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A ROBUST NUMERICAL SCHEME FOR SINGULARLY PERTURBED PARABOLIC REACTION-DIFFUSION PROBLEMS

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ABSTRACT. In this article, we study the numerical solution of singularly perturbed parabolic reaction-diffusion initial-boundary-value problems. To solve these problems, we develop a numerical scheme which combines the cubic spline scheme and the classical finite difference scheme for the spatial derivatives, and backward difference scheme for the time derivative. To resolve the boundary layers, we use the piecewise uniform Shishkin mesh for the spatial discretization. Stability analysis and error estimates are obtained. The proposed method is applied to a test problem, which shows the parameter-uniform convergent results.

Key Words Singularly perturbed parabolic problem, boundary layers, cubic spline, piecewise-uniform Shishkin mesh, uniform convergence.

Subject Classification: AMS 65M06, CR G1.8

1. INTRODUCTION

Consider the singularly perturbed parabolic initial-boundary value problem (IBVP) in the domain $\Omega = (0,1) \times (0,T]$:

$$(1.1)\qquad \begin{cases} Lu(x,t) \equiv \dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} + b(x,t)u(x,t) = f(x,t), & (x,t) \in \Omega \\ u(x,0) = s(x), \quad \text{on } S_x = \{(x,0) : 0 \le x \le 1\}, \\ u(0,t) = a_0(t), \quad \text{on } S_0 = \{(0,t) : 0 \le t \le T\}, \\ u(1,t) = a_1(t), \quad \text{on } S_1 = \{(0,t) : 0 \le t \le T\}, \end{cases}$$

where $0 < \varepsilon \ll 1$ is a small parameter, and b, f are sufficiently smooth functions with $b(x,t) \ge \beta > 0$ on $\overline{\Omega}$. Under suitable continuity and compatibility conditions on the data, the IBVP (1.1) has a unique solution $u(x,t)$. Boundary layers occur in the solution when $\varepsilon \to 0$. These boundary layers are neighbors of the boundaries of the domain, where the solution varies rapidly, while away from the layers the solution

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changes slowly, and smoothly. Away from any corner of the domain a boundary layer of either regular or parabolic type may occur. A boundary layer is said to be of parabolic type, if the characteristics of the reduced equation corresponding to $\varepsilon = 0$ are parallel to the boundary, and of regular type, if these characteristics are not parallel to the boundary. A boundary layer near to a corner is said to be of corner type. These types of problems include the linearized Burgers' equation which arises in oil reservoir simulation [2].

Numerical treatment of the IBVP (1.1) is difficult because of the presence of boundary layers in its solution. In particular, classical finite difference methods fail to yield satisfactory numerical results on uniform meshes, and to obtain stability one has to reduce the spatial step-size in relation with ε. The same is true for finite element methods in the case of uniform mesh and polynomial basis functions. Basically, by these methods one cannot obtain ε-uniform error estimates. When regular layers are present it is possible to obtain an ε-uniform method by constructing an appropriately fitted finite difference operator (*i.e.,* finite difference scheme with fitting factor) on uniform meshes. Indeed, Shishkin [12] proved that this approach is not possible if a parabolic boundary layer is present, more details can be obtained from the book of Miller et al. [6]. One can also refer the books of Farrell et al. [3], and Roos et al. [11] for further results related to the theory and numerics of singularly perturbed parabolic problems.

The main goal of this paper is to provide an ε-uniform numerical method for the IBVP (1.1). In this method, the time derivative is replaced by the backward difference scheme, and the spatial derivative is replaced by an hybrid scheme, which is a combination of the cubic spline and the classical central difference scheme (more details about the hybrid scheme can be found in the articles [9, 10] for ordinary differential equations). The proposed scheme is parameter-uniform convergent of order $O(N_x^{-2}\ln^2 N_x + N_t^{-1})$. Truncation errors are derived, stability analysis is carried out; and ε-uniform error estimates are obtained.

There are various numerical methods exist in the literature for singularly perturbed parabolic PDEs. To cite a few: Stynes and O'Riordan [13] presented a uniformly convergent finite element method for these types of problems using exponential basis functions. In [5], finite element method of exponentially fitted lumped schemes were given. Farrell et al. [4] proposed numerical methods for IBVPs of the form (1.1). The authors proposed two higher-order time accurate schemes for the parabolic IBVP (1.1) in [1]. In [8], we developed an efficient hybrid numerical scheme for singularly perturbed parabolic IBVPs with interior layer.

The paper is organized in the following way: Section 2 presents some results regarding the analytical solution of the IBVP (1.1). Section 3 deals with the Shishkin mesh, and the numerical scheme. Section 4 provides the truncation error, stability analysis and error estimate for the numerical solution. Section 5 carries some numerical results, and the paper ends with conclusions.

Through out this paper C denotes a generic positive constant independent of ε, the meshes (x_i, t_j), and the step sizes h_i, τ_j. The norm $||\cdot||$ denotes the supremum norm.

2. ANALYTICAL BEHAVIOR OF THE SOLUTION

In this section, we present some bounds for the analytical solution $u(x,t)$ of (1.1) and its derivatives. The proof of the theorems and more details can be found in the article by Miller et. al. [7].

