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UNIFORMLY CONVERGENT SECOND-ORDER NUMERICAL METHOD FOR SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this work, we propose a uniformly-convergent second order Richardson extrapolation technique for solving singularly perturbed delay differential equation. First, we solve the equation using the upwind finite difference scheme on piecewise-uniform Shishkin mesh, then apply the Richardson extrapolation technique on the computed solution fixing the transition parameter. Error estimates are obtained. Numerical example permits us to confirm in practise the theoretical result.

Key Words: Singular perturbation problems, delay differential equations, boundary layer, upwind scheme, Richardson extrapolation, Shishkin mesh.

AMS Subject Classification: 65L10, 65L12, CR G1.7

1. INTRODUCTION

In this article, we consider the following singularly perturbed delay differential equation

(1.1)
$$(P_\varepsilon)\begin{cases} L_\varepsilon u(x) \equiv -\varepsilon u''(x) + a(x)u'(x-\delta) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1), \\ u(x) = \gamma(x), \quad -\delta \le x \le 0, \\ u(1) = \lambda\,, \end{cases}$$

where $0 < \varepsilon \ll 1$ is a small parameter and the delay parameter δ is such that $0 < \delta < 1$, which is of $O(\varepsilon)$. The functions $a(x), b(x), f(x)$ and $\gamma(x)$ are sufficiently smooth functions and λ is a constant. It is also assumed that $b(x) \ge \beta > 0, \forall x \in \overline{\Omega}$. When $\delta = 0$, the equation (1.1) reduces to a singularly perturbed differential equation. Depending upon the sign of $a(x)$, *i.e.*, if $a(x) > 0$ (or $a(x) < 0$), a boundary layer is located at right (or left) end of the domain. The layer is maintained for sufficiently small δ with $\delta \ne 0$ and $\delta = o(\varepsilon)$. Lange et al. ([3], [4]) provided an asymptotic approach to BVPs of the type (1.1). By considering several examples, they have shown that the effect of the small delay on the solution cannot be neglected. The argument for small delay problems are widespread in many mathematical models of

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biophysics and mechanics where delay term plays an important role in modelling real-life phenomena [9].

The solution of (1.1) has steep layers which is difficult to approximate efficiently by most of the numerical methods using uniform grid [2]. In this context, one may think of solving the above problem with a suitably chosen nonuniform grid. If the presence, location, and thickness of a boundary layer is known a priori, then highly appropriate nonuniform grids can be generated. It is well known that the classical upwind scheme used to solve (P_ε) yields solutions that are first–order accurate measured in the maximum norm. Here our main aim is to construct a simple postprocessing method on the Shishkin mesh converts the almost first-order convergence of upwinding to almost second order convergence.

Richardson extrapolation is a well-known postprocessing technique where two computed solutions are approximated by an average quantity to provide better approximation. This technique is widely used because of its simple construction and can be extended to problems in more than one dimension. This technique was examined in [7] where singularly perturbed convection diffusion equation was considered. Here we shall apply the Richardson extrapolation to delay differential equations on a Shishkin mesh to reduce the nodal errors from $O(N^{-1}\ln N)$ to $O(N^{-2}\ln^2 N)$.

The paper is organized as follows. In Section 2, we establish the maximum principle, stability result and some a priori estimates on the solution and its derivatives of the BVP (1.1). Section 3 presents upwind finite difference scheme and the convergence of the standard upwind method where we obtain the error *i.e.*, $O(N^{-1}\ln N)$ which is optimal for the BVP of type (1.1). In Section 4, we apply the Richardson extrapolation on the computed solution and prove that the extrapolated solution is almost second order convergent measured in maximum norm. Finally, a numerical example is provided in Section 5 to illustrate the applicability of the present method with maximum point-wise error and the rate of convergence are shown in terms of tables and figures.

Through out this paper C (sometimes subscripted) will denote the generic positive constant independent of mesh size and the perturbation parameters ε, δ and N (the dimension of the discrete problem) which can take different values at different places, even in the same argument. But the subscripted C is a fixed constant. Throughout this paper we shall assume that

$$\varepsilon \leq C_1 N^{-1} \tag{1.2}$$

as is generally the case of discretization of convection-dominated problems. Here $\|\cdot\|$ denotes the maximum norm over $\overline{\Omega}$.

2. CONTINUOUS PROBLEM

To tackle with the delay argument, using Taylor's series expansion up to two terms, we have

$$u'(x-\delta) = u'(x) - \delta u''(x). \tag{2.1}$$

Now, using (2.1) in (1.1), we obtain the following BVP:

$$\begin{cases} L_\varepsilon u(x) \equiv -\big(\varepsilon + \delta a(x)\big)u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1) \\ u(x) = \gamma(x), \quad -\delta \le x \le 0, \\ u(1) = \lambda\,. \end{cases} \tag{2.2}$$

Without loss of generality, here we assume that $2\alpha^* \ge a(x) \ge 2\alpha > 0$ and $\big(\varepsilon + \delta a(x)\big) > 0, \forall x \in \overline{\Omega}$. Under these assumptions, the solution of the problem (2.2) has a unique solution and it exhibits layer behavior on the right side of the domain at $x = 1$.

2.1. Properties of the solution and its derivatives.

Lemma 2.1 (Maximum Principle)**.** *Let v be a smooth function satisfying $v(0) \ge 0$, $v(1) \ge 0$ and $L_\varepsilon v(x) \ge 0$ $\forall x \in \ \Omega$, then $v(x) \ge 0$, $\forall x \in \ \overline{\Omega}$.*

Proof. Let $x^* \in \overline{\Omega}$ be such that $v(x^*) = \min\limits_{x \in \overline{\Omega}} v(x)$ and assume that $v(x^*) < 0$. Clearly $x^* \notin \{0, 1\}$ and $v'(x^*) = 0$ and $v''(x^*) \ge 0$. Now consider

$$L_\varepsilon v(x^*) \equiv -\big(\varepsilon + \delta a(x^*)\big)v''(x^*) + a(x^*)v'(x^*) + b(x^*)v(x^*) < 0,$$

which contradicts our assumption. Hence $v(x) \ge 0$, $\forall x \in \ \overline{\Omega}$. ■

An immediate consequence of the maximum principle is the following stability estimate.

