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AN ASYMPTOTIC HYBRID DIFFERENCE SCHEME FOR SINGULARLY PERTURBED THIRD AND FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS SOURCE TERM

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ABSTRACT. We consider Singularly perturbed Boundary-Value Problems (BVPs) for third and fourth order Ordinary Differential Equations (ODEs) with a discontinuous source term and a small positive parameter multiplying the highest derivative. Because of the type of Boundary Conditions (BCs) imposed on these equations these problems can be transformed into weakly coupled systems. In this system, the first equations does not have the small parameter but the second contains it. In this paper a computational method named as "An asymptotic hybrid finite difference scheme" for solving these systems is presented. In this method we first find an zero order asymptotic approximation to the solution and then the system is decoupled by replacing the first component of the solution by this approximation in the second equation. Then the second equation is independently solved by a hybrid finite difference method. Numerical experiments support our theoretical results.

Key Words Singularly perturbed problem, discontinuous source term, third order differential equation,Fourth order differential equation,asymptotic expansion approximation,hybrid finite difference scheme,self-adjoint, boundary value problem, fitted mesh.

Subject Classification: AMS 65L10 CR G1.7

1. INTRODUCTION

Singularly Perturbed Differential Equations appear in several branches of applied mathematics. Analytical and numerical treatment of these equations have drawn much attention of many researchers [1, 2, 5]. In general classical numerical methods fail to produce good approximations for these equations. Hence one has to go for non-classical methods. A good number of articles have been appearing in the past three decades on non-classical methods which cover mostly second order equations. But only a few authors have developed numerical methods for singularly perturbed higher order differential equations. Singularly perturbed higher order problems are classified on the basis that how the order of the original differential equation is affected if one

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sets $\varepsilon = 0$ [5]. Here ε is a small positive parameter multiplying the highest derivative of the differential equation. We say that the Singular Perturbation Problem (SPP) is of convection- diffusion type if the order of the DE is reduced by one, whereas it is called reaction-diffusion type if the order is reduced by two.

In this paper the second type is considered. For the analytical treatment SPBVPs for the higher-order non-linear ODEs which have important applications in Fluid dynamics, one may refer [3, 13]. In this paper we consider the following two problems.

Third Order Singularly Perturbed Boundary Value Problems [14]**.** Find $y \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega) \cap \mathcal{C}^3(\Omega^- \cup \Omega^+)$ such that

$$-\varepsilon y'''(x) + b(x)y'(x) + c(x)y(x) = f(x),\ x\ \in \Omega^- \cup \Omega^+, \tag{1.1}$$

$$y(0) = p,\ y'(0) = q,\ y'(1) = r, \tag{1.2}$$

where $b(x), c(x)$ are smooth functions on $\overline{\Omega}$ satisfying the following conditions.

$$b(x) \geq \beta > 0, \tag{1.3}$$

$$0 \geq c(x) \geq -\gamma, \gamma > 0, \tag{1.4}$$

$$\beta - \theta\gamma \geq \eta > 0, \quad \text{for some} \ \ \theta \ \ \text{arbitrarily close to 2, for some } \eta. \tag{1.5}$$

Fourth Order Singularly Perturbed Boundary Value Problems [13]**.** Find $y \in \mathcal{C}^2(\overline{\Omega}) \cap \mathcal{C}^3(\Omega) \cap \mathcal{C}^4(\Omega^- \cup \Omega^+)$ such that

$$-\varepsilon y^{(iv)}(x) + b(x)y''(x) - c(x)y(x) = -f(x),\ x\ \in \Omega^- \cup \Omega^+, \tag{1.6}$$

$$y(0) = p,\ y(1) = q,\ y''(0) = -r,\ y''(1) = -s, \tag{1.7}$$

where $b(x), c(x)$ are smooth functions on $\overline{\Omega}$ satisfying the following conditions.

$$b(x) \geq \beta > 0, \tag{1.8}$$

$$0 \geq c(x) \geq -\gamma, \gamma > 0, \tag{1.9}$$

$$\beta - \theta\gamma \geq \eta > 0, \quad \text{for some} \ \ \theta \ \ \text{arbitrarily close to 2 for some } \eta. \tag{1.10}$$

For both problems defined above $\Omega = (0,1)$, $\Omega^- = (0,d)$, $\Omega^+ = (d,1)$, $d \in \Omega$ ε is a small positive parameter. It is assumed that f is sufficiently smooth on $\overline{\Omega} \setminus \{d\}$. Further it is assumed that $f(x)$ and its derivatives have right and left limits at $x = d$. It is convenient to introduce the notation for jump at d for any function w as $[w](d) = w(d+) - w(d-)$.

