_i) = \Phi(x_i) \pm (V-v)(x_i)$, where

$$\Phi(x_i) = \begin{cases} C(1-x_i) N^{-2} & \text{for} \quad i = 0 \\ C\dfrac{1+x_i}{1+d} N^{-2} & \text{for} \quad i = 1, \ldots, N/2-1 \\ C\dfrac{2-x_i}{2-d} N^{-2} & \text{for} \quad i = N/2+1, \ldots, N-1 \\ C x_i N^{-2} & \text{for} \quad i = N. \end{cases}$$

Then, we have

$$\begin{aligned}
B_0^*\Psi^{\pm}(x_0) &\equiv C(\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} + \frac{b(x_0)}{2})N^{-2} - C(\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} - \frac{b(x_0)}{2})(1-x_1)N^{-2} \\
&\quad \pm B_0^*(V-v)(x_0) \\
&\geq Cb(x_0)N^{-2} + C(\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} - \frac{b(x_0)}{2})x_1 N^{-2} \pm CN^{-2} > 0, \\
L^N\Psi^{\pm}(x_i) &\leq 0, \quad \text{for} \quad x_i \in \Omega^N \\
B_N^*\Psi^{\pm}(x_N) &\equiv C(\frac{2\varepsilon}{H_2^2} + \frac{a(x_N)}{H_2} + \frac{b(x_N)}{2} + \frac{2}{H_2} + \frac{2a(x_N)}{\varepsilon} - \frac{b(x_N)H_2}{2})N^{-2} \\
&\quad -C(\frac{2\varepsilon}{H_2^2} + \frac{a(x_N)}{H_2} - \frac{b(x_N)}{2})x_{N-1}N^{-2} \pm B_N^*(V-v)(x_N) \\
&\geq C(b(x_N) + \frac{2}{H_2} + \frac{2a(x_N)}{\varepsilon} - \frac{b(x_N)H_2}{2})N^{-2} \\
&\quad +C(\frac{2\varepsilon}{H_2^2} + \frac{a(x_N)}{H_2} - \frac{b(x_N)}{2})H_2 N^{-2} \pm CN^{-2} > 0.
\end{aligned}$$

Applying Lemma 3.2, we get $\Psi^{\pm}(x_i) \geq 0$. Thus, we get the required result. □

Lemma 3.5. *At each mesh point x_i, the layer component of the error satisfies the estimate*

$$|(W-w)(x_i)| \leq CN^{-2}(\ln N)^2, \quad \textit{for} \quad i = 0, \ldots, N.$$

Proof. As given in Remark 3.1, we have

$$\begin{aligned}
|B_0^*(W-w)(x_0)| &\leq CH_1^2|w^{(3)}(x_0)| \leq \frac{C\sigma^2 N^{-2}}{\varepsilon^2} \leq CN^{-2}(\ln N)^2, \\
|B_N^*(W-w)(x_N)| &\leq CH_2^2|w^{(3)}(x_N)| \leq \frac{C\sigma^2 N^{-2}}{\varepsilon^2} \leq CN^{-2}(\ln N)^2.
\end{aligned}$$

Since $|U(x_{N/2})| \leq C$ and with (3.19), we have $|W(x_{N/2})| \leq C$ and for $x_i \in \Omega^N \cap (0, d-\sigma)$ and $x_i \in \Omega^N \cap (d+\sigma, 1)$

$$|(W-w)(x_i)| \leq |W(x_i)| + |w(x_i)| \leq CN^{-2} + Ce^{-\alpha\sigma/\varepsilon} \leq CN^{-2}. \tag{3.20}$$