Theorem 2.1. [7] *Assume that the coefficients of the parabolic PDE, and the initial and boundary conditions given in (1.1) are sufficiently smooth, and satisfy the necessary compatibility conditions stated in Theorem 3 of* [7]*. Then, the IBVP (1.1) has a unique solution $u(x,t) \in \mathcal{C}^4_\lambda(\overline{\Omega})$, where*

$$\mathcal{C}^4_\lambda(\overline{\Omega}) = \left\{ u : \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \in \mathcal{C}^0_\lambda(\overline{\Omega}), \text{ for all non-negative integers } i,\ j \text{ with } 0 \le i + 2j \le 4 \right\},$$

here $\mathcal{C}^0_\lambda(\overline{\Omega})$ is the set of Hölder continuous functions. Furthermore, the derivatives of the solution u satisfy, for all non-negative integers i, j, such that $0 \le i + 2j \le 4$,

$$\left\| \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \right\| \le C\varepsilon^{-i/2}.$$

Proof. The proof can be found in [7]. ■

We shall decompose the solution u as $u = v + w$, where v, w are respectively the smooth and singular components. The smooth part is further decomposed into $v = v_0 + \varepsilon v_1$, where

$$\begin{cases} b(x,t)v_0 + \dfrac{\partial v_0}{\partial t} = f(x,t), & \Omega \\ v_0(x,0) = s(x), & \text{on } S_x, \end{cases}$$

and

$$\begin{cases} Lv_1 = \dfrac{\partial^2 v_0}{\partial x^2}, \quad \Omega \\ v_1(x,0) = 0, \quad \text{on } S_x, \\ v_1(0,t) = 0, \quad v_1(1,t) = 0, \quad \text{on } S_0, S_1. \end{cases}$$

Thus, v satisfies the following IBVP:

$$\begin{cases} Lv = f, \quad \text{in } \Omega, \\ v(x,0) = s(x), \quad \text{on } S_x, \\ v(0,t) = v_0(0,t), \quad v(1,t) = v_0(1,t), \quad \text{on } S_0, S_1. \end{cases}$$

The singular component w is the solution of the IBVP

$$\begin{cases} Lw = 0, \quad \text{in } \Omega, \\ w(x,0) = 0, \quad \text{on } S_x, \\ w(0,t) = a_0(t) - v_0(0,t), \quad w(1,t) = a_1(t) - v_0(1,t), \quad \text{on } S_0, S_1. \end{cases}$$

Further, we decompose the singular component w into left and right components as $w = w_\ell + w_r$, where w_ℓ and w_r respectively, satisfy the following problems:

$$\begin{cases} Lw_\ell = 0, \quad \text{in } \Omega, \\ w_\ell(x,0) = 0, \quad \text{on } S_x, \\ w_\ell(0,t) = a_0(t) - v_0(0,t), \quad w_\ell(1,t) = 0, \quad \text{on } S_0, S_1, \end{cases}$$

and

$$\begin{cases} Lw_r = 0, \quad \text{in } \Omega, \\ w_r(x,0) = 0, \quad \text{on } S_x, \\ w_r(0,t) = 0, \quad w_r(1,t) = a_1(t) - v_0(1,t), \quad \text{on } S_0, S_1. \end{cases}$$

The smooth and singular components v, and w satisfy the following bounds.

Theorem 2.2. [7] *Let $u(x,t)$ be the solution of the IBVP* (1.1)*. And assume that the coefficients of the parabolic PDE, and the initial and boundary conditions given in* (1.1) *are sufficiently smooth, and satisfy the necessary compatibility conditions. Then, for all non-negative integers i, j, such that $0 \le i + 2j \le 4$, we have*

$$\left\| \frac{\partial^{i+j} v}{\partial x^i \partial t^j} \right\| \le C(1 + \varepsilon^{1-i/2}), \quad \forall (x,t) \in \Omega,$$

$$\left| \frac{\partial^{i+j} w_\ell}{\partial x^i \partial t^j} \right| \le C\varepsilon^{-i/2} e^{-x/\sqrt{\varepsilon}}, \text{ and } \left| \frac{\partial^{i+j} w_r}{\partial x^i \partial t^j} \right| \le C\varepsilon^{-i/2} e^{-(1-x)/\sqrt{\varepsilon}}.$$

3. THE DISCRETE PROBLEM

Here, in this section, we present the piecewise-uniform Shishkin mesh for the spatial discretization of the domain. And, we derive the difference scheme which is a combination of the cubic spline scheme and the classical central difference scheme for the spatial derivatives, and the backward difference (implicit-Euler) scheme for the time derivative.

3.1. **Discretization of the Domain.** Consider the domain $\Omega = (0,1) \times (0,T]$. First, we present the Shishkin mesh for the spatial part: Let $\overline{D} = [0,1]$ be the spatial domain, which is divided into three subintervals as $\overline{D} = [0,\sigma) \cup [\sigma, 1-\sigma] \cup (1-\sigma, 1]$ for some σ such that $0 < \sigma \le 1/4$. On the subintervals $[0,\sigma), (1-\sigma, 1]$ a uniform mesh with $N_x/4$ mesh-intervals are placed, where $[\sigma, 1-\sigma]$ has a uniform mesh with $N_x/2$ mesh-intervals. It is obvious that the mesh is uniform when $\sigma = 1/4$, and it is fitted to the problem by choosing σ be the following function of N_x, ε

$$\sigma = \min\left\{\frac{1}{4}, \sigma_0 \sqrt{\varepsilon} \ln N_x\right\},$$

where $\sigma_0 \geq 2$ is a constant.