Motivated by the papers [13, 14] a computational method is suggested for the above problems. Because of the type of the BCs imposed one can transform the problem into a weakly coupled system of DEs. Then one obtains a zero order asymptotic expansion approximation for the solution of the problem. Then the first component of the solution appearing in the second equation is replaced by its zero order asymptotic expansion approximation. Then the system gets decoupled. Then second equation can be solved independently. Infact, the second equation was solved by earlier authors [13, 14] by using FMM (Fitted Mesh Method) on Shishkin mesh and the order of convergence obtained by them is of $O(\sqrt{\varepsilon} + N^{-1}\ln N)$. In the present paper we apply hybrid finite difference scheme on Shishkin meshes and obtained higher order convergence for small values of parameter ε.

Throughout this paper, C denotes a generic positive constant that is independent of parameter (ε), N, the dimension of the discrete problem. In the following we use the norm $\| w \|_D = \sup_{x\in D} | w(x) |$.

2. ASYMPTOTIC EXPANSION APPROXIMATION

As mentioned above zero- order asymptotic expansion for the solution of the problem (1.1–1.2) and (1.6–1.7) are obtained. The SPBVP (1.1–1.2) can be transformed into an equivalent problem of the form

$$\begin{cases} y_1'(x) - y_2(x) = 0, & x \in (0,1], \\ -\varepsilon y_2''(x) + b(x)y_2(x) + c(x)y_1(x) = f(x), & x \in \Omega^- \cup \Omega^+, \end{cases} \tag{2.1}$$

$$y_1(0) = p, \quad y_2(0) = q, \quad y_2(1) = r, \tag{2.2}$$

where $\overline{y} = (y_1, y_2)^T$, $y_1 \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega) \cap \mathcal{C}^3(\Omega^- \cup \Omega^+)$, $y_2 \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega) \cap \mathcal{C}^2(\Omega^- \cup \Omega^+)$.

Similarly the SPBVP (1.6–1.7) can be transformed into an equivalent problem of the form

$$\begin{cases} -y_1''(x) - y_2(x) = 0, & x \in \Omega, \\ -\varepsilon y_2''(x) + b(x)y_2(x) + c(x)y_1(x) = f(x), & x \in \Omega^- \cup \Omega^+, \end{cases} \tag{2.3}$$

$$y_1(0) = p, \quad y_1(1) = q, \quad y_2(0) = r, \quad y_2(1) = s, \tag{2.4}$$

where $\overline{y} = (y_1, y_2)^T$, $y_1 \in \mathcal{C}^2(\overline{\Omega}) \cap \mathcal{C}^3(\Omega) \cap \mathcal{C}^4(\Omega^- \cup \Omega^+)$, $y_2 \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega) \cap \mathcal{C}^2(\Omega^- \cup \Omega^+)$.

Remark 2.1. Hereafter, only the above systems are only considered instead of BVPS (1.1–1.2) and (1.6–1.7) with the conditions (1.3–1.5) and (1.8–1.10). The condition (1.4) and (1.9) are imposed to ensure that systems (2.1–2.2) and (2.3-2.4) respectively quasi-monotone [13, 14]. The conditions (1.5)and (1.10) are sufficient to establish the maximum principle for (2.1–2.2) and (2.3–2.4). This, in turn, can be used to derive the stability result, error estimates etc.

Motivated by [13, 14] we can construct an asymptotic expansion approximation for the solution of the system (2.1–2.2) and (2.3–2.4). Find a continuous functions u_{01} on $\overline{\Omega}$ such that

$$b(x)u_{01}'(x) + c(x)u_{01}(x) = f(x), \forall x \in \Omega^- \cup \Omega^+ \cup \{1\}, \quad u_{01}(0) = p.$$

Then find

$$u_{02}(x) = \frac{f(x) - c(x)u_{01}(x)}{b(x)}, \quad x \in (\Omega^- \cup \Omega^+ \cup \{1\}). \tag{2.5}$$