Lemma 2.2. *If u is the solution of the boundary value problem* (2.2)*, then*

$$\|u\| \ \le \ \frac{1}{\beta}\|f\| + \max\{|u(0)|, |\lambda|\}. \tag{2.3}$$

Proof. Let us consider the following barrier function

$$\psi^{\pm}(x) = \beta^{-1}\|f\| + \max\{|u(0)|, |\lambda|\} \pm u(x).$$

It easy to show that $\psi^{\pm}(x)$ is non–negative at $x = 0, 1$. Now from (2.2)

$$\begin{aligned} L_\varepsilon \psi^{\pm}(x) &= -(\varepsilon + \delta a(x))\big(\psi^{\pm}(x)\big)'' + a(x)\big(\psi^{\pm}(x)\big)' + b(x)\psi^{\pm}(x) \\ &= b(x)\Big[\beta^{-1}\|f\| + \max\{|\gamma(0)|, |\lambda|\}\Big] \pm L_\varepsilon u(x) \\ &\ge \big[\|f\| \pm f(x)\big] + b(x)\max\{|\gamma(0)|, |\lambda|\} \ge 0. \end{aligned}$$

Thus by applying the maximum principle we conclude that $\psi^{\pm}(x) \geq 0, \forall x \in \overline{\Omega}$ which is the required result. ■

Lemma 2.3. *Let $0 \leq k \leq 5$ be a positive integer. Assume that $f \in \mathcal{C}^k[0,1]$, then the derivatives $u^{(k)}$ of the solution u of (2.2) satisfy the following bound*

$$|u^{(k)}| \leq C\left[1 + (\varepsilon + \delta\alpha)^{-k} \exp\left(\frac{-\alpha(1-x)}{\varepsilon + \delta\alpha}\right)\right], \tag{2.4}$$

where C depends on $\|a\|$, $\|a'\|$, $\|b\|$, $\|b'\|$ and on the boundary conditions.

Proof. For $0 \leq k \leq 3$, the detailed proofs are given in [6]. By differentiating (2.2) twice and following the same argument given in [6], one can obtain the result for $k = 4, 5$. ■

2.2. **Solution decomposition.** To show that a numerical method is ε–uniform convergent, some prior information of the solution are required. Let us decompose the solution of (2.2) into smooth and singular parts as follows:

$$u(x, \varepsilon) = v(x, \varepsilon) + w(x, \varepsilon).$$

Now $v(x, \varepsilon)$ can be written in an asymptotic expansion as

$$v(x, \varepsilon) = v_0(x) + (\varepsilon + \delta\alpha)v_1(x) + (\varepsilon + \delta\alpha)^2 v_2(x),$$

where v_0, v_1 and v_2 are defined as the solution of the following differential equations:

$$\begin{cases} a(x)v_0'(x) + b(x)v_0(x) = f(x), & v_0(0) = u(0), \\ a(x)v_1'(x) + b(x)v_1(x) = (\varepsilon + \delta a(x))v_0''(x)/(\varepsilon + \delta\alpha), & v_1(0) = 0, \\ L_\varepsilon v_2(x) = (\varepsilon + \delta a(x))v_1''(x)/(\varepsilon + \delta\alpha), & v_2(0) = 0, v_2(1) = 0. \end{cases} \tag{2.5}$$

Hence the smooth component v satisfy the BVP

$$L_\varepsilon v(x) = f(x), \quad v(0) = v_0(0) + (\varepsilon + \delta\alpha)v(0) = u(0),\ v(1) = u(1), \tag{2.6}$$

and the layer component satisfies

$$L_\varepsilon w(x) = 0, \quad w(0) = 0,\ |w(1)| \leq C. \tag{2.7}$$

Lemma 2.4. *Let $u(x)$ be the solution of (2.2) and $u(x) = v(x) + w(x)$. For sufficiently small ε and $0 \leq k \leq 4$, the derivatives of $v(x)$ and $w(x)$ satisfy the following bounds:*

$$\|v^{(k)}\| \leq C\big(1 + (\varepsilon + \delta\alpha)^{3-k}\big), \tag{2.8}$$

$$|w^{(k)}(x)| \leq C\big(\varepsilon + \delta\alpha\big)^{-k} \exp\left(\frac{-\alpha(1-x)}{\varepsilon + \delta\alpha}\right), \quad x \in \overline{\Omega}. \tag{2.9}$$

Proof. From (2.5), we observe that v_0 is independent of ε and δ. Similarly, v_1 involves the quantity $(\varepsilon+\delta a(x))/(\varepsilon+\delta\alpha)$ which is bounded above as $a(\cdot)$ is bounded in Ω. Again, v_2 is the solution of the class of problem similar to (2.2) and hence, using Lemma 2.3 for $0 \le k \le 3$, we have the bounds

$$\|v^{(k)}\| \le C\big(1+(\varepsilon+\delta\alpha)^{3-k}\big).$$

For $k=4$, one can write $\big((\varepsilon+\delta a(x))v''(x)\big)'' = \big(b(x)v(x)+a(x)v'(x)-f(x)\big)''$ which leads to the bound $\|v^{(4)}\| \le C(\varepsilon+\delta\alpha)^{-1}$.

To obtain the required bound on $w(x)$ and its derivatives, we consider the barrier function

$$\psi^{\pm}(x) = C\exp\left(\frac{-\alpha(1-x)}{\varepsilon+\delta\alpha}\right) \pm w(x).$$

It is easy to show that $\psi^{\pm}$ is nonnegative at $x=0,1$. Using (2.2), we have

$$L_\varepsilon\psi^{\pm}(x) = \left(\frac{C\alpha}{\varepsilon+\delta\alpha}\right)\left[a(x) - \frac{\alpha(\varepsilon+\delta a(x))}{\varepsilon+\delta\alpha} + \frac{\varepsilon+\delta\alpha}{C\alpha}b(x)\right] \ge 0.$$

Thus applying the maximum principle of Lemma 2.1, we conclude that $\psi^{\pm}(x) \ge 0$. From this result, we can obtain the required pointwise bound on $w(x)$ and its derivatives. ∎

3. DISCRETE PROBLEM

To solve the BVP (P_ε) numerically, we shall use the piecewise-uniform Shishkin mesh [1], [5]. This mesh has a transition point $1-\tau$, where

$$\tau = \min\left\{\frac{1}{2}, \frac{2(\varepsilon+\delta\alpha)}{\beta}\ln N\right\}.$$

Now divide the subinterval $[0, 1-\tau]$ into $N/2$ equal subdivisions of width H, where $H = 2(1-\tau)/N$. Similarly divide the subinterval $[1-\tau, 1]$ into $N/2$ equal subdivisions of width h, where $h = 2\tau/N = 4(\varepsilon+\delta\alpha)\ln N/\beta N$. Then the Shishkin mesh $\Omega_{N,\tau} = \{x_i\}_{i=0}^{N}$, where $x_0 = 0, x_N = 1$ and the mesh width $h_i := x_i - x_{i-1}$ satisfy $h_i = H$ for $i = 1,\dots,N/2$ and $h_i = h$ for $i = N/2+1,\dots,N$. Here the piecewise-uniform mesh is entirely determined by the two chosen parameters N and τ.