Here, we use the multi-index notation $N = (N_x, N_t)$, where N_t denotes the number of mesh elements in the t-direction. We shall introduce the meshes in the spatial and temporal variables respectively as $\omega^{N_x} : 0 = x_0 < x_1 < \cdots < x_{N_x} = 1$, and $\omega^{N_t} : 0 = t_0 < t_1 < \cdots < t_{N_t} = T$. Let the meshes in $\overline{\Omega}$ be the tensor product of the one-dimensional meshes ω^{N_x} and ω^{N_t}, and denote it by $\overline{\Omega}_\varepsilon^{N_x,N_t}$. Further, let $h_i = x_{i+1} - x_i$ be the mesh diameter in the spatial dimension, and $\tau_j = t_{j+1} - t_j$, with $\hbar_i = (h_{i-1} + h_i)/2$, $h = 4\sigma/N_x$, $H = 2(1-2\sigma)/N_x$, $k = \max_{j=1,\ldots,N_t} \tau_j$. The domain is discretized in the x-direction with Shishkin mesh and uniform mesh in the t-direction.

3.2. **The Difference Scheme.** We discretize the IBVP (1.1) in the outer region $[\sigma, 1-\sigma]$ by the backward difference in time, and the central difference in space, as

$$\left(\frac{U_i^{j+1} - U_i^j}{k}\right) - \frac{\varepsilon}{\hbar_i}\left[\left(\frac{U_{i+1}^{j+1} - U_i^{j+1}}{h_i}\right) - \left(\frac{U_i^{j+1} - U_{i-1}^{j+1}}{h_{i-1}}\right)\right] + b_i^{j+1}U_i^{j+1} = f_i^{j+1}.$$

After rearranging the terms, we obtain the following, for $i = N_x/4, \ldots, 3N_x/4$, $j = 0, \ldots, N_t - 1$:

$$\left(-\frac{\varepsilon}{h_{i-1}\hbar_i}\right)U_{i-1}^{j+1} + \left(\frac{1}{k} + \frac{2\varepsilon}{h_i h_{i-1}} + b_i^{j+1}\right)U_i^{j+1} + \left(-\frac{\varepsilon}{h_i\hbar_i}\right)U_{i+1}^{j+1} - \frac{U_i^j}{k} = f_i^{j+1}. \tag{3.1}$$

In the boundary layer regions, *i.e.,* in the subintervals $[0, \sigma)$ and $(1-\sigma, 1]$ the time derivative is replaced by the backward difference, and the spatial derivative is replaced by the cubic spline scheme

$$\varepsilon M_i^{j+1} = \left(\frac{U_i^{j+1} - U_i^j}{k}\right) + b_i^{j+1}U_i^{j+1} - f_i^{j+1}. \tag{3.2}$$

Using equation (3.2) to obtain the values of M_{i-1}^{j+1}, and M_{i+1}^{j+1}, and substituting these values in the following cubic spline relation

$$\left(\frac{h_{i-1}}{6}\right)M_{i-1}^{j+1} + \left(\frac{h_{i-1} + h_i}{3}\right)M_i^{j+1} + \left(\frac{h_i}{6}\right)M_{i+1}^{j+1} = \left(\frac{U_{i+1}^{j+1} - U_i^{j+1}}{h_i}\right) - \left(\frac{U_i^{j+1} - U_{i-1}^{j+1}}{h_{i-1}}\right),$$

we obtain the difference scheme, for $i = 1, \ldots, N_x/4 - 1$ and $3N_x/4 + 1, \ldots, N_x - 1$, $j = 0, \ldots, N_t - 1$:

(3.3)
$$\begin{cases} \left(\frac{h_{i-1}}{6k} + \frac{h_{i-1}}{6}b_{i-1}^{j+1} - \frac{\varepsilon}{h_{i-1}}\right)U_{i-1}^{j+1} + \left(\left(\frac{h_{i-1} + h_i}{3k}\right) + \left(\frac{h_{i-1} + h_i}{3}\right)b_i^{j+1} + \right. \\ \left. +\varepsilon\left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right)\right)U_i^{j+1} + \left(\frac{h_i}{6k} + \frac{h_i}{6}b_{i+1}^{j+1} - \frac{\varepsilon}{h_i}\right)U_{i+1}^{j+1} - \frac{h_{i-1}}{6k}U_{i-1}^j - \left(\frac{h_{i-1} + h_i}{3k}\right)U_i^j - \\ -\frac{h_i}{6k}U_{i+1}^j = \frac{h_{i-1}}{6}f_{i-1}^{j+1} + \left(\frac{h_{i-1} + h_i}{3}\right)f_i^{j+1} + \frac{h_i}{6}f_{i+1}^{j+1}. \end{cases}$$

Combining (3.1) and (3.3), we obtain the following difference scheme

(3.4)
$$\begin{cases} L^{N_x,N_t}[U_i^{j+1}] \equiv \left[r^-_{i,j+1}U^{j+1}_{i-1} + r^c_{i,j+1}U^{j+1}_i + r^+_{i,j+1}U^{j+1}_{i+1}\right] + \left[p^-_i U^j_{i-1} + p^c_i U^j_i + p^+_i U^j_{i+1}\right] \\ \qquad\qquad = q^-_i f^{j+1}_{i-1} + q^c_i f^{j+1}_i + q^+_i f^{j+1}_{i+1}, \quad \text{for } i = 1,\dots,N_x-1,\ j = 0,\dots,N_t-1, \\ U^{j+1}_0 = a^{j+1}_0, \quad U^{j+1}_{N_x} = a^{j+1}_1, \quad \text{for } j = 0,\dots,N_t-1, \\ U^0_i = s_i, \quad \text{for } i = 1,\dots,N_x-1, \end{cases}$$