Applying the procedure adopted in [6, Theorem 1], we have

$$|L^N(W-w)(x_i)| \leq \begin{cases} Ch^2\varepsilon^{-2}\exp(-(d-x_i)\alpha_1/\varepsilon), & \text{for} \quad i = N/4+1, \ldots, N/2-1, \\ Ch^2\varepsilon^{-2}\exp(-(x_i-d)\alpha_2/\varepsilon), & \text{for} \quad i = N/2+1, \ldots, 3N/4-1 \end{cases}$$

and at the mesh point $x_{N/2} = d$,

$$|L^N W(x_{N/2}) - \frac{hf_{N/2-1}}{2\varepsilon - ha_{N/2-1}} - \frac{hf_{N/2+1}}{2\varepsilon + ha_{N/2-1}}| \leq Ch^2\varepsilon^{-2}.$$

Consider the two mesh functions $\Phi^{\pm}(x_i) = \Psi(x_i) \pm (W - w)(x_i)$, where

$$\Psi(x_i) = CN^{-2} + \frac{C\sigma^2 N^{-2}}{\varepsilon^2} \begin{cases} 1 + (d-\sigma) - x_i, & \text{for} \quad x_i = d - \sigma, \\ 1 + x_i - (d-\sigma), & \text{for} \quad x_i \in \Omega^N \cap (d-\sigma, d), \\ 1 + (d+\sigma) - x_i, & \text{for} \quad x_i \in \Omega^N \cap (d, d+\sigma), \\ 1 + x_i - (d-\sigma), & \text{for} \quad x_i = d + \sigma. \end{cases}$$

Note that

$$\begin{aligned} B_0^*\Psi(d-\sigma) &\geq Cb(x_0)N^{-2} + \frac{C\sigma^2N^{-2}}{\varepsilon^2}(b(x_0) + H_1(\frac{2\varepsilon}{H_1^2} - \frac{a(x_0)}{H_1} - \frac{b(x_0)}{2})) > 0, \\ L^N\Psi(x_i) &\leq \begin{cases} -\frac{C\sigma^2N^{-2}}{\varepsilon^2}\alpha_1 - \beta\Psi(x_i), \\ -\frac{C\sigma^2N^{-2}}{\varepsilon^2}\alpha_2 - \beta\Psi(x_i), \end{cases} < \begin{cases} 0, & x_i \in \Omega^N \cap (d-\sigma, d) \\ 0, & x_i \in \Omega^N \cap (d, d+\sigma) \end{cases} \\ B_N^*\Psi(d+\sigma) &\geq C(b(x_N) + \frac{2}{H_2} + \frac{2a(x_N)}{\varepsilon} - \frac{b(x_N)H_2}{2})N^{-2} + \frac{C\sigma^2N^{-2}}{\varepsilon^2}(b(x_N) \\ &\quad + \frac{2}{H_2} + \frac{2a(x_N)}{\varepsilon} - \frac{b(x_N)H_2}{2} + H_2(\frac{2\varepsilon}{H_2^2} + \frac{a(x_N)}{H_2} - \frac{b(x_N)}{2})) > 0 \end{aligned}$$

and $L^N\Psi^{\pm}(x_{N/2}) < 0$. Applying Lemma 3.2 to $\Phi^{\pm}(x_i)$, over the interval $[d-\sigma, d+\sigma]$, we get $|(W-w)(x_i)| \leq CN^{-2}(\ln N)^2$. Thus, we get the required result. □

Theorem 3.6. *Let u be the solution of* (2.1), (2.2) *and U be the corresponding numerical solution generated by the difference scheme* (3.9), (3.10) *or* (3.15), (3.16). *Then for each i, $0 \leq i \leq N$, we have*

$$|(U-u)(x_i)| \leq CN^{-2}(\ln N)^2.$$

Proof. The proof of the theorem follows immediately, if one applies the above Lemmas 3.4 and 3.5 to $U - u = (V - v) + (W - w)$. In the case of difference scheme (3.15), (3.16) we obtain the same bounds using lemmas analogous to Lemmas 3.4 and 3.5. □