For a mesh function Z_j, we define the following difference operators:

$$D^+Z_j = \frac{Z_{j+1}-Z_j}{h_{j+1}},\ \ D^-Z_j = \frac{Z_j - Z_{j-1}}{h_j},$$

$$D^+D^-Z_j = \frac{2}{h_j+h_{j+1}}\left(\frac{Z_{j+1}-Z_j}{h_{j+1}} - \frac{Z_j-Z_{j-1}}{h_j}\right).$$

The upwind finite difference scheme for (2.2) takes the form

(3.1)

$$(P_\varepsilon^N)\begin{cases} L_\varepsilon^N U_i^N \equiv -(\varepsilon+\delta a_i)D^+D^-U_i^N + a_iD^-U_i^N + b_iU_i^N = f_i\,, & 1 \le i \le N-1,\\ U_0^N = \gamma(0) = \gamma_0, \quad U_N^N = \lambda\,, \end{cases}$$

where U_i^N denotes the approximation of $u(x_i)$, $a_i = a(x_i)$ and b_i, f_i are defined in a similar fashion. Equation (3.1) can be expressed in the following form of system of algebraic equations

$$\begin{cases} -r_i^- U_{i-1}^N + r_i^c U_i^N - r_i^+ U_{i+1}^N = f_i, \quad i = 1, \ldots, N-1, \\ U_0^N = \gamma_0, \quad U_N^N = \lambda\,, \end{cases} \tag{3.2}$$

where

$$r_i^- = \frac{2(\varepsilon + \delta a_i)}{h_i(h_i + h_{i+1})} - \frac{a_i}{h_i}, \quad r_i^c = \frac{2(\varepsilon + \delta a_i)}{h_i h_{i+1}} + \frac{a_i}{h_i} + b_i, \quad r_i^+ = \frac{2(\varepsilon + \delta a_i)}{h_{i+1}(h_i + h_{i+1})}\ .$$

One can easily see that

$$r_i^- > 0,\ r_i^+ > 0 \quad \text{and} \quad r_i^c + r_i^- + r_i^+ \geq 0, \quad \text{for} \quad i = 1, \ldots, N-1, \tag{3.3}$$

which imply the stiffness matrix is an M–matrix.

Lemma 3.1 (Discrete maximum principle)**.** *The system of equations $L_\varepsilon^N V_j = F_j$ for given V_0 and V_N has a unique solution. If $L_\varepsilon^N V_j < L_\varepsilon^N Z_j$ for $1 \leq j \leq N-1$ with $V_0 < Z_0$ and $V_N < Z_N$, then $V_j < Z_j$ for $1 \leq j \leq N$.*

Proof. From (3.3), it is clear that the matrix associated with L_ε^N is an irreducible M-matrix and therefore has a positive inverse. Hence, the result follows. ■

3.1. **Convergence of upwind scheme.** The local truncation error of the difference scheme (3.1) at the node x_i is given by

$$\tau_i = L_\varepsilon^N U_i^N - (L_\varepsilon u)(x_i), \tag{3.4}$$

where u and U_i^N denote the exact solution of (2.2) and (3.1) respectively.

In order to obtain a bound of the local truncation error, we require the following lemma.

Lemma 3.2. *For any $\psi \in \mathcal{C}^3(\overline{\Omega})$,*

$$\begin{aligned} \left| \left(D^- - \frac{d}{dx} \right) \psi(x_i) \right| &\leq \frac{1}{(x_{i+1} - x_i)} \int_{x_{i-1}}^{x_i} \big(x_{i-1} - s \big) \psi''(s) ds, \\ \left| \left(D^+ D^- - \frac{d^2}{dx^2} \right) \psi(x_i) \right| &\leq \frac{1}{(x_{i+1} - x_{i-1})} \left[\frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} \big(x_{i+1} - s \big)^2 \psi'''(s) ds - \right. \\ &\left. - \frac{1}{h_i} \int_{x_{i-1}}^{x_i} \big(s - x_{i-1} \big)^2 \psi'''(s) ds \right]. \end{aligned}$$

Proof. The complete proof of this lemma is given in Lemma 4.1 of [5]. ■

Using Taylor series expansion, the truncation error (3.4) can be expressed as

$$\tau_i = \frac{-2(\varepsilon + \delta a_i)}{h_i + h_{i+1}} \left[\frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} (s - x_{i+1})^2 u'''(s) ds - \frac{1}{h_i} \int_{x_{i-1}}^{x_i} (s - x_{i-1})^2 u'''(s) ds \right]$$

$$+\frac{a_i}{h_i}\int_{x_{i-1}}^{x_i}(x_{i-1}-s)u''(s)ds, \tag{3.5}$$

from which we obtain the bound

$$|\tau_i| < (\varepsilon+\delta\alpha)\int_{x_{i-1}}^{x_{i+1}}|u'''(s)|ds + C\int_{x_{i-1}}^{x_{i+1}}|u''(s)|ds. \tag{3.6}$$

Theorem 3.3. *Let u be the solution of (P_ε) and U^N be the solution of (P_ε^N) computed on $\Omega_{N,\tau}$. Then we have the following ε–uniformly convergence result*

$$|U^N(x_i)-u(x_i)| \le CN^{-1}\ln N, \quad \forall x_i \in [0,1].$$

Proof. Let us decompose the solution U^N as $U^N = V^N + W^N$, where V^N is the solution of the BVP

$$L_\varepsilon^N V^N = f, \quad V^N(0)=v(0),\ V^N(1)=v(1), \tag{3.7}$$

and W^N is the solution of the following BVP

$$L_\varepsilon^N W^N = 0, \quad W^N(0)=w(0),\ W^N(1)=w(1). \tag{3.8}$$

Now the error can be written as $(U^N-u)(x_i) = (V^N-v)(x_i)+(W^N-w)(x_i)$. We will find out the error of the individual components in the two different region $[0,1-\tau]$ and $[1-\tau,1]$. Now

$$\begin{aligned} L_\varepsilon^N(V^N-v)(x_i) &= f(x_i) - L_\varepsilon^N v(x_i) \\ &= L_\varepsilon v(x_i) - L_\varepsilon^N v(x_i) \\ &= (L_\varepsilon - L_\varepsilon^N)v(x_i) \\ &= -(\varepsilon+\delta a(x_i))\left(\frac{d^2}{dx^2} - D^+D^-\right)v(x_i) + a(x_i)\left(\frac{d}{dx}-D^-\right)v(x_i). \end{aligned} \tag{3.9}$$

Now using Lemma 3.2 in (3.9), we have

$$L_\varepsilon^N(V^N-v)(x_i) \le (x_{i+1}-x_{i-1})\left(\frac{\varepsilon+\delta a(x_i)}{3}\right)\|v'''\| + \frac{a(x_i)}{2}\|v''\|. \tag{3.10}$$

Since $(x_{i+1}-x_{i-1}) \le 2N^{-1}$ and using the bounds of $\|v'''\|$ and $\|v''\|$ given in (2.8) , we obtain that

$$L_\varepsilon^N(V^N-v)(x_i) \le CN^{-1}, \quad \forall x_i \in \Omega_{N,\tau}. \tag{3.11}$$

Applying the discrete maximum principle to the function $(V^N-v)(x_i)$, we have

$$|(V^N-v)(x_i)| \le \beta^{-1}\max_{1\le i\le N-1}|L_\varepsilon^N(V^N-v)(x_i)|,$$

and hence

$$|(V^N-v)(x_i)| \le CN^{-1}. \tag{3.12}$$

Now we have to find the error in the singular component. Without loss of generality, assume that $\tau = 2(\varepsilon + \delta\alpha)\ln N/\beta < 1/2$. Applying the same argument as we did for the regular part, we will reach at

$$|L_\varepsilon^N(W^N - w)(x_i)| \le (x_{i+1} - x_{i-1})\left(\frac{\varepsilon + \delta a(x_i)}{3}\right)\|w'''\| + \frac{a(x_i)}{2}\|w''\|\,.$$