where, for $i = 1,\dots,N_x/4-1$ and $3N_x/4+1,\dots,N_x-1$

(3.5)
$$\begin{cases} r^-_{i,j+1} = \dfrac{h_{i-1}}{6k} + \dfrac{h_{i-1}}{6} b^{j+1}_{i-1} - \dfrac{\varepsilon}{h_{i-1}}, \quad r^c_{i,j+1} = \left(\dfrac{h_{i-1}+h_i}{3k}\right) + \left(\dfrac{h_{i-1}+h_i}{3}\right) b^{j+1}_i + \\ +\varepsilon\left(\dfrac{1}{h_{i-1}} + \dfrac{1}{h_i}\right), \quad r^+_{i,j+1} = \dfrac{h_i}{6k} + \dfrac{h_i}{6} b^{j+1}_{i+1} - \dfrac{\varepsilon}{h_i}, \\ p^-_i = -\dfrac{h_{i-1}}{6k}, \quad p^c_i = -\left(\dfrac{h_{i-1}+h_i}{3k}\right), \quad p^+_i = -\dfrac{h_i}{6k}, \\ q^-_i = \dfrac{h_{i-1}}{6}, \quad q^c_i = \left(\dfrac{h_{i-1}+h_i}{3}\right), \quad q^+_i = \dfrac{h_i}{6}, \end{cases}$$

and for $i = N_x/4, \dots 3N_x/4$

$$\text{(3.6)} \qquad \begin{cases} r^-_{i,j+1} = -\dfrac{\varepsilon}{h_{i-1}\hbar_i}, \quad r^c_{i,j+1} = \dfrac{1}{k} + \dfrac{2\varepsilon}{h_i h_{i-1}} + b^{j+1}_i, \quad r^+_{i,j+1} = -\dfrac{\varepsilon}{h_i \hbar_i}, \\ p^-_i = 0, \quad p^c_i = -\dfrac{1}{k}, \quad p^+_i = 0, \\ q^-_i = 0, \quad q^c_i = 1, \quad q^+_i = 0. \end{cases}$$

4. **ERROR ANALYSIS**

Here, we derive the truncation error for the numerical scheme, and carry out the stability analysis. Finally, we obtain the ε-uniform error estimate.

Lemma 4.1. *Assume that N_x is sufficiently large and $16\sigma_0^2 N_x^{-2}\ln^2 N_x[1+k\beta^*] < 6k$, where $\beta^* = \max_{0\le i\le N_x,\, 0\le j\le N_t} b(x_i,t_j)$. Then, for $i = 1,\dots,N_x$, $j = 0,\dots,N_t-1$, we have*

$$r^-_{i,j+1} < 0,\ r^+_{i,j+1} < 0,\ |r^c_{i,j+1}| - |r^-_{i,j+1}| - |r^+_{i,j+1}| \ > \ 0.$$

Proof. For $i = N_x/4,\dots,3N_x/4$, from (3.1), we can easily see that $r^-_{i,j+1} < 0$ and $r^+_{i,j+1} < 0$. And, also $|r^c_{i,j+1}| - |r^-_{i,j+1}| - |r^+_{i,j+1}| \ge 0$.

Now, for $r^-_{i,j+1} < 0$, $i = 1,\dots,N_x/4-1$ and $i = 3N_x/4+1,\dots,N_x-1$, from (3.3), we have to prove that

$$r^-_{i,j+1} = \left(\frac{h_{i-1}}{6k} + \frac{h_{i-1}}{6} b^{j+1}_{i-1} - \frac{\varepsilon}{h_{i-1}}\right) \ < \ 0,$$

i.e., we require that

$$h_{i-1}^2 < 6\varepsilon\left(\frac{1}{k}+\max(b(x_i,t_j))\right)^{-1}$$

$$\sigma_0 < N_x(4\ln N_x)^{-1}\left[\frac{1}{6k}+\frac{\max(b(x_i,t_j))}{6}\right]^{-1/2},$$

this is true from our assumption on N_x, *i.e.*, from $16\sigma_0^2 N_x^{-2}\ln^2 N_x[1+k\beta^*] < 6k$.

Similarly, it can be shown that $r_{i,j+1}^+ < 0$ for $i = 1,\ldots,N_x/4-1$ and $i = 3N_x/4+1,\ldots,N_x-1$.

For $i = 1,\ldots,N_x/4-1$ and $i = 3N_x/4+1,\ldots,N_x$, we have

$$\begin{aligned}|r_{i,j+1}^c| - |r_{i,j+1}^-| - |r_{i,j+1}^+| &= \left(\frac{h_{i-1}+h_i}{2k}\right)+\left(\frac{h_{i-1}+h_i}{3}\right)b_i^{j+1}+\left(\frac{h_{i-1}b_{i-1}^{j+1}+h_i b_{i+1}^{j+1}}{6}\right)\\ &> \min\{h_{i-1},\, h_i\}\left[\frac{1}{k}+\min\{b(x_i,t_j)\}\right] > 0.\end{aligned}$$

Hence, we have obtained the required result. ■

The following lemma provides the stability result for a general numerical scheme for the IBVP (1.1). The proof of this lemma can be found in the book of Roos et al. [11]. Here, we are only stating the result.