4. ANALYSIS ON DERIVATIVE APPROXIMATION

In this section, we approximate the scaled derivative $\varepsilon u'$ of the solution of the problem (2.1), (2.2) by the scaled centred discrete derivative $\varepsilon D^0U(x_i)$ at all internal points x_i, $i = 1, \ldots, N-1$. We note that for $i = 1, \ldots, N-1$, the error $e(x_i) \equiv U(x_i) - u(x_i)$, satisfies the equations $L^Ne(x_i) + b(x_i)e(x_i) = b(x_i)e(x_i) +$ truncation error, where, by Theorem 3.6, $b(x_i)e(x_i) = O(N^{-2}(\ln N)^2)$. In the proofs of the following lemmas and theorems, we use the above equation. Hence the analysis carried out in [13, §3.5] and [10] can be applied immediately with a slight modification where ever necessary. Therefore, proofs for some lemmas are omitted; for some of the theorems short proves are given.

Lemma 4.1. *At each mesh point $x_i \in \Omega^N$ and for all $x \in \bar{\Omega}_i = [x_i, x_{i+1}]$, we have*

$$|\varepsilon(D^0 u(x_i) - u'(x))| \le CN^{-2} \ln^2 N,$$

where $u(x)$ is the solution of (2.1), (2.2).

Proof. Using the technique adopted in [13, Lemma 3.13], [10] and applying the argument separately on each of the subintervals $\Omega^N \cap (0, d)$ and $\Omega^N \cap (d, 1)$, the present theorem can be proved. □

Lemma 4.2. *Let v and V be the exact and discrete regular components of the solutions of* (2.1), (2.2) *and* (3.9), (3.10) *respectively. Then for all $x_i \in \Omega^N$, we have*

$$|\varepsilon D^0 (V - v)(x_i)| \le CN^{-2}.$$

Proof. Using the technique adopted in [13, Lemma 3.14] and applying the argument separately on each of the subinterval $\Omega^N \cap (0, d - \sigma]$ and $\Omega^N \cap [d + \sigma, 1)$, we can prove that

$$|\varepsilon D^+ e(x_i)| \le CN^{-2}. \tag{4.1}$$

Similarly we can prove that $|\varepsilon D^- e(x_i)| \le CN^{-2}$.

Now we prove that $|\varepsilon D^+ e(x_i)| \le CN^{-2}$ for all $x_i \in \Omega^N \cap (d - \sigma, d + \sigma)$. Using the result from (4.1), we have $|\varepsilon D^+ e(x_{N/4})| \le CN^{-2}$. To prove the result for $N/4 + 1 \le i \le N/2 - 1$, $N/2 + 1 \le i \le 3N/4 - 1$, we rewrite the relation $\tau(x_i) = L^N e(x_i)$, in the form, $\varepsilon D^+ e(x_j) - \varepsilon D^+ e(x_{j-1}) + \frac{1}{4}(x_{j+1} - x_{j-1}) a(x_j)(D^+ e(x_j) + D^+ e(x_{j-1})) = \frac{1}{2}(x_{j+1} - x_{j-1})[\tau(x_j) + b(x_j) e(x_j)]$. Summing and rearranging, we obtain

$$\begin{aligned} |\varepsilon D^+ e(x_i)| \;\le\; & |\varepsilon D^+ e(x_{N/4})| + \frac{1}{2} \sum_{j=N/4+1}^{i} (x_{j+1} - x_{j-1})[|\tau(x_j)| + |b(x_j)||e(x_j)|] \\ & + \frac{1}{4} \Big| \sum_{j=N/4+1}^{i} (x_{j+1} - x_{j-1}) a(x_j)(D^+ e(x_j) + D^+ e(x_{j-1})) \Big|. \end{aligned}$$