From the bound of the derivative of $w(x)$, we have

$$(\varepsilon + \delta a(x_i))|w'''(x)| \le |w''(x)|\,.$$

On simplification,

$$|L_\varepsilon^N(W^N - w)(x_i)| \le C(x_{i+1} - x_{i-1})\|w''\|\,.$$

Now using the bound of $w''(x)$ given in (2.9), we have

$$|w''(x_i)| \le C(\varepsilon + \delta\alpha)^{-2}\exp(-\alpha(1-x)/(\varepsilon + \delta\alpha))\,.$$

Substituting $1 - x = 2(\varepsilon + \delta\alpha)\phi(t)$, where the mesh generating function ϕ is piecewise continuously differentiable for some arbitrary $t \in [x_{i-1}, x_{i+1}]$ on the fine part of the mesh as defined in [8], we obtain

$$|L_\varepsilon^N(W^N - w)(x_i)| \le C(\varepsilon + \delta\alpha)^{-2}\int_{x_{i-1}}^{x_{i+1}} \exp(-2\phi(t))\phi'(t)dt.$$

Again, we have $x_i - x_{i-1} = h_i = 2(\varepsilon + \delta\alpha)N^{-1}\ln N$. Therefore,

$$e^{-\phi(t)} \le e^{-(1-x_{i-1})/2(\varepsilon+\delta\alpha)} \le e^{-(1-x_i)/2(\varepsilon+\delta\alpha)}e^{h_i/2(\varepsilon+\delta\alpha)} \le Ce^{-(1-x_i)/2(\varepsilon+\delta\alpha)}.$$

Hence

$$|L_\varepsilon^N(W^N - w)(x_i)| \le C(\varepsilon + \delta\alpha)^{-1}N^{-1}\ln N\int_{x_{i-1}}^{x_{i+1}} e^{-(1-x_i)/2(\varepsilon+\delta\alpha)}dx.$$

Thus, by applying the discrete maximum principle given in Lemma 3.1, we obtain

$$|W^N - w(x_i)| \le CN^{-1}\ln N\,. \tag{3.13}$$

Now combining the error bounds of the smooth component given in (3.12) and the singular component given in (3.13), we get the required result. ■

Before we move to extrapolation analysis, we provide here two more lemmas which are needed in the next section.

For $i = 0, 1, \ldots, N$, we define the mesh functions on $\Omega_{N,\tau}$ as

$$S_0 = \bar{S}_0 = 1, \quad S_i = \prod_{j=1}^{i}\left(1 + \frac{\alpha h_j}{\varepsilon + \delta\alpha}\right), \quad \bar{S}_j = \prod_{j=1}^{i}\left(1 + \frac{\alpha h_j}{2(\varepsilon + \delta\alpha)}\right). \tag{3.14}$$

Lemma 3.4. *There exist a positive constant C_2 such that for $i = 1, \dots, N-1$,*

$$L_\varepsilon^N S_i \geq \frac{C_2}{\varepsilon + \delta\alpha + \alpha h_i} S_i \quad \text{and} \quad L_\varepsilon^N \bar{S}_i \geq \frac{C_2}{2(\varepsilon + \delta\alpha) + \alpha h_i} \bar{S}_i \,. \tag{3.15}$$

Moreover, we can get a positive constant C_3 such that for which $i = N/2+1, \dots, N-1$, we have

$$L_\varepsilon^N S_i \geq C_3(\varepsilon + \delta\alpha)^{-1} S_i \quad \text{and} \quad L_\varepsilon^N \bar{S}_i \geq C_3(\varepsilon + \delta\alpha)^{-1} \bar{S}_i \,. \tag{3.16}$$

Proof. Define $\widehat{S}_i = \prod_{j=1}^{i} \left(1 + \frac{\bar{\alpha} h_j}{\varepsilon + \delta\alpha}\right)$, where $\bar{\alpha}$ can be α or $\alpha/2$. Now

$$\begin{aligned} L_\varepsilon^N \widehat{S}_i &= -\frac{2(\varepsilon + \delta a_i)}{h_i + h_{i+1}} \left[\frac{\widehat{S}_{i+1} - \widehat{S}_i}{h_{i+1}} - \frac{\widehat{S}_i - \widehat{S}_{i-1}}{h_i}\right] + a_i \left[\frac{\widehat{S}_i - \widehat{S}_{i-1}}{h_i}\right] + b_i \widehat{S}_i \\ &\geq \left(\frac{\varepsilon + \delta a(x_i)}{\varepsilon + \delta\alpha + \bar{\alpha} h_i}\right)\left(a_j \bar{\alpha} - 2\bar{\alpha}^2 \widehat{S}_j\right) \\ &\geq \frac{C_2}{\varepsilon + \delta\alpha + \bar{\alpha} h_i} \widehat{S}_j \,. \end{aligned}$$

for some C_2, since $a(x) \geq 2\alpha \geq 2\bar{\alpha}$ and $h = O(\varepsilon)$. Hence the required results (3.15) and (3.16) are obtained. ∎

Lemma 3.5. *Let $\Omega_{N,\tau} = \{x_i\}_{i=0}^N$ with $h_i = x_i - x_{i-1}$. Then there exist a positive constant C_4 such that*

$$N^{-1} \leq \prod_{j=N/2+1}^{N} \left(1 + \frac{\alpha h_j}{2(\varepsilon + \delta\alpha)}\right)^{-1} \leq C_4 N^{-1} \quad \text{and} \quad \prod_{j=N/2+1}^{N} \left(1 + \frac{\alpha h_j}{\varepsilon + \delta\alpha}\right)^{-1} \leq C_4 N^{-2}. \tag{3.17}$$

Proof. We have

$$\prod_{j=N/2+1}^{N} \left(1 + \frac{\alpha h_j}{2(\varepsilon + \delta\alpha)}\right)^{-1} = \left(1 + \frac{2 \ln N}{N}\right)^{-N/2} \leq C_4 N^{-1} \,.$$

Using the inequality $\ln(1+x) \leq x - x^2/4$ for $0 \leq x \leq 1$, we have

$$\ln\left[\prod_{j=N/2+1}^{N} \left(1 + \frac{\alpha h_j}{2(\varepsilon + \delta\alpha)}\right)^{-1}\right] = \ln\left(1 + \frac{2 \ln N}{N}\right)^{-N/2} \geq -\ln N + \frac{\ln^2 N}{2N},$$

and hence the first inequality holds. The proof of the second inequality is given in [10]. ∎

4. RICHARDSON EXTRAPOLATION TECHNIQUE

The main goal of this paper is to improve the accuracy of the upwind scheme which can be done by extrapolation. The solution U^N is computed on the mesh $\Omega_{N,\tau}$. Now we can solve the same problem on $\Omega_{2N,\tau} = \{\widetilde{x}_i\}_{i=0}^{2N}$ with $2N$ number of subintervals by keeping the same transition point $1-\tau$. Thus on $\Omega_{2N,\tau}$: $\widetilde{x}_i - \widetilde{x}_{i-1} =$