Lemma 4.2. [11] *Consider the IBVP* (1.1). *Assume that we solve this IBVP by applying some numerical scheme, and the difference scheme (excluding the initial and boundary conditions) can be written as*

$$(L_{h,\tau}u_{h,\tau})^{j+1} := A\hat{u}^{j+1} - Du^j = w^j, \quad \text{for } j = 0,\ldots,N-1, \tag{4.1}$$

where $u^j = (u_0^j,\ldots,u_M^j)^T$, $\hat{u}^{j+1} = (u_1^{j+1},\ldots,u_{M-1}^{j+1})^T$, w^j *is a vector independent of the computed solution, and* A *and* D *are matrices. Suppose also that* A *is an* M*-matrix, and* $D \geq 0$.

Let y *and* z *be two mesh functions, such that* $y^j = (y_0^j,\ldots,y_M^j)^T$, *and* $z^j = (z_0^j,\ldots,z_M^j)^T$ *for each* j. *Assume that* $|(L_{h,\tau}y)^{j+1}| \leq (L_{h,\tau}z)^{j+1}$, *for* $j = 0,\ldots,N-1$, *and* $|y| \leq z$ *on the boundary* $S_x \cup S_0 \cup S_1$. *Then,* $|y| \leq z$ *on* $\overline{\Omega}_\varepsilon^{N_x,N_t}$.

Proof. The proof of this lemma can be found in [11]. ■

Corollary 4.3. If a difference scheme satisfies the hypotheses of Lemma 4.2, then it satisfies the discrete maximum principle.

Lemma 4.4. *Consider the IBVP* (1.1), *and the numerical scheme given in* (3.4). *Assume that the statement given in Lemma* 4.2 *holds true for the difference scheme* (3.4). *Then, the difference operator defined in* (3.4) *satisfies the discrete maximum principle.*

Proof. The difference scheme (3.4) can be written in the form of (4.1) with $A = (a_{ij})$ and $D = (d_{ij})$ as

$$\begin{cases} \text{for } i = 1, \ldots, N_x/4 - 1 \text{ and } 3N_x/4 + 1, \ldots, N_x - 1 \\ a_{i,i-1} = \dfrac{h_{i-1}}{6k} + \dfrac{h_{i-1}}{6} b_{i-1}^{j+1} - \dfrac{\varepsilon}{h_{i-1}}, \quad a_{i,i} = \left(\dfrac{h_{i-1} + h_i}{3k}\right) + \left(\dfrac{h_{i-1} + h_i}{3}\right) b_i^{j+1} + \\ +\varepsilon \left(\dfrac{1}{h_{i-1}} + \dfrac{1}{h_i}\right), \quad a_{i,i+1} = \dfrac{h_i}{6k} + \dfrac{h_i}{6} b_{i+1}^{j+1} - \dfrac{\varepsilon}{h_i}, \\ d_{i,i-1} = \dfrac{h_{i-1}}{6k}, \quad d_{i,i} = \left(\dfrac{h_{i-1} + h_i}{3k}\right), \quad d_{i,i+1} = \dfrac{h_i}{6k}, \\ \text{for } i = N_x/4, \ldots 3N_x/4 \\ a_{i,i-1} = -\dfrac{\varepsilon}{h_{i-1}\hbar_i}, \quad a_{i,i} = \dfrac{1}{k} + \dfrac{2\varepsilon}{h_i h_{i-1}} + b_i^{j+1}, \quad a_{i,i+1} = -\dfrac{\varepsilon}{h_i \hbar_i}, \\ d_{i,i-1} = 0, \quad d_{i,i} = \dfrac{1}{k}, \quad d_{i,i+1} = 0. \end{cases}$$

Lemma 4.1 shows that the matrix A is an M-matrix. And the matrix $D \geq 0$. Therefore, Corollary 4.3 implies that the difference operator (3.4) satisfies the discrete maximum principle. ∎

Theorem 4.5. *Let u and U be respectively the continuous and the numerical solutions of the IBVPs* (1.1)*, and* (3.4)*. Then, we have the following ε-uniform error estimate*

$$\sup_{0<\varepsilon\leq 1} ||U - u||_{\overline{\Omega}_\varepsilon^{N_x,N_t}} \;\leq\; C(N_x^{-2} \ln^2 N_x + N_t^{-1}).$$

Proof. As like in the continuous case, we decompose the numerical solution by

$$U = V + W,$$

where V is the solution of the nonhomogeneous problem

$$L^{N_x,N_t} V_i^j = q_i^- f_{i-1}^j + q_i^c f_i^j + q_i^+ f_{i+1}^j, \quad V = v, \text{ on the boundary,}$$

and W satisfies the homogeneous problem

$$\begin{cases} L^{N_x,N_t} W_i^j = 0, \\ W_0^j = a_0^j - (v_0)_0^j, \quad W_{N_x}^j = a_1^j - (v_0)_{N_x}^j \\ W_i^0 = 0. \end{cases}$$

The error can be written as

$$U - u = (V - v) + (W - w).$$

The smooth component $(V-v)$ of the error is estimated by a classical argument. From the differential and difference equations, we can obtain that

$$\begin{aligned} L^{N_x,N_t}(V-v) &= f - L^{N_x,N_t}v \\ &= (L - L^{N_x,N_t})v \\ &= -\varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta^2_{hyb}\right)v + \left(\frac{\partial}{\partial t} - D_t^-\right)v, \end{aligned}$$

where $\delta^2_{hyb}(\cdot)$ is the hybrid scheme for the second-order spatial derivative $\partial^2(\cdot)/\partial x^2$.