We now bound each term separately. We have already bounded the first term. We know that $|\tau(x_j)| \le CN^{-2}$ and so the second term is also bounded by CN^{-2}. To bound the last term we observe that

$$\begin{aligned} (x_{j+1} - x_{j-1}) a(x_j) D^+ e(x_j) = & \Big(\frac{x_{j+1} - x_{j-1}}{x_{j+1} - x_j} a(x_j) e(x_{j+1}) - \frac{x_j - x_{j-2}}{x_j - x_{j-1}} a(x_{j-1}) e(x_j)\Big) \\ & - \frac{x_{j+1} - x_{j-1}}{x_{j+1} - x_j}(a(x_j) - a(x_{j-1})) e(x_j) - \Big(\frac{x_{j+1} - x_{j-1}}{x_{j+1} - x_j} - \frac{x_j - x_{j-2}}{x_j - x_{j-1}}\Big) a(x_{j-1}) e(x_j) \end{aligned}$$

and

$$\begin{aligned} (x_{j+1} - x_{j-1}) a(x_j) D^+ e(x_{j-1}) = & \Big(\frac{x_{j+1} - x_{j-1}}{x_j - x_{j-1}} a(x_j) e(x_j) - \frac{x_j - x_{j-2}}{x_{j-1} - x_{j-2}} a(x_{j-1}) e(x_{j-1})\Big) \\ & - \frac{x_{j+1} - x_{j-1}}{x_j - x_{j-1}}(a(x_j) - a(x_{j-1})) e(x_{j-1}) - \Big(\frac{x_{j+1} - x_{j-1}}{x_j - x_{j-1}} - \frac{x_j - x_{j-2}}{x_{j-1} - x_{j-2}}\Big) a(x_{j-1}) e(x_{j-1}). \end{aligned}$$

We now sum both side of these expressions. We observe that the terms in the first bracket on the right hand side of each expression telescope and that the last bracket on the right-hand side is non-zero only for $j = N/4 + 1$. It follows that

$$\sum_{j=N/4+1}^{i}(x_{j+1}-x_{j-1})a(x_j)D^+e(x_j) = (\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_i}a(x_i)e(x_{i+1}) - \frac{x_{N/4+1}-x_{N/4-1}}{x_{N/4+1}-x_{N/4}}a(x_{N/4})$$

$$e(x_{N/4+1})) - \sum_{j=N/4+1}^{i}\frac{x_{j+1}-x_{j-1}}{x_{j+1}-x_j}(a(x_j)-a(x_{j-1}))e(x_j) - (1-\frac{H_1}{h})a(x_{N/4})e(x_{N/4-1})$$

and

$$\sum_{j=N/4+1}^{i}(x_{j+1}-x_{j-1})a(x_j)D^+e(x_{j-1}) =$$

$$\left(\frac{x_{i+1}-x_{i-1}}{x_i-x_{i-1}}a(x_i)e(x_i) - \frac{x_{N/4+1}-x_{N/4-1}}{x_{N/4}-x_{N/4-1}}a(x_{N/4})e(x_{N/4})\right)$$

$$-\sum_{j=N/4+1}^{i}\frac{x_{j+1}-x_{j-1}}{x_j-x_{j-1}}(a(x_j)-a(x_{j-1}))e(x_{j-1}) - \left(1-\frac{h}{H_1}\right)a(x_{N/4})e(x_{N/4})).$$

Using the method of proof given in [13, Lemma 3.14] and the fact that $|e(x_j)| \leq CN^{-2}$ and $|a(x_j)-a(x_{j-1})| \leq \| a' \| (x_j - x_{j-1})$, we get for all i, $N/4+1 \leq i \leq N/2-1$, $N/2+1 \leq i \leq 3N/4-1$, $|\varepsilon D^+e(x_i)| \leq CN^{-2}$. Thus we have proved that $|\varepsilon D^+e(x_i)| \leq CN^{-2}$, $x_i \in \Omega^N$. Similarly we can prove that $|\varepsilon D^-e(x_i)| \leq CN^{-2}$, $x_i \in \Omega^N$. This implies that

$$|\varepsilon D^0e(x_i)| \equiv |\frac{\varepsilon(D^+ + D^-)e(x_i)}{2}| \leq CN^{-2}, \quad x_i \in \Omega^N, \text{ which completes the proof.}$$

□

Lemma 4.3. *Let w be the singular component of the solution of* (2.1), (2.2) *and W the discrete singular component of the solution of* (3.9), (3.10).