Further let $\overline{v}_{l0} = (v_{l01}, v_{l02})$, $\overline{v}_{r0} = (v_{r01}, v_{r02})$, be the left layer corrections given by

$$v_{l01} = -\sqrt{\frac{\varepsilon}{b(0)}} v_{l02},$$

$$v_{l02} = k_1 e^{-x\sqrt{\frac{b(0)}{\varepsilon}}}, \quad x \in \{0\} \cup \Omega^-,$$

$$v_{r01} = -\sqrt{\frac{\varepsilon}{b(d)}} v_{r02},$$

$$v_{r02} = k_2 e^{-(x-d)\sqrt{\frac{b(d)}{\varepsilon}}}, \quad x \in \Omega^+ \cup \{1\},$$

and let $\overline{w}_{l0} = (w_{l01}, w_{l02})$, $\overline{w}_{r0} = (w_{r01}, w_{r02})$, be the right layer corrections given by

$$w_{l01} = -\sqrt{\frac{\varepsilon}{b(0)}} w_{l02},$$

$$w_{l02} = k_3 e^{-x\sqrt{\frac{b(0)}{\varepsilon}}}, \quad x \in \{0\} \cup \Omega^-,$$

$$w_{r01} = -\sqrt{\frac{\varepsilon}{b(d)}} w_{r02},$$

$$w_{r02} = k_4 e^{-(x-d)\sqrt{\frac{b(d)}{\varepsilon}}}, \quad x \in \Omega^+ \cup \{1\},$$

$$v_{01} = \begin{cases} v_{l01}, & x \in \Omega^- \\ v_{r01}, & x \in \Omega^+, \end{cases} \quad \text{and}$$

$$w_{01} = \begin{cases} w_{l01}, & x \in \Omega^- \\ w_{r01}, & x \in \Omega^+. \end{cases}$$

Similarly for v_{02} and w_{02} on $\Omega^- \cup \Omega^+$. Now define

$$y_{1as} = u_{01} + v_{01} + w_{01}, \quad x \in \Omega^- \cup \Omega^+,$$

$$y_{2as} = u_{02} + v_{02} + w_{02}, \quad x \in \Omega^- \cup \Omega^+.$$

The constants k_1, k_2, k_3 and k_4 are determined by imposing the following boundary conditions [13, 14]:

$$y_{2as}(0) = y_2(0), \quad y_{2as}(1) = y_2(1), \quad y_{2as}(d-) = y_2as(d+).$$

Similarly one can construct an asymptotic expansion for the solution of the BVP (2.3–2.4). Infact, for this problem u_{01} is solution of the BVP

$$-u_{01}'' - u_{02} = 0 \tag{2.6}$$

$$b(x)u_{02}(x) + c(x)u_{01}(x) = f(x), \quad x \in (\Omega^- \cup \Omega^+). \tag{2.7}$$

and

$$u_{01}(0) = p \quad u_{01}(1) = q \quad u_{01}(d-) = u_{01}(d+) \quad u'_{01}(d-) = u'_{01}(d+) \tag{2.8}$$

Note: One can see that there are strong layers at $x = 1$ as well as at $x = d$ for the solution component y_2.

Theorem 2.2. *The zero-order asymptotic expansion approximation* $\overline{y}_{as}$ *of the solution* $\overline{y}(x)$ *of* (2.1)–(2.2) *satisfies the inequality*

$$| y_i(x) - y_{i,as}(x) | \leq C\sqrt{\varepsilon}, \qquad x \in \overline{\Omega}, \qquad i = 1, 2.$$

In particular we have (proof of Theorem 3.2, [14]*)*

$$| y_1(x) - u_{01}(x) | \leq C\sqrt{\varepsilon}.$$

3. SOME ANALYTICAL AND NUMERICAL RESULTS FOR SPBVP FOR SECOND-ORDER REACTION-DIFFUSION EQUATION WITH A DISCONTINUOUS SOURCE TERMS

We consider the BVP

$$-\varepsilon y_2^{*''}(x) + b(x)y_2^*(x) = f(x) - c(x)u_{01}(x), \qquad x \in \Omega^- \cup \Omega^+, \tag{3.1}$$

$$y_2^*(0) = r, \; y_2^*(1) = s. \tag{3.2}$$

where $b(x), f(x), c(x)$ are same functions as given in the BVP (2.1–2.2)and u_{01} is the solutions of the initial value problem (2.1).