Using the bounds on the solution and its derivatives, we have

$$\begin{aligned} |L^{N_x,N_t}(V-v)| &\le \left|\varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta^2_{hyb}\right)v\right| + \left|\left(\frac{\partial}{\partial t} - D_t^-\right)v\right| \\ &\le \begin{cases} C\left[\varepsilon h^2\left\|\frac{\partial^4 v}{\partial x^4}\right\| + k\left\|\frac{\partial^2 v}{\partial t^2}\right\|\right], & \forall\ x_i \ne \sigma,\ x_i \ne 1-\sigma, \\ C\left[\varepsilon(H-h)\left\|\frac{\partial^3 v}{\partial x^3}\right\| + k\left\|\frac{\partial^2 v}{\partial t^2}\right\|\right], & \text{for } x_i = \sigma,\ x_i = 1-\sigma. \end{cases} \end{aligned}$$

Using the estimates for the derivatives of v from Theorem 2.2, we obtain that

$$|L^{N_x,N_t}(V-v)| \le \begin{cases} C\left[N_x^{-2} + N_t^{-1}\right], & \forall\ x_i \ne \sigma,\ x_i \ne 1-\sigma, \\ C\left[\sqrt{\varepsilon}N_x^{-1} + N_t^{-1}\right], & \text{for } x_i = \sigma,\ x_i = 1-\sigma. \end{cases}$$

Define the following function

$$\phi(x_i,t_j) = C\left[\frac{\sigma}{\sqrt{\varepsilon}}N_x^{-2}\theta(x_i) + (1+t_j)N_x^{-2} + t_jN_t^{-1}\right],$$

where

$$\theta(x) = \begin{cases} \frac{x}{\sigma}, & 0 \le x \le \sigma \\ 1, & \sigma \le x \le 1-\sigma, \\ \frac{1-x}{\sigma}, & 1-\sigma \le x \le 1. \end{cases}$$

Then, for all $(x_i,t_j) \in \overline{\Omega}_\varepsilon^{N_x,N_t}$, we have

$$0 \le \phi(x_i,t_j) \le C(N_x^{-2} + N_t^{-1}),$$

and

$$L^{N_x,N_t}\phi(x_i,t_j) \ge \begin{cases} C(N_x^{-2} + N_t^{-1}), & \text{if } x_i \ne \sigma,\ x_i \ne 1-\sigma, \\ C(\sqrt{\varepsilon}N_x^{-1} + N_t^{-1}), & \text{if } x_i = \sigma,\ x_i = 1-\sigma. \end{cases}$$

Introducing the barrier functions

$$\psi^{\pm}(x_i,t_j) = \phi(x_i,t_j) \pm (V-v)(x_i,t_j)$$

it follows that for each point $(x_i,t_j) \in \Omega_\varepsilon^{N_x,N_t}$, we have

$$L^{N_x,N_t}\psi(x_i,t_j) \ge 0,$$

and at each point on the boundary $(x_i, t_j) \in \Gamma_\varepsilon^{N_x,N_t}(= \overline{\Omega}_\varepsilon^{N_x,N_t} \setminus \Omega_\varepsilon^{N_x,N_t})$

$$\psi^{\pm}(x_i, t_j) = \phi(x_i, t_j) \geq 0.$$

Thus from the discrete maximum principle (Lemma 4.4), we obtain

$$|(V - v)(x_i, t_j)| \leq C(N_x^{-2} \ln^2 N_x + N_t^{-1}). \tag{4.2}$$

To estimate the error in the singular component, we rewrite W as

$$W = W_\ell + W_r,$$

where W_ℓ and W_r are respectively defined by

$$L^{N_x,N_t} W_\ell = 0, \quad \text{in } \Omega_\varepsilon^{N_x,N_t}, \ (W_\ell)_0^j = a_0^j - (v_0)_0^j, \quad (W_\ell)_{N_x}^j = 0, \ (W_\ell)_i^0 = 0,$$

and

$$L^{N_x,N_t} W_r = 0, \quad \text{in } \Omega_\varepsilon^{N_x,N_t}, \ (W_r)_0^j = 0, \quad (W_r)_{N_x}^j = a_1^j - (v_0)_{N_x}^j, \ (W_r)_i^0 = 0,$$

The error can be written as

$$W - w = (W_\ell - w_\ell) + (W_r - w_r).$$

From the differential and difference equations, it is easily seen that the truncation error for W is

$$L^{N_x,N_t}(W - w) = L^{N_x,N_t}(W_\ell - w_\ell) + L^{N_x,N_t}(W_r - w_r). \tag{4.3}$$

Now, consider the first part $(W_\ell - w_\ell)$

$$|L^{N_x,N_t}(W_\ell - w_\ell)| \leq \left| \varepsilon \left(\frac{\partial^2}{\partial x^2} - \delta_{hyb}^2 \right) w_\ell \right| + \left| \left(\frac{\partial}{\partial t} - D_t^- \right) w_\ell \right|.$$

First, consider the case, where $x_i \in (0, \sigma)$:

$$\begin{aligned} |L^{N_x,N_t}(W_\ell - w_\ell)| &\leq C \left[\varepsilon h^2 \left\| \frac{\partial^4 w_\ell}{\partial x^4} \right\| + k \left\| \frac{\partial^2 w_\ell}{\partial t^2} \right\| \right] \\ &\leq C \left[\varepsilon h^2 \varepsilon^{-2} + k \right] \\ &\leq C \left[N_x^{-2} \ln^2 N_x + N_t^{-1} \right]. \end{aligned}$$