When $\sigma = \dfrac{2\varepsilon}{\alpha}\ln N$, we have

$$|W(x_i)| \leq \begin{cases} C(1-x_i)N^{-2}, & x_i \in \Omega^N \cap (0, d-\sigma] \\ C(3-x_i)N^{-2}, & x_i \in \Omega^N \cap [d-\sigma, 1) \end{cases} \tag{4.2}$$

and for $x_i \in \Omega^N \cap (0, d-\sigma]$ and $x_i \in \Omega^N \cap [d-\sigma, 1)$

$$|\varepsilon D^0W(x_i)| \leq CN^{-2}. \tag{4.3}$$

Proof. To prove (4.2), use the barrier functions

$$\Psi^{\pm}(x_i) = CN^{-2} + \begin{cases} |W(d-\sigma)|\dfrac{1-x_i}{1-(d-\sigma)} & x_i \in \Omega^N \cap (0, d-\sigma) \\ |W(d+\sigma)|\dfrac{3-x_i}{3-(d+\sigma)} & x_i \in \Omega^N \cap (d+\sigma, 1) \end{cases} \pm |W(x_i)|$$

to get the required result.

Finally, to prove (4.3), we use (4.2) and the procedure followed in [13, Lemma 3.15] to get $|\varepsilon D^+W(x_i)| \leq CN^{-2}$. Similarly it can be proved that $|\varepsilon D^-W(x_i)| \leq CN^{-2}$. This implies $|\varepsilon D^0W(x_i)| \leq \varepsilon(|D^+W(x_i)| + |D^-W(x_i)|)/2 \leq CN^{-2}$. □

Lemma 4.4. *Let w and W be the exact and discrete singular components of the solutions of* (2.1), (2.2) *and* (3.9), (3.10) *respectively. Then for all $x_i \in \Omega^N$, we have*

$$|\varepsilon D^0(W-w)(x_i)| \leq CN^{-2}\ln^2 N.$$

Proof. Consider the case $\sigma = \frac{2\varepsilon}{\alpha}\ln N$. For all $x_i \in \Omega^N \cap (0, d-\sigma]$ and $x_i \in \Omega^N \cap [d+\sigma, 1)$ using triangle inequality we have

$$|\varepsilon D^0(W-w)(x_i)| \leq |\varepsilon(D^0W - w')(x_i)| + |\varepsilon(D^0w - w')(x_i)|.$$

By Lemma 4.1, it is obvious to see that the second term is bounded. To bound the first term, first we consider $|\varepsilon(D^+W - w')(x_i)|$ and using triangle inequality, we write it as $|\varepsilon(D^+W - w')(x_i)| \leq |\varepsilon D^+W(x_i)| + |\varepsilon w'(x_i)| \leq CN^{-2}$. For $x_i = d-\sigma$ and $x_i = d+\sigma$, we use $L^NW(d-\sigma) = 0$, $L^NW(d+\sigma) = 0$, the estimate obtained at the points $(d-\sigma)$, $(d+\sigma)$ and the proof of Lemma 3.5, to obtain

$$\begin{aligned}\varepsilon D^+W(x_{N/4}) &\leq (1 + CN^{-1}\ln N)CN^{-2} + CN^{-2} \leq CN^{-2},\\ \varepsilon D^+W(x_{3N/4-1}) &\leq (1 + CN^{-1}\ln N)CN^{-2} + CN^{-2} \leq CN^{-2}.\end{aligned}$$