3.1. Analytical Results.

Theorem 3.1. *If* (y_1, y_2) *and* y_2^* *are solutions of the BVPs* (2.1–2.2) *and* (3.1–3.2) *respectively, and* u_{01} *is the solution of the reduced problem* (2.1)*, then* [14]

$$| y_2(x) - y_2^*(x) | \leq C\sqrt{\varepsilon}, \quad x \in \overline{\Omega}.$$

Remark 3.2. A similar statement can be made for the BVP (2.3–2.4)

3.2. **Numerical Results.** Consider the following reaction-diffusion BVP (3.1–3.2). Zhongdi Cen [3] presented a hybrid scheme for a class of SPBVPs of convection-diffusion type for second order ODEs with discontinuous convection coefficient. We now apply this scheme for the above problem (3.1–3.2). On $\Omega^- \cup \Omega^+$ a piecewise uniform mesh of N mesh intervals is constructed as follows. The interval $\overline{\Omega}^-$ is subdivided into the three subintervals.

$$[0, \tau_1], \; [\tau_1, d - \tau_1] \qquad \text{and} \qquad [d - \tau_1, d]$$

for some τ_1 that satisfies $0 < \tau_1 \leq \dfrac{d}{4}$. On $[0, \tau_1]$ and $[d - \tau_1, d]$ a uniform mesh with $\dfrac{N}{8}$ mesh intervals is placed, while on $[\tau_1, d - \tau_1]$ has a uniform mesh with $\dfrac{N}{4}$ mesh intervals. The subintervals $[d, d+\tau_2], [d+\tau_2, 1-\tau_2], [1-\tau_2, 1]$ of $\overline{\Omega}^+$ are treated analogously for some τ_2 satisfying $0 < \tau_2 \le \dfrac{1-d}{4}$. The interior points of the mesh are denoted by

$$\Omega_\varepsilon^N = \left\{ x_i : 1 \le i \le \frac{N}{2} - 1 \right\} \cup \left\{ x_i : \frac{N}{2} + 1 \le i \le N-1 \right\}.$$

Clearly $x_{N/2} = d$ and $\overline{\Omega}_\varepsilon^N = \{x_i\}_0^N$. Note that this mesh is a uniform mesh when $\tau_1 = \dfrac{d}{4}$ and $\tau_2 = \dfrac{1-d}{4}$. It is fitted to the singular perturbation problem (3.1–3.2) by choosing τ_1 and $\tau_2$to be the following functions of N and ε

$$\tau_1 = \min\left\{ \frac{d}{4}, 2\sqrt{\varepsilon/\beta}\ln N \right\} \qquad \text{and} \qquad \tau_2 = \min\left\{ \frac{1-d}{4}, 2\sqrt{\varepsilon/\beta}\ln N \right\}.$$

On the piecewise-uniform mesh $\overline{\Omega}_\varepsilon^N$ a standard centered finite difference operator is used. Then the mesh widths are

$$h_i = \begin{cases} H_1 = 8\tau/N, \quad i = 1, \ldots, N/8, \quad i = (5N/8)+1, \ldots, N, \\ H_2 = 4(d-2\tau)/N, i = (N/8)+1, \ldots, N/4, \quad i = (3N/4)+1, \ldots, 5N/8, \\ h = 8\tau/N, \quad i = (N/4)+1, \ldots, (3N/4) \end{cases}$$

Our discretization is similar to that of [14] and [13] in those they use the central difference approximation

$$L_c^N \equiv -\varepsilon\delta^2 y_{2,i}^* + b(x_i) y_{2,i}^* = f(x_i) - c(x_i) u_{01}(x_i) = \bar{f}_i, \qquad \forall x_i \in \Omega_\varepsilon^N, \tag{3.3}$$

$$y_{2,0}^* = q, \qquad y_{2,n}^* = r, \qquad L_\tau \quad \text{at} \quad x = N/2, \tag{3.4}$$

where

$$\delta^2 y_{2,i}^* = \left(\frac{y_{2,i+1}^* - y_{2,i}^*}{x_{i+1} - x_i} - \frac{y_{2,i}^* - y_{2,i-1}^*}{x_i - x_{i-1}} \right) \frac{2}{x_{i+1} - x_{i-1}},$$

In the point $x_{N/2} = d$ we shall use the difference operator L_τ;

$$L_\tau y_{2,N/2}^* = \frac{-y_{2,N/2+2}^* + 4y_{2,N/2+1}^* - 3y_{2,N/2}^*}{2h} - \frac{y_{2,N/2-2}^* - 4y_{2,N/2-1}^* + 3y_{2,N/2}^*}{2h} = 0 \tag{3.5}$$