$H/2$ for $\widetilde{x}_i \in [0, 1-\tau]$ and $\widetilde{x}_i - \widetilde{x}_{i-1} = h/2$ for $\widetilde{x}_i \in [1-\tau, 1]$. From Theorem 3.3, we know that

(4.1)
$$U^N(x_i) - u(x_i) = C_5 N^{-1} \ln N + R_N(x_i) = C_5 N^{-1} \frac{\alpha\tau}{2(\varepsilon + \delta\alpha)} + R_N(x_i),\ \forall x_i \in \Omega_{N,\tau},$$

where C_5 is a fixed constant and the remainder R_N is of $o(N^{-1} \ln N)$. Similarly, one can have

$$U^{2N}(x_i) - u(x_i) = C_5 (2N)^{-1} \frac{\alpha\tau}{2(\varepsilon + \delta\alpha)} + R_{2N}(x_i), \quad \forall x_i \in \Omega_{2N,\tau}, \tag{4.2}$$

where U^{2N} denote the solution of discrete problem (P_ε^N) on $\Omega_{2N,\tau}$ and the remainder R_{2N} is of $o(N^{-1} \ln N)$. Eliminating the first terms (which is of $O(N^{-1})$) from the above two equations, we obtain

$$u(x_i) - (2U^{2N}(x_i) - U^N(x_i)) = R_N(x_i) - R_{2N}(x_i) = o(N^{-1} \ln N), \quad \forall x_i \in \Omega_{N,\tau}. \tag{4.3}$$

That is, $(2U^{2N}(x_i) - U^N(x_i))$ is a better approximation of $u(x_i)$ which is more accurate than $U^N(x_i)$ or $U^{2N}(x_i)$ on $\Omega_{N,\tau}$. Hence we use the extrapolation formula

$$(2U^{2N}(x_i) - U^N(x_i)) \quad \text{for } x_i \in \Omega_{N,\tau}. \tag{4.4}$$

From the decomposition of the solution U^N of (P_ε^N) on $\Omega_{N,\tau}$ given in (3.7) and (3.8), one can set

$$U^N - u = (V^N - v) + (W^N - w) \quad \text{and} \quad U^{2N} - u = (V^{2N} - v) + (W^{2N} - w).$$

Let $\xi(x) = a(x) v''(x)/2$, $\forall x \in [0, 1]$. In the following lemma, we find the bound of the truncation error of v.

Lemma 4.1. *Under the assumption (1.2) and for all $x_i \in (0, 1)$, we have*

$$L_\varepsilon^N (V^N - v)(x_i) = \xi(x_i) h_i + O(H^2).$$

Proof. From (2.8), we know that $|v'''| \leq C$. Now using the assumption $\varepsilon \leq N^{-1} \leq H$ and Lemma 3.2, we can prove the above lemma. ■

For all $x \in (0, 1)$, define the function $E(x)$ as the solution of the BVP

$$L_\varepsilon E(x) = \xi(x), \quad E(0) = E(1) = 0. \tag{4.5}$$

One can decompose E as $E = \eta + \vartheta$, where η, ϑ are the smooth and layer parts of E. Now using Lemma 2.4 of [6], we have the following bounds:

$$|\eta^{(k)}(x)| \leq C(1 + (\varepsilon + \delta\alpha)^{2-k}), \quad 0 \leq k \leq 3, \tag{4.6}$$

$$|\vartheta^{(k)}(x)| \leq C(\varepsilon + \delta\alpha)^{-k}) \exp(-\alpha(1-x)/(\varepsilon + \delta\alpha)), \quad 0 \leq k \leq 3, \tag{4.7}$$

with

$$L_\varepsilon \eta(x) = \xi(x) \text{ and } L_\varepsilon \vartheta(x) = 0, \quad \eta(0) = \vartheta(0) = 0, \quad \eta(1) = \vartheta(1) = 0. \tag{4.8}$$

In the following lemma, we determine the error in the smooth part V^N.

Lemma 4.2. *We have*

$$V^N(x_i) - v(x_i) = HE(x_i) + O(N^{-2}), \quad \forall x_i \in [0, 1-\tau].$$

Proof. Fix $x_i \in (0,1)$. Now from the truncation error, we have

$$L_\varepsilon^N \eta(x_i) = L_\varepsilon \eta(x_i) + L_\varepsilon^N \eta(x_i) - L_\varepsilon \eta(x_i) = L_\varepsilon \eta(x_i) + O(H).$$

So

$$HL_\varepsilon^N \eta(x_i) = H\xi(x_i) + O(H^2).$$

Using the fact that $h_j \le H$, Lemma 4.1 yields

$$L_\varepsilon^N (V^N - v - H\eta)(x_i) = \begin{cases} O(H^2) \\ (h-H)\xi(x_i) + O(H^2) \end{cases}$$

$$(4.9) \qquad = \begin{cases} O(H^2), & x_i \in (0, 1-\tau) \\ O(H), & x_i \in (1-\tau, 1). \end{cases}$$

Define the mesh function

$$Z_i = C_6 \left[N^{-2}(1+x_i) + H\bar{S}_i \prod_{j=1}^{N} \left(1 + \frac{\alpha h_j}{2(\varepsilon+\delta\alpha)}\right)^{-1} \right], \quad \text{for } i = 0, \dots, N,$$

where the constant C_6 has to be determined later. Then

$$L_\varepsilon^N Z_i \ge C_6 \left[2\alpha N^{-2} + H \prod_{j=1}^{N} \left(1 + \frac{\alpha h_j}{2(\varepsilon+\delta\alpha)}\right)^{-1} L_\varepsilon^N \bar{S}_i \right], \quad \text{for } i = 1, \dots, N-1.$$

Using (3.15) and $\varepsilon \le N^{-1}$ for $0 < i \le N/2$, it follows that

$$(4.10) \qquad L_\varepsilon^N Z_i \ge C_6 \alpha N^{-2} \ge C_6 \alpha H^2/4,$$

and for $0 < i \le N/2$,

$$(4.11) \qquad L_\varepsilon^N Z_i \ge C_6 C_3 H(\varepsilon+\delta\alpha)^{-1} \prod_{j=1}^{N} \left(1 + \frac{\alpha h_j}{2(\varepsilon+\delta\alpha)}\right)^{-1} \ge C_6 C_3 H.$$

It is easy to check that $Z_0 \ge 0 = |V^N(0) - v(0) - H\eta(0)|$ and $Z_N \ge C_5 = |V^N(1) - v(1) - H\eta(1)|$. If we choose sufficiently large C_5, then Z_i plays the role of a barrier function for $\pm[V^N(x_i) - v(x_i) - H\eta(x_i)]$. Hence

$$Z_i \ge V^N(x_i) - v(x_i) - H\eta(x_i), \quad \forall x_i \in (0, 1-\tau).$$

But for $i = 1, \dots, N/2$, we have from (3.17),

$$(4.12) \quad Z_i \le C_6 \left[2\alpha N^{-2} + H \prod_{j=1}^{N} \left(1 + \frac{\alpha h_j}{2(\varepsilon+\delta\alpha)}\right)^{-1} \le C(N^{-2} + hN^{-1}) \right] \le CN^{-2}.$$

Thus, for $i = 1, \dots, N/2$,

$$|V^N(x_i) - v(x_i) - HE(x_i)| \;\le\; |V^N(x_i) - v(x_i) - H\eta(x_i)| + |H\vartheta(x_i)|$$

$$(4.13) \qquad \leq \quad Z_i + 2N^{-1}|\vartheta(x_i)| \leq CN^{-2}.$$

Now we are in a position to show that extrapolation improves the accuracy of V^N on $(0, 1-\tau]$.