Now consider $x_i \in (d-\sigma, d) \cup (d, d+\sigma)$. For convenience we introduce the notation $\hat{e}(x_i) = (W-w)(x_i)$ and $\hat{\tau}(x_i) = L^N\hat{e}(x_i)$. We have already established that

$$|\hat{e}(x_i)| \leq CN^{-2}(\ln N)^2 \quad \text{and} \quad |\hat{\tau}(x_i)| \leq C\sigma^2\varepsilon^{-2}N^{-2}e^{-\alpha(d-x_i)/\varepsilon}. \tag{4.4}$$

We write the equation $\hat{\tau}(x_i) = L^N\hat{e}(x_i)$ in the form

$$\varepsilon D^+(\hat{e}(x_j) - \hat{e}(x_{j-1})) + \frac{1}{2}a(x_j)(x_{j+1} - x_{j-1})D^0\hat{e}(x_j) = \frac{1}{2}(x_{j+1} - x_{j-1})[\hat{\tau}(x_j) - b(x_j)\hat{e}(x_j)].$$

Summing and rearranging gives

$$\begin{aligned}\varepsilon D^+\hat{e}(x_i) &= \varepsilon D^+\hat{e}(x_{3N/4-1}) + \frac{1}{2}\sum_{j=i+1}^{3N/4-1}[a(x_j)(\hat{e}(x_{j+1}) - \hat{e}(x_{j-1})) - h[\hat{\tau}(x_j) - b(x_j)\hat{e}(x_j)]]\\ &\leq \varepsilon D^+\hat{e}(x_{3N/4-1}) + a(x_{3N/4-1})\hat{e}(x_{3N/4}) - a(x_i)\hat{e}(x_{i+1})\\ &\quad + a(x_{3N/4-1})\hat{e}(x_{3N/4-1}) - a(x_i)\hat{e}(x_i)\\ &\quad - \frac{1}{2}\sum_{j=i+1}^{3N/4-1}[(a(x_j) - a(x_{j-1}))\hat{e}(x_j) + (a(x_j) - a(x_{j-1}))\hat{e}(x_{j-1}) - h\hat{\tau}(x_j) + b(x_j)\hat{e}(x_j)].\end{aligned}$$

Hence using the result at the point $x_{3N/4}$ and (4.4), we have

$$\varepsilon D^+\hat{e}(x_i) \leq CN^{-2}(\ln^2 N + \frac{\sigma}{\varepsilon}\frac{\alpha h/\varepsilon}{1 - e^{-\alpha h/\varepsilon}}).$$

But $y = \alpha h/\varepsilon = 4N^{-1}\ln N$ and $B(y) = \frac{y}{1-e^{-y}}$ are bounded and it follows that $|\varepsilon D^{+}\hat{e}(x_i)| \leq CN^{-2}\ln^2 N$ as required. Thus, we have

$$|\varepsilon D^0 e(x_i)| \equiv |\frac{\varepsilon(D^+ + D^-)e(x_i)}{2}| \leq CN^{-2}\ln^2 N.$$

□

Theorem 4.5. *Let u be the solution of* (2.1), (2.2) *and U the corresponding numerical solution generated by the difference scheme* (3.9), (3.10) *or* (3.15), (3.16)*. Then for each i, $1 \leq i \leq N-1$, we have*

$$\sup_{0<\varepsilon\leq 1} \| \varepsilon(D^0U(x_i) - u') \|_{\overline{\Omega}_i} \leq CN^{-2}\ln^2 N,$$

where C is independent of ε and N.