We set

$$L^N y_{2,i}^* = \begin{cases} L_c^N y_{2,i}^* & \text{for } i \ne N/2 \\ L_\tau^N y_{2,i}^* & \text{for} \quad i = N/2. \end{cases} \tag{3.6}$$

and

$$\bar{f}_i = \begin{cases} f_i & \text{for} \quad i \ne N/2 \\ & \text{for} \quad i = N/2 \end{cases}$$

Then our scheme reads: Find $y_{2,i}^N \in R^{N+1}$ with

$$L^N y_{2,i}^* N = \bar{f}_i \quad i = 1, 2, \ldots, N-1$$
$$y_{2,0}^* = y_2^*(0), \quad y_{2,N}^* = y_2^*(1).$$

The matrix associated with L^N is not an M-matrix. We transform the equation so that the new equation has a monotonicity property. From equations (3.1–3.2) we can get

$$y^*_{2,N/2-2} = \left(f_{N/2-1} - b_{N/2-1}y^*_{2,N/2-1} - \frac{\varepsilon}{h}\frac{y^*_{2,N/2} - y^*_{2,N/2-1}}{h} + \frac{\varepsilon}{h^2}y^*_{2,N/2-1}\right)\frac{h^2}{\varepsilon}$$
$$y^*_{2,N/2+2} = \left(f_{N/2+1} - b_{N/2+1}y^*_{2,N/2+1} + \frac{\varepsilon}{h}\frac{y^*_{2,N/2+1} - y^*_{2,N/2}}{h} + \frac{\varepsilon}{h^2}y^*_{2,N/2+1}\right)\frac{h^2}{\varepsilon}.$$

Inserting the expressions for $y^*_{2,N/2+2}$ and $y^*_{2,N/2-2}$ in (3.5) gives

$$\begin{aligned} L^N_\upsilon y^*_{2,N/2} &= \left(\left(2 + \frac{h^2}{\varepsilon}b_{N/2+1}\right)y^*_{2,N/2+1} - 4y^*_{2,N/2} + (2 + \frac{h^2}{\varepsilon}b_{N/2-1})y^*_{2,N/2-1}\right)\frac{1}{h} \\ &= (f_{N/2+1} + f_{N/2-1})\frac{h}{2\varepsilon} \end{aligned}$$

We define the discrete linear operators

$$L^N_H y^{*N}_{2,i} = \begin{cases} L^N_c y^{*N}_{2,i} & \text{for } i \neq N/2 \\ L^N_\upsilon y^{*N}_{2,i} & \text{for } i = N/2. \end{cases} \tag{3.7}$$

and

$$\bar{f}_{Hi} = \begin{cases} \bar{f}_i & \text{for } i \neq N/2 \\ f_{N/2+1} + f_{N/2-1}\frac{h}{2\varepsilon} & \text{for } i = N/2 \end{cases}$$

Clearly we have a system of equations

$$L^N_H y^{*N}_{2,i} = \bar{f}_{Hi}, \quad \text{for } i = 1, 2, \ldots, N+1, \quad y^*_{2,0} = y^*_2(0), \quad y^*_{2,N} = y^*_2(1). \tag{3.8}$$

Then we have

Theorem 3.3. *The error in using the scheme* (3.7) *to solve the problem* (3.1)–(3.2) *at the inner grid points* $\{x_i, i = 1, 2, \ldots, N-1\}$ *satisfies*

$$\mid y^*_2(x_i) - y^*_{2,i} \mid \leq C(N^{-1}\ln N)^2.$$

Proof: Using the procedure adopted in [3] we can derive the required result.

4. ERROR ESTIMATE

Theorem 4.1. *Let* (y_1, y_2) *be the solution of* (2.1)–(2.2). *Further, let* $y^*_{2,i}$ *be the numerical solution of* (3.1)–(3.2) *obtained by the scheme* (3.3)–(3.4). *Then*

$$\mid y_2(x_i) - y^*_{2,i} \mid \leq C[(N^{-1}\ln N)^2 + \sqrt{\varepsilon}].$$

Proof: The result of the present theorem follows from the inequality

$$\mid y_2(x_i) - y^*_{2,i} \mid \leq \mid y_2(x_i) - y^*_2(x_i) \mid + \mid y^*_2(x_i) - y^*_{2,i} \mid$$

and Theorems 3.1 and 3.3.

Remark 4.2. A similar statement is true for the BVP (2.3–2.4). In [13, 14] the authors applied FMM on Shishkin mesh and obtained an error estimate of order $(O\sqrt{\varepsilon} + N^{-1}\ln N)$. From the above result it is obvious that the present method has improved the earlier results.