Lemma 4.3. *For all $x_i \in [0, 1-\tau]$,*

$$|v(x_i) - (V^{2N}(x_i) - V^N(x_i))| \leq CN^{-2}.$$

Proof. From Lemma 4.2, we have for $x_i \in [0, 1-\tau]$

$$V^N(x_i) - v(x_i) = HE(x_i) + O(N^{-2}).$$

Note that the subinterval mesh widths of $\Omega_{2N,\tau}$ is half of $\Omega_{N,\tau}$ and the function $E(x)$ defined in (4.5) depend only on τ. Hence

$$V^{2N}(x_i) - v(x_i) = \frac{H}{2}E(x_i) + O(N^{-2}).$$

So from the above two relations we can conclude that

$$v(x_i) - (V^{2N}(x_i) - V^N(x_i)) = -2(V^{2N} - v)(x_i) + (V^N - v)(x_i) = O(N^{-2}),$$

for $1 \leq i \leq N/2$. ■

Now in the next lemma, we show the error of $V^N(x_i)$ after extrapolation on $[1-\tau, 1]$.

Lemma 4.4. *For all $x_i \in [1-\tau, 1]$, we have for some constant C*

$$|v(x_i) - (V^{2N}(x_i) - V^N(x_i))| \leq CN^{-2}\ln N.$$

Proof. Let us define the function $G(x)$ on $[1-\tau, 1]$ as the solution of the BVP

$$L_\varepsilon G(x) = 0, \quad G(1-\tau) = 1, G(1) = 0.$$

On $\Omega_{N,\tau}$, define a discrete approximation G^N of G by

$$L_\varepsilon^N G^N(x_i) = 0, \quad G^N(1-\tau) = 1, \, G^N(1) = 0.$$

From the convergence of upwind scheme given in Theorem 3.3, we have

$$(4.14) \qquad |G(x_i) - G^N(x_i)| \leq CN^{-1}\ln N, \quad \text{for } N/2 \leq i \leq N.$$

Define $\omega(x_i) = V^N(x_i) - v(x_i) - (HE(1-\tau))G^N(x_i)$ for $N/2 \leq i \leq N$. Then $\omega(x_{N/2}) = O(N^{-2})$, $\omega(x_N) = 0$ and $L_\varepsilon^N \omega(x_i) = O(h_i) + O(H^2) = O(N^{-2}\ln N)$. Now using the barrier function $C(1+x_i)N^{-2}$ we can get

$$(4.15) \qquad |\omega(x_i)| \leq CN^{-2}\ln N, \quad \text{for } N/2 \leq i \leq N.$$

Observe that $|E(1-\tau)| \leq C$, so we obtain

$$(4.16) \quad V^N(x_i) - v(x_i) = (HE(1-\tau))G^N(x_i) + O(N^{-2}\ln N), \quad \text{for } N/2 \leq i \leq N.$$

Similarly, on the mesh $\Omega_{2N,\tau}$, one has

$$V^{2N}(\tilde{x}_i) - v(\tilde{x}_i) = (H/2)E(1-\tau)G^N(\tilde{x}_i) + O(N^{-2}\ln N), \quad \text{for } N \le i \le 2N. \tag{4.17}$$

But $\tilde{x}_{2i} = x_i$ for $N/2 \le i \le N$. Now combining (4.16) and (4.17), we obtain

$$v(x_i) - (2V^{2N} - V^N)(x_i) = 2(v - V^{2N})(x_i) - (v - V^N)(x_i) = O(N^{-2}\ln N),$$

for $N \le i \le 2N$. ∎

Now in the following lemmas, we shall apply the extrapolation on W^N in $\Omega_{N,\tau}$. First we will find the error $W^N(x_i) - w(x_i)$ in both the subintervals $[0, 1-\tau]$ and $[1-\tau, 1]$ separately.

Lemma 4.5. *For all* $x_i \in [0, 1-\tau]$,

$$|w(x_i) - (W^{2N}(x_i) - W^N(x_i))| \le CN^{-2}.$$

Proof. Following the same procedure as done in [7] for one parameter convection diffusion problem, we can prove this lemma. ∎

Now let us define $F(x)$ on $[1-\tau, 1]$ as the solution of the following BVP:

$$L_\varepsilon F(x) = \frac{-2}{\alpha} a(x) w''(x), \quad F(1-\tau) = F(1) = 0. \tag{4.18}$$

The following lemma shows the error of the singular component in the inner region of $\Omega_{N,\tau}$.

Lemma 4.6. *We have*

$$W^N(x_i) - w(x_i) = (N^{-1}\ln N)F(x_i) + O(N^{-2}\ln N), \quad \forall x_i \in [1-\tau, 1].$$

Proof. From the bound of the layer part, we know that

$$|F^{(k)}(x)| \le C(\varepsilon + \delta\alpha)^{-k} e^{-\alpha(1-x)/(\varepsilon+\delta\alpha)}, \quad \text{for } 0 \le k \le 3.$$

By using Taylor series expansion at $x = x_i \in [1-\tau, 1]$, we get

$$L_\varepsilon^N F(x_i) = L_\varepsilon F(x_i) + O\left((\varepsilon + \delta\alpha)^{-1} N^{-1} \ln N e^{-\alpha(1-x)/(\varepsilon+\delta\alpha)}\right).$$

Now for some $\tilde{x}_1 \in (x_i, x_{i+1})$ and $\tilde{x}_2, \tilde{x}_3 \in (x_i, x_{i+1})$, we have

$$\begin{aligned} L_\varepsilon^N(W^N - w)(x_i) &= \frac{(\varepsilon+\delta\alpha)}{4!}[w^{(4)}(\tilde{x}_1) + w^{(4)}(\tilde{x}_2)]h^2 + \frac{a(x_i)}{2}\left[\frac{w'''(\tilde{x}_3)}{3}h^2 + w''(x_i)h\right] \\ &= -\frac{2}{\alpha}((\varepsilon+\delta\alpha)^{-1}N^{-1}\ln N)a(x_i)w''(x_i) + \\ &+ O\left((\varepsilon+\delta\alpha)^{-1}N^{-2}\ln^2 N e^{-\alpha(1-x)/(\varepsilon+\delta\alpha)}\right). \end{aligned} \tag{4.19}$$