Proof. From triangular inequality we have $|\varepsilon(D^0U(x_i) - u'(x))| \leq |\varepsilon D^0(U-u)(x_i)| + |\varepsilon(D^0u(x_i) - u'(x))|$. From Lemma 4.1 we get $|\varepsilon(D^0u(x_i) - u'(x))| \leq CN^{-2}\ln^2 N$. To bound $|\varepsilon D^0(U-u)(x_i)|$, it can be written as

$$|\varepsilon D^0(U-u)(x_i)| \leq |\varepsilon D^0(V-v)(x_i)| + |\varepsilon D^0(W-w)(x_i)| \leq CN^{-2}\ln^2 N,$$

by Lemma 4.2 and Lemma 4.4. In the case of difference scheme (3.15), (3.16) we obtain the same bound using lemmas analogous to Lemmas 4.2, 4.3 and 4.4. □

5. NUMERICAL RESULTS

In this section, an example is given to illustrate the difference schemes discussed in this paper.

$$\begin{aligned} &\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad x \in \Omega^- \cup \Omega^+ \\ &-u'(0) = 1/4, \quad u(1) + \varepsilon u'(1) = 1, \end{aligned} \tag{5.1}$$

where,

$$a(x) = \begin{cases} -4, & x \leq 0.5, \\ 4, & x \geq 0.5, \end{cases} \quad b(x) = 1, \quad 0 \leq x \leq 1 \quad \text{and} \quad f(x) = \begin{cases} -2, & x \leq 0.5, \\ -1, & x \geq 0.5. \end{cases}$$

Let U^N be the numerical approximation for the exact solution $u(x)$ on the mesh Ω^N and N is the number of mesh point. For all integers N, satisfying $N, 2N \in R_N = [32, 64, 128, 256, 512, 1024]$ and for a finite set of values $\varepsilon \in R_\varepsilon = [10^{-12}, 10^{-1}]$, we compute the maximum pointwise two-mesh differences for the solution and the scaled first derivative respectively as $E_\varepsilon^N = \| U^N - \bar{U}^{2N} \|$ and $D_\varepsilon^N = \| \varepsilon(D^0U^N - \bar{D}^0U^{2N}) \|$. From these values the $\varepsilon-$uniform maximum pointwise two-mesh differences $E^N = \max_{\varepsilon\in R_\varepsilon} E_\varepsilon^N$, $D^N = \max_{\varepsilon\in R_\varepsilon} D_\varepsilon^N$, are formed for each available value of N satisfying $N, 2N \in R_N$. Approximations to the $\varepsilon-$uniform order of convergence are defined, for all $N, 4N \in R_N$, by $p^N = \log_2 \frac{E^N}{E^{2N}}$ and $s^N = \log_2 \frac{D^N}{D^{2N}}$. The results of the above procedure are summarized in Table 1 and Table 2 for the solution and its scaled first derivative generated by the difference schemes (3.9), (3.10) and (3.15), (3.16) respectively. In

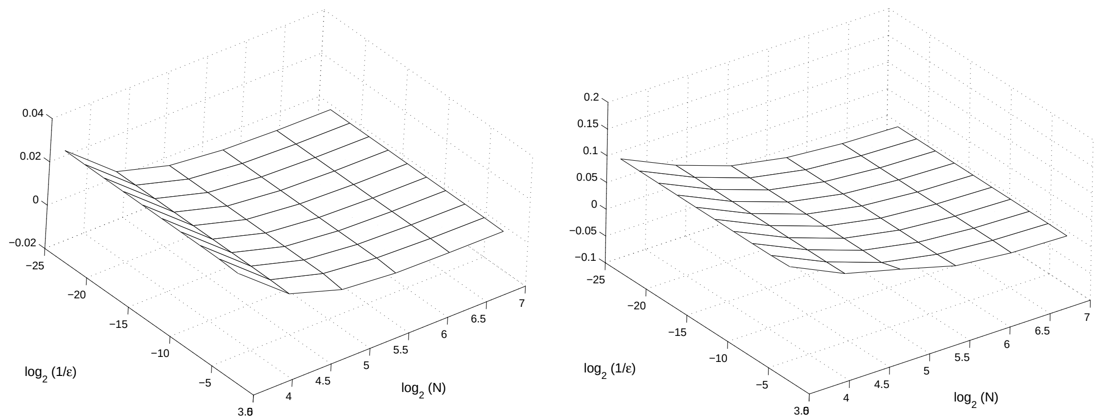

FIGURE 1. Maximin pointwise error E_ε^N and D_ε^N as a function of N and ε to the problem (5.1) generated by the difference scheme (3.9), (3.10) to the solution and the scaled derivative $\varepsilon u'$ respectively.