5. NUMERICAL EXPERIMENTS

In this section, two examples are given to illustrate the computational methods discussed in this paper.

Example 5.1. Consider the singularly perturbed BVP with discontinuous source term:

$$-\varepsilon y'''(x) + 4y'(x) - y(x) = \begin{cases} 0.7 & x < 0.5 \\ -0.6 & x \geq 0.5 \end{cases}, \quad x \in \Omega,$$
$$y(0) = 1, \quad y'(0) = 0, \quad y'(1) = 0,$$

and the corresponding system

$$y_1'(x) - y_2(x) = 0,$$
$$-\varepsilon y_2''(x) + 4y_2(x) - y_1(x) = \begin{cases} 0.7 & x < 0.5 \\ -0.6 & x > 0.5 \end{cases}, x \in \Omega,$$
$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_2(1) = 0.$$

For this problem

$$u_{01}(x) = \begin{cases} -0.7 + 1.7e^{x/4}, & x \in \{0\} \cup \Omega^- \\ 0.6 + 1.7e^{x/4} - 1.3e^{-(0.5-x)/4}, & x \in \Omega^+ \cup \{0.5, 1\} \end{cases},$$

Example 5.2. Consider the singularly perturbed BVP with discontinuous source term:

$$-\varepsilon y^{iv}(x) + 4y''(x) - y(x) = \begin{cases} 0.7 & x < 0.5 \\ -0.6 & x \geq 0.5 \end{cases}, \quad x \in \Omega,$$
$$y(0) = 1, \quad y(1) = 1, \quad y''(0) = 0, \quad y''(1) = 0,$$

and the corresponding system

$$-y_1''(x) - y_2(x) = 0,$$
$$-\varepsilon y_2''(x) + 4y_2(x) - y_1(x) = \begin{cases} 0.7 & x < 0.5 \\ -0.6 & x > 0.5, \end{cases} \quad x \in \Omega,$$
$$y_1(0) = 1, \quad y_1(1) = 1, \quad y_2(0) = 0, \quad y_2(1) = 0,$$

For this problem

$$u_{01}(x) = \begin{cases} -0.7 + C_1 e^{x/2} + C_2 e^{-x/2}, & x \in \{0\} \cup \Omega^- \\ 0.6 + C_3 e^{x/2} + C_4 e^{-x/2}, & x \in \Omega^+ \cup \{0.5, 1\}, \end{cases}$$

where

$$C_1 = 1.7 - C_2$$

$$C_2 = \frac{1.7 - 0.4e^{-1/2} + C_4(e^{-1} + e^{-1/2})}{1 + e^{-1/2}}$$

$$C_3 = (0.4 - C_4 e^{-1/2})e^{-1/2}$$

$$C_4 = \frac{1.3(e^{-0.5/2} + e^{-0.5/2}) + 1.7(e^{-1/2} - e^{1/2}) + 0.8e^{-1/2} - 3.4}{2(e^{-1} - 1)}$$

6. DISCUSSION

From the tables it is obvious that the scheme presented in this paper gives better results when compared to the method presented in the corresponding literature. The rate of convergence r^N are computed using the following formula:

$$r^N = \log_2 \left(\frac{E^N}{E^{2N}} \right).$$

where $E^N = ||y_2^* - y_{2,i}^{*I}||_\infty$ and $y_{2,i}^{*I}$ denotes the piecewise linear interpolant of y_2^*.

In Table 1, we present values E^N, r^N for the first derivative of the solution of the BVP given in Example 5.1 and Table 2 gives the second derivative of the solution of the BVP given in Example 5.2.

TABLE 1. Values of E^N and r^N for the first derivative of the solution y of the Example 5.1 for $\varepsilon = 2^{-1} - 2^{-30}$.

Number of mesh points N					
	64	128	256	512	1024
E^N	1.0062e-02	3.9832e-03	1.7131e-03	8.0103e-04	3.9850e-04
r^N	1.3369e+00	1.2173e+00	1.0967e+00	1.0073e+00	

TABLE 2. Values of E^N and r^N for the second derivative of the solution y of the Example 5.2 for $\varepsilon = 2^{-1} - 2^{-30}$.

Number of mesh points N					
	64	128	256	512	1024
E^N	2.3088e-01	9.1401e-02	3.9309e-02	1.8381e-02	9.1442e-03
r^N	1.3369e+00	1.2174e+00	1.0966e+00	1.0073e+00	

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