Hence from the above two equations, we have

$$(N^{-1}\ln N)L_\varepsilon^N F(x_i) = (N^{-1}\ln N)L_\varepsilon F(x_i) + O\left((\varepsilon+\delta\alpha)^{-1}N^{-2}\ln^2 N e^{-\alpha(1-x)/(\varepsilon+\delta\alpha)}\right)$$

$$= L_\varepsilon^N(W^N - w)(x_i) + O\left((\varepsilon+\delta\alpha)^{-1}N^{-2}\ln^2 N e^{-\alpha(1-x)/(\varepsilon+\delta\alpha)}\right).$$

That is, $\forall x \in (1-\tau, 1)$,

$$|L_\varepsilon^N(W^N - w - (N^{-1}\ln N)F)(x_i)| \le C_7(\varepsilon+\delta\alpha)^{-1}N^{-2}\ln^2 N e^{-\alpha(1-x)/(\varepsilon+\delta\alpha)}, \tag{4.20}$$

for some constant C_7. For $i = N/2, \dots, N$, consider the mesh function

$$\Gamma_i = C_8\left[N^{-2}(1+x_i) + (N^{-2}\ln^{-2} N)S_i \prod_{j=1}^{N}\left(1+\frac{\alpha h_j}{2(\varepsilon+\delta\alpha)}\right)^{-1}\right],$$

where C_8 has to be chosen later. From (3.15), we have

$$L_\varepsilon^N\Gamma_i \ge C_3C_8(\varepsilon+\delta\alpha)^{-1}N^{-2}\ln^2 N\prod_{j=1}^{N}\left(1+\frac{\alpha h_j}{2(\varepsilon+\delta\alpha)}\right)^{-1}. \tag{4.21}$$

For $i \ge N/2$, we can have

$$\begin{aligned} e^{-\alpha(1-x_{i+1})/(\varepsilon+\delta\alpha)} &= e^{\alpha h_{i+1}/(\varepsilon+\delta\alpha)}\prod_{j=i+1}^{N} e^{-\alpha h_j/(\varepsilon+\delta\alpha)} \\ &\le e^{4n^{-1}\ln N}\prod_{j=i+1}^{N}\left(1+\frac{\alpha h_j}{\varepsilon+\delta\alpha}\right)^{-1} \\ &\le e^4\prod_{j=i+1}^{N}\left(1+\frac{\alpha h_j}{\varepsilon+\delta\alpha}\right)^{-1}, \end{aligned} \tag{4.22}$$

provided $C_8 \ge e^4C_7/C_3$. Hence from (4.21) and (4.22), we can get

$$L_\varepsilon^N\Gamma_i \ge |L_\varepsilon^N(W^N - w - (N^{-1}\ln N)F)(x_i)| \quad \text{for } x_i \in (1-\tau, 1).$$

It is easy to show that $\Gamma_N \ge 0 = |W^N(1) - w(1) - (N^{-1}\ln N)F(1)|$. From Lemma 4.5, we have $|(w - W^N)(1-\tau)| \le C_9N^{-2}$. So $\Gamma_{N/2} \ge C_8N^{-2} = |(W^N - w - (N^{-1}\ln N)F)(1-\tau)|$ provided $C_8 \ge C_9$. Now choosing $C_8 = \max\{e^4C_7/C_3, C_9\}$, we can show that Γ_i is a barrier function for $\pm[W^N(x_i) - w(x_i) - (N^{-1}\ln N)F(x_i)]$ for $x_i \in [1-\tau, 1]$. Hence applying the discrete maximum principle, we can conclude that

$$|W^N(x_i) - w(x_i) - (N^{-1}\ln N)F(x_i)| \le \Gamma_i \le C_8N^{-2}\ln^2 N.$$

In the next lemma, we will obtain the second order error bound for the extrapolation of the singular component.

Lemma 4.7. *For some constant C and $\forall x \in [1-\tau, 1]$,*

$$|w(x_i) - (2W^{2N}(x_i) - W^N(x_i))| \le CN^{-2}\ln^2 N.$$

Proof. From Lemma 4.6, we have

$$\begin{aligned} W^N(x_i) - w(x_i) &= N^{-1}\ln NF(x_i) + O(N^{-2}\ln^2 N) \\ &= N^{-1}(\tau\alpha/(\varepsilon+\delta\alpha))F(x_i) + O(N^{-2}(\tau\alpha/(\varepsilon+\delta\alpha))^2). \end{aligned} \tag{4.23}$$

Similarly,

$$W^{2N}(x_i) - w(x_i) = N^{-1}(\tau\alpha/(\varepsilon + \delta\alpha))F(x_i) + O(N^{-2}(\tau\alpha/(\varepsilon + \delta\alpha))^2). \tag{4.24}$$

Replacing the first term from (4.23) and (4.24), we have

$$w(x_i) - (2W^{2N} - W^N)(x_i) = 2(w - W^{2N})(x_i) - (w - W^N)(x_i) = O(N^{-2}\ln^2 N),$$

Hence the desired result is obtained. ■

We will state the main result of this paper in the following theorem, which provides the $\varepsilon-$ uniform second order convergent result of the Richardson extrapolation technique applied to the solution of the BVP (2.2).

Theorem 4.8. *Assume that $\varepsilon \leq N^{-1}$, then there exist a positive constant C such that*

$$|u(x_i) - (2U^{2N}(x_i) - U^N(x_i))| \leq CN^{-2}\ln^2 N, \quad \forall x_i \in \Omega_{N,\tau}.$$

Proof. For each $x_i \in \Omega_{N,\tau}$, we have

$$|u(x_i) - (2U^{2N}(x_i) - U^N(x_i))| \leq |[v - (2V^{2N} - V^N)(x_i)]| + |[w - (2W^{2N} - W^N)(x_i)]|.$$

Combining the results of Lemmas 4.3 and 4.4 for the first term and the results of Lemmas 4.5 and 4.7 for the second term of the above equation, we will obtain the required result. ■

5. NUMERICAL RESULTS

In this section, to validate the theoretical results, we apply the proposed numerical scheme to the following delay differential equation.

$$\begin{cases} -\varepsilon u''(x) + u'(x-\delta) + u(x) = 0, \quad x \in \Omega = (0,1), \\ u(x) = 1, \quad -\delta \leq x \leq 0, \\ u(1) = -1. \end{cases} \tag{5.1}$$

The solution of this BVP exhibits a boundary layer at $x = 1$. The exact solution is given by $u(x) = C_1 \exp(m_1 x) - C_2 \exp(m_2 x)$, where

$$m_{1,2} = \frac{1 \mp \sqrt{1 + 4(\varepsilon + \delta)}}{2(\varepsilon + \delta)}, C_1 = \frac{1 + \exp(m_2)}{\exp(m_2) - \exp(m_1)}, C_2 = \frac{\exp(m_1) + 1}{\exp(m_2) - \exp(m_1)}.$$