Figs. 1, 2, the maximum pointwise errors are plotted as a function of N and ε. Note that for all value of ε the error decreases steadily with increasing N.

TABLE 1. Values of E^N, p^N and D^N, s^N generated by the difference scheme (3.9), (3.10) to the solution u and the scaled derivative $\varepsilon u'$ respectively.

ε	Number of mesh points N					
	32	64	128	256	512	1024
E^N	3.1044e-2	1.1365e-2	4.2467e-3	1.4911e-3	5.2567e-4	1.9785e-4
p^N	1.4497	1.4202	1.5100	1.5041	1.4098	-
D^N	1.1538e-1	6.8749e-2	3.2979e-2	1.3578e-2	5.0634e-3	1.7833e-3
s^N	0.7470	1.0598	1.2803	1.4231	1.5056	-

TABLE 2. Values of E^N, p^N and D^N, s^N generated by the difference scheme (3.15), (3.16) to the solution u and the scaled derivative $\varepsilon u'$ respectively.

ε	Number of mesh points N					
	32	64	128	256	512	1024
E^N	3.1108e-2	1.1379e-2	4.2506e-3	1.4916e-3	5.2630e-4	1.9785e-4
p^N	1.4509	1.4206	1.5108	1.5029	1.4115	-
D^N	1.1546e-1	6.8774e-2	3.2986e-2	1.3580e-2	5.0637e-3	1.7827e-3
s^N	7.4746e-1	1.0600	1.2804	1.4232	1.5061	-

6. CONCLUSION

A singularly perturbed convection-diffusion problem with the discontinuous convection coefficient and source term was examined. Due to the presence of non-smooth

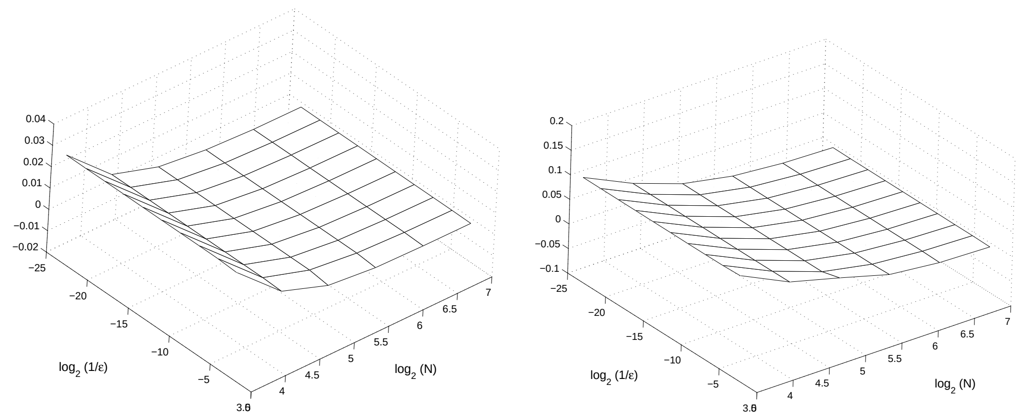

FIGURE 2. Maximin pointwise error E_ε^N and D_ε^N as a function of N and ε to the problem (5.1) generated by the difference scheme (3.15), (3.16) to the solution and the scaled derivative $\varepsilon u'$ respectively.

data and because of the sign pattern of the coefficient of the first derivative, an interior layer appears in the solution. Two hybrid finite difference schemes with Shishkin mesh was constructed for solving this problem which generates an $\varepsilon-$uniform convergent numerical approximation not only to the solution but also to the scaled first derivative of the solution. Numerical results were presented, which are in agreement with the theoretical results.

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