For $x_i \in (\sigma, 1)$, we use the following trick to obtain the bound

$$\begin{aligned} \left| \varepsilon \left(\frac{\partial^2}{\partial x^2} - \delta_{hyb}^2 \right) w_\ell \right| &\leq \varepsilon C \max_{x_{i-1} \leq x \leq x_{i+1}} \left| \frac{\partial^2 w_\ell}{\partial x^2} \right| \\ &\leq (\varepsilon C \varepsilon^{-1}) \exp(-x_{i-1}/\sqrt{\varepsilon}) \\ &\leq C \exp(-\sigma/\sqrt{\varepsilon}) \\ &\leq C \exp(-\sigma_0 \ln N_x) \\ &\leq C(N_x^{-\sigma_0}). \end{aligned}$$

Similarly, we can show that

$$|L^{N_x,N_t}(W_r - w_r)| \le \begin{cases} C\left[N_x^{-2}\ln^2 N_x + N_t^{-1}\right], & \forall\ x_i \in (1-\sigma, 1), \\ CN_x^{-\sigma_0}, & \forall\ x_i \in (0, 1-\sigma), \end{cases}$$

Combining the estimates for $(W_\ell - w_\ell)$, and $(W_r - w_r)$, we will obtain

$$|L^{N_x,N_t}(W - w)| \ \le\ C\left[N_x^{-2}\ln^2 N_x + N_t^{-1} + N_x^{-\sigma_0}\right].$$

We have assumed that $\sigma_0 \ge 2$, thus, we have

$$|L^{N_x,N_t}(W - w)| \ \le\ C\left[N_x^{-2}\ln^2 N_x + N_t^{-1}\right].$$

Using the discrete maximum principle (Lemma 4.4), we can show that

$$|W - w| \ \le\ C\left[N_x^{-2}\ln^2 N_x + N_t^{-1}\right]. \tag{4.4}$$

Finally, combining the estimates obtained in (4.2) and (4.4), we obtain the required error estimate. ■

5. NUMERICAL RESULTS

In this section, we shall implement the proposed scheme to a test problem studied by various researchers earlier. The numerical results are presented in terms of maximum point-wise errors, and rate of convergence.

Here, in the numerical experiments, we have taken $N_t = O((N_x/\ln N_x)^2)$, mainly to obtain the error estimate of order $O(N_x^{-2}\ln^2 N_x)$; and $\sigma_0 = 3$. The numerical results are obtained at time $T = 1$.

Example 5.1. [7] Consider the following parabolic initial-boundary-value problem:

$$u_t(x,t) - \varepsilon u_{xx}(x,t) + u(x,t) = f(x,t), \quad (x,t) \in (0,1) \times (0,1] \tag{5.1}$$

The right-hand side source term, initial and boundary conditions are calculated from the exact solution

$$u(x,t) = \left(t + \frac{x^2}{2\varepsilon}\right)\operatorname{erfc}\left(\frac{x}{2\sqrt{\varepsilon t}}\right) - \sqrt{\frac{t}{\pi\varepsilon}}\, x e^{-x^2/4\varepsilon t}.$$

The exact solution is used to calculate the maximum nodal error, more precisely, we determine the maximum error as

$$E_\varepsilon^{N_x,N_t} = \max_{\overline{\Omega}_\varepsilon^{N_x,N_t}} ||u(x_i,t_j) - U_i^j||, \quad \text{and} \quad E^{N_x,N_t} = \max_\varepsilon E_\varepsilon^{N_x,N_t},$$

where $u(x,t)$ denotes the exact solution, and U_i^j stands for the numerical solution obtained by using N_x, N_t mesh intervals in the domain $\overline{\Omega}_\varepsilon^{N_x,N_t}$. In addition, the rate of convergence is calculated by

$$p = \log_2\left(\frac{E_\varepsilon^{N_x,N_t}}{E_\varepsilon^{2N_x,\widetilde{N}_t}}\right), \quad \text{and} \quad p_{uni} = \log_2\left(\frac{E^{N_x,N_t}}{E^{2N_x,\widetilde{N}_t}}\right).$$

where $\widetilde{N}_t = \text{round}(2N_x/\ln 2N_x)^2$.

Further, we have calculated the normalized flux

$$F_\varepsilon(x,t) = \sqrt{\varepsilon}\,\frac{\partial u(x,t)}{\partial x},$$

and its numerical approximation

$$F_\varepsilon^{N_x}(x,t) = \sqrt{\varepsilon} D_x^+ U(x,t).$$

The errors in the normalized fluxes have been calculated as

$$Q_\varepsilon^{N_x} = \max_{0\le t\le T} ||F_\varepsilon(0,t) - F_\varepsilon^{N_x}(0,t)||, \quad \text{and} \quad Q^{N_x} = \max_\varepsilon Q_\varepsilon^{N_x},$$

and the rate of convergence is determined from

$$q = \log_2\left(\frac{Q_\varepsilon^{N_x}}{Q_\varepsilon^{2N_x}}\right), \quad \text{and} \quad q_{uni} = \log_2\left(\frac{Q^{N_x}}{Q^{2N_x}}\right).$$

The maximum point-wise errors of the solution, and the normalized flux and the respective rates of convergence are presented in Tables 1, and 2. The maximum point-wise errors are plotted in loglog scale in Figure 1.

The numerical results given in Tables 1-2, and Figure 1 reveal that the proposed method performs well, and produces second-order (up to a logarithmic factor) ε-uniform numerical results.