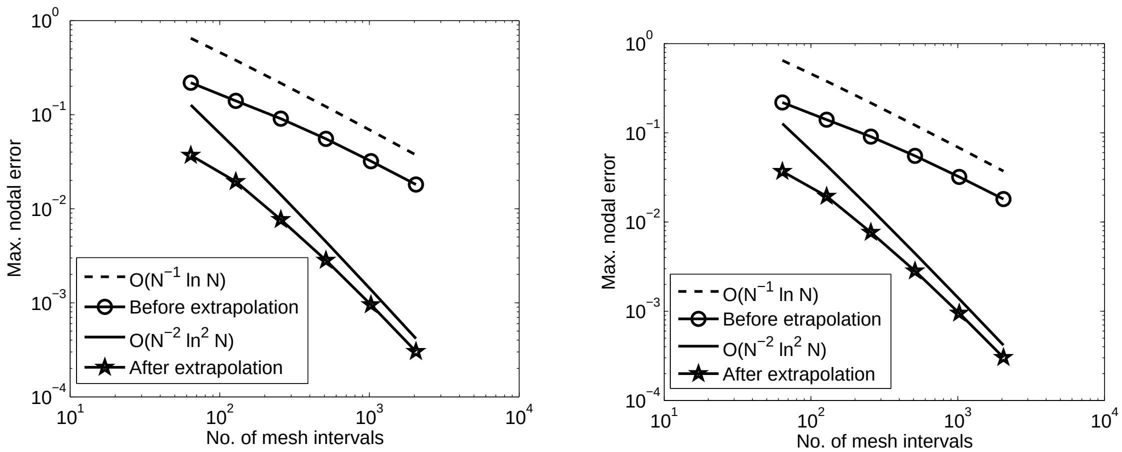
For any value of N, the maximum pointwise errors E_ε^N are calculated by $E_\varepsilon^N = \|u - U^N\|$ where u is the exact solution and U^N is the numerical solution. We use the double mesh method to compute the rate of convergence as $p^N = \log_2\left(E_\varepsilon^N / E_\varepsilon^{2N}\right)$.

In Table 1, we present the maximum pointwise error before and after extrapolation for a range of values of ε and N and the corresponding order of convergence. From these computational results, one can observe the ε–uniform convergence of the scheme before and after extrapolation. In Figure 1, the number of mesh intervals

TABLE 1. Maximum point-wise errors E_ε^N and the rate of convergence p^N before and after extrapolation

ε	Number of intervals N						
		32	64	128	256	512	1024
	before	1.4785e-3	7.4410e-4	3.7334e-4	1.8698e-4	9.3568e-5	4.6804e-5
	rate	0.9905	0.9950	0.9976	0.9988	0.9994	
$1e-0$	after	9.7307e-6	2.4617e-6	6.1892e-7	1.5518e-7	3.8843e-8	9.7491e-9
	rate	1.9829	1.9918	1.9958	1.9982	1.9943	
	before	8.6354e-2	5.6101e-2	3.4222e-2	2.0195e-2	1.1593e-2	6.5208e-3
	rate	0.6222	0.7131	0.7609	0.8008	0.8301	
$1e-2$	after	7.3194e-3	3.0500e-3	1.1405e-3	3.9703e-4	1.3022e-4	4.1064e-5
	rate	1.2629	1.4192	1.5223	1.6083	1.6650	
	before	8.7003e-2	5.6261e-2	3.4226e-2	2.0163e-2	1.1576e-2	6.5127e-3
	rate	0.6289	0.7170	0.7634	0.8006	0.8298	
$1e-4$	after	7.9395e-3	3.1993e-3	1.1760e-3	4.0513e-4	1.3199e-4	4.1408e-5
	rate	1.3113	1.4439	1.5374	1.6179	1.6725	
	before	8.7014e-2	5.6265e-2	3.4227e-2	2.0163e-2	1.1576e-2	6.5126e-3
	rate	0.6290	0.7171	0.7634	0.8006	0.8298	
$1e-6$	after	7.9572e-3	3.2064e-3	1.1791e-3	4.0641e-4	1.3254e-4	4.1636e-5
	rate	1.3113	1.4433	1.5366	1.6165	1.6706	
$1e-8$	before	8.7015e-2	5.6265e-2	3.4227e-2	2.0163e-2	1.1576e-2	6.5126e-3
to	rate	0.6290	0.7171	0.7634	0.8006	0.8298	
$1e-10$	after	7.9574e-3	3.2065e-3	1.1791e-3	4.0641e-4	1.3254e-4	4.1624e-5
	rate	1.3113	1.4433	1.5367	1.6165	1.6709	

and the maximum point-wise errors are presented using loglog scale along with the theoretical rate of convergence. Clearly these figures show that the computed errors decreases approximately at the same rates which is proved theoretically and the rate of convergence of the upwind scheme is doubled after extrapolation.



(a) *For* $\varepsilon = 1e - 4$. (b) *For* $\varepsilon = 1e - 8$.

FIGURE 1. Loglog plot of the maximum point-wise errors before and after extrapolation.

6. CONCLUSION

In this article, we applied the Richardson extrapolation technique for singularly perturbed delay differential equations on Shishkin mesh. First, we solve the BVP (1.1) by upwind finite difference scheme to get U^N and carried out the convergence analysis to obtain the optimal error of order $O(N^{-1} \ln N)$. By keeping the transition parameter fixed on the Shishkin mesh, we calculate U^{2N}. Then the Richardson extrapolation is implemented. We have shown both theoretically and computationally that after extrapolation, the rate of convergence of the upwind scheme is raised from almost first order *i.e*, $O(N^{-1} \ln N)$ to almost second order *i.e*, $O(N^{-2} \ln^2 N)$.

REFERENCES

[1] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan and G.I. Shishkin, *Robust Computational Techniques for Boundary Layers*, Chapman & Hall/CRC Press, Boca Raton, FL, 2000.

[2] R. B. Kellogg and A. Tsan, Analysis of some difference approximations for a singular perturbation problem without turning point, *Math. Comp.*, **32**(144):1025–1039, 1978.

[3] C. G. Lange and R. M. Miura, Singular perturbation analysis of boundary-value problems for differential-difference equations, *SIAM J. Appl. Math.*, **42**(3):502–531, 1982.

[4] C. G. Lange and R. M. Miura, Singular perturbation analysis of boundary-value problems for differential-difference equations. V. small shifts with layer behavior, *SIAM J. Appl. Math.*, **54**(1):249–272, 1994.

[5] J. J. H. Miller, E. O'Riordan and G. I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific, Singapore, 1996.

[6] J. Mohapatra and S. Natesan, Uniform convergence analysis of finite difference scheme for singularly perturbed delay differential equation on an adaptively generated grid, 2008 (submitted for publication).

[7] M. C. Natividad and M. Stynes, Richardson extrapolation for a convection-diffusion problem using shishkin mesh, *Appl. Numer. Math.*, **45**:315–329, 2003.

[8] H. G. Roos and T. Linβ, Sufficient conditions for uniform convergence on layer-adapted grids, *Computing*, **63**:27–45, 1999.

[9] R. B. Stein, Some models of neuronal variability, *Biophysical Journal*, **7**:37–68, 1967.

[10] M. Stynes and H. G. Roos, The midpoint upwind scheme, *Appl. Numer. Math.*, **23**:361–374, 1997.