6. **CONCLUSIONS**

In this article, we proposed a numerical method for singularly perturbed parabolic initial-boundary-value problem with parabolic boundary layers. The time derivative is discretized by the backward difference scheme, and the spatial derivative is discretized by the hybrid scheme, which is a combination of cubic spline (for the boundary layer regions) and classical finite difference scheme (for the outer region). Truncation errors are obtained, and the stability analysis is carried out via. the discrete maximum principle. Further, we derived the ε-uniform error estimates for the numerical solution. To validate the theoretical results, a test problem is solved numerically.

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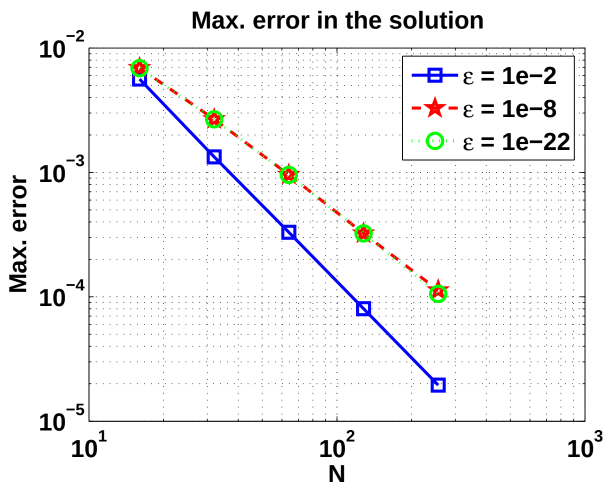
TABLE 1. Maximum point-wise errors $E_\varepsilon^{N_x,N_t}$, rate of convergence p and ε uniform errors E^{N_x,N_t} and rate of convergence p_{uni} for Example 5.1.

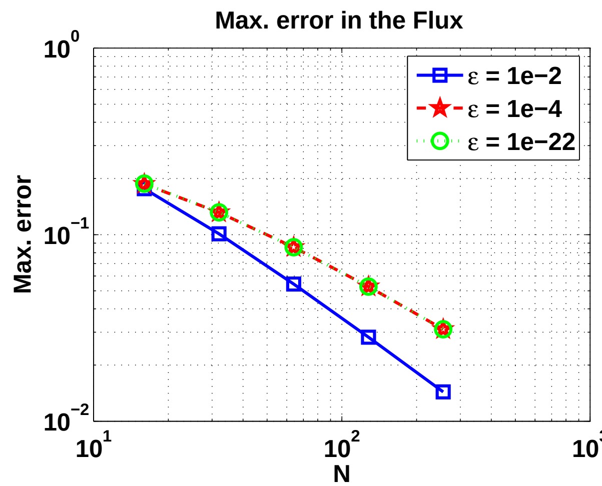
ε	Number of mesh points N_x					
	16	32	64	128	256	512
1e-0	6.7199e-03 1.9674	1.7184e-03 1.9811	4.3525e-04 1.9828	1.1012e-04 1.9815	2.7885e-05 1.9796	7.0707e-06
1e-2	1.8144e-03 1.8131	5.1634e-04 1.8570	1.4254e-04 1.8341	3.9977e-05 1.8273	1.1265e-05 1.8226	3.1848e-06
1e-4	2.0440e-02 1.1274	9.3563e-03 1.5444	3.2077e-03 1.6020	1.0567e-03 1.6493	3.3687e-04 1.6548	1.0698e-04
1e-6	2.0446e-02 1.1276	9.3573e-03 1.5444	3.2080e-03 1.6020	1.0568e-03 1.6492	3.3690e-04 1.6548	1.0699e-04
1e-24	2.0446e-02 1.1273	9.3594e-03 1.5448	3.2080e-03 1.5942	1.0625e-03 1.6571	3.3690e-04 1.6548	1.0700e-04
E^{N_x,N_t} p_{uni}	2.0446e-02 1.1273	9.3594e-03 1.5445	3.2085e-03 1.5944	1.0625e-03 1.6569	3.3693e-04 1.6548	1.0700e-04

TABLE 2. Maximum point-wise errors for the normalized flux $Q_\varepsilon^{N_x}$, rate of convergence q and ε uniform errors Q^{N_x} and rate of convergence q_{uni} for Example 5.1.

ε	Number of mesh points N_x					
	16	32	64	128	256	512
1e-0	2.5972e-01 1.0503	1.2541e-01 1.0335	6.1265e-02 1.0189	3.0233e-02 1.0101	1.5011e-02 1.0052	7.4786e-03
1e-2	1.1384e-01 0.8718	6.2209e-02 0.9300	3.2651e-02 0.9631	1.6749e-02 0.9809	8.4858e-03 0.9902	4.2717e-03
1e-4	2.0401e-01 0.3939	1.5526e-01 0.5158	1.0859e-01 0.6218	7.0565e-02 0.7057	4.3265e-02 0.7675	2.5415e-02
1e-6	2.0326e-01 0.3930	1.5479e-01 0.5152	1.0831e-01 0.6215	7.0401e-02 0.7055	4.3172e-02 0.7674	2.5363e-02
1e-24	2.0325e-01 0.3929	1.5479e-01 0.5152	1.0830e-01 0.6215	7.0400e-02 0.7055	4.3171e-02 0.7674	2.5362e-02
Q^{N_x} q_{uni}	2.0401e-01 0.3939	1.5526e-01 0.5158	1.0859e-01 0.6218	7.0565e-02 0.7057	4.3265e-02 0.7675	2.5415e-02



(a) *Solution.*



(b) *Normalized flux.*

FIGURE 1. Loglog plot of the maximum error for Example 5.1.