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TERMINAL BOUNDARY CONDITION FOR SINGULARLY PERTURBED TWO-POINT BOUNDARY VALUE PROBLEMS

Awoke Andargie[1] and Y.N.Reddy[2]

[1]Department of Mathematics, Bahir Dar University,

P.O.Box 79, Ethiopia, awoke248@yahoo.com

[2]Department of Mathematics, National Institute of Technology,

Warangal 506 004, India

Abstract: A terminal boundary condition for Singularly Perturbed two-Point Boundary value Problems (with left and right layer) is presented. By using a terminal point, the original second order problem is partitioned in to inner and outer region problems. An implicit terminal boundary condition at the terminal point is determined from the outer region problem. The outer region problem with the implicit boundary condition is solved and produces a condition for the inner region problem. The modified inner region problem (using the transformation) is solved as a two-point boundary value problem. We used Chawla's fourth order finite difference method to solve both the inner and outer region problems. The proposed method is iterative on the terminal point. To demonstrate the applicability of the method, we solved seven singular perturbation problems.

Key words: Singular perturbation problems, Finite Differences, Terminal Boundary Condition, Terminal point

1. INTRODUCTION

The numerical treatment of singular perturbation problems is far from the trivial, because of the boundary layer behavior of solutions. Singular perturbation problems appear in varies areas of applied mathematics, science and engineering, like fluid mechanics (boundary layer theory). A wide variety of papers and books are available, describing varies techniques for solving singular perturbation problems, among these one can refer Bellman [1], Bender and Orsazag [2], Hinch [5], Kadalbajo and Reddy [6-7], Kevorkian and Cole [8], O'Malley [10], Nayfah [8-9] and Van Dyke [13]. Several authors published papers on solving SSP by dividing the interval (domain decomposition) of definition (the domain of definition of the problem) into non-overlapping subintervals called outer and inner regions, among these; we mention Vigo-aguiar and Natesan [14], Wang [15] and Chakravarthy and Reddy [3].

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In the present paper, the method of a Terminal Boundary Condition for Singularly Perturbed two point Boundary value Problems with the boundary layer at the left and right end is presented. The method consists of the following steps: (1) The original second order problem is divided in to two problems, an inner region and an outer region problem using a terminal point. (2) An implicit terminal boundary condition at the terminal point is determined from the outer region problem. (3) The outer region problem with the implicit boundary condition is solved. (4) Using the stretching transformation, the modified inner region problem is solved as a two- point boundary value problem. Finally, we combine the solutions of both the inner region and outer region problems to get the approximate solution of the original problem.

The present method is iterative on the terminal point. We repeat the process (numerical scheme) for various choices of the terminal point, until the solution profiles do not differ materially from iteration to iteration.

2. Left Boundary Layer Problems

Consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \ , \quad 0 \le x \le 1 \tag{1}$$

with $y(0) = \alpha$ (2a)

and $y(1) = \beta$; (2b)

where ε is a small positive parameter $(0 < \varepsilon << 1)$ and α, β are known constants. We assume that $a(x), b(x)$ and f(x) are sufficiently continuously differentiable functions in [0,1]. Further more, we assume that $a(x) \ge M > 0$ throughout the interval [0, 1], where M is some positive constant. Under these assumptions, (1) has a unique solution $y(x)$ which in general, displays a boundary layer of width O(ε) at x=0 for small values of ε.

As mentioned the method consists of the following steps:

Step 1: Dividing the original problem in to two regions, an inner region and outer region problem. Let $x_p \, (0 < x_p << 1)$ be the terminal point or width or thickness of the boundary layer (inner region), then the inner and outer region problems are defined on $0 \le x \le x_p$ and $x_p \le x \le 1$ respectively.

***Step 2:* Determining the terminal boundary condition**

By using Taylor's expansion, we have

$$y'(x - x_p) \approx y'(x) - x_p y''(x) \tag{3}$$

Using (3) in to (1), we get

$$\varepsilon y'(x) - \varepsilon y'(x - x_p) + x_p a(x) y'(x) + x_p b(x) y(x) = x_p f(x) \tag{4}$$

Again, we approximate

$$y'(x - x_p) = \frac{y(x) - y(x - x_p)}{x_p} \tag{5}$$

Substituting (5) in (4), we get

$$\varepsilon x_p y'(x)-\varepsilon x_p y(x)+\varepsilon y(x-x_p)+x_p^2 a(x)y'(x)+x_p^2 b(x)y(x)=x_p^2 f(x) \tag{6}$$

Evaluating (6) at $x=x_p$, we get:

$$qy(x_p)+ry'(x_p)=s \tag{7}$$

Where, $q=x_p(x_p b(x_p)-\varepsilon)$ (8a)

$$r=x_p(\varepsilon+x_p a(x_p)) \tag{8b}$$

$$s=x_p^2 f(x_p)-\varepsilon y(0) \tag{8c}$$

Equation (7) which is in explicit form is taken as the terminal boundary condition at $x=x_p$ ***(the terminal point).***

Step 3: Solving the **outer region** problem

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x)=f(x)\ ,\quad x_p\le x\le 1 \tag{9}$$

With $qy(x_p)+ry'(x_p)=s$ (10a)

and $y(1)=\beta$; (10b)

From the solution o f the outer region problem we get the value of $y(x_p)$ .Let us denote it by $y(x_p)=\gamma$

Step 4: Solving the **inner region** problem:

To solve the inner region problem, we take the transformation

$$t=\frac{x}{\varepsilon} \tag{11}$$

By using (11), we transform equations (1) with

$$y(x)=y(t\varepsilon)=Y(t) \tag{12a}$$

$$y'(x)=\frac{y'(t\varepsilon)}{\varepsilon}=\frac{Y'(t)}{\varepsilon} \tag{12b}$$

$$y''(x)=\frac{y''(t\varepsilon)}{\varepsilon^2}=\frac{Y''(t)}{\varepsilon^2} \tag{12c}$$

$$a(x)=a(t\varepsilon)=A(t) \tag{12d}$$

$$b(x)=b(t\varepsilon)=B(t) \tag{12e}$$

$$f(x)=f(t\varepsilon)=f(t) \tag{12f}$$

to obtain the **new inner region** problem of the form:

$$Y''(t)+A(t)Y'(t)+\varepsilon B(t)Y(t)=\varepsilon H(t),\ 0\le t\le t_p \tag{13}$$

with $y(0)=\alpha$ (14a)

and $Y(t_p)=y(x_p)=\gamma$ where $t_p=\dfrac{x_p}{\varepsilon}$ (14b)

Solution of the original problem

To solve the two-point boundary value problems given in equations (9)-(10) [outer region problem] and (13)-(14) [inner region problem), we used Chawla's [4] fourth-order finite difference method. In fact, any standard analytic or numerical method can be used. Finally, we combine the solutions of both the inner region defined on $0 \le x \le x_p$ and outer region defined on $x_p \le x \le 1$ problems to get the approximate solution of the original problem.

We repeat the process (numerical scheme) for various choices of x_p (the terminal point), until the solution profiles do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criterion, namely

$$\left| y^{m+1}(x) - y^{m}(x) \right| \le \sigma \quad 0 \le x \le x_p \tag{15}$$

Where $y^m(x)$ = the solution for the m[th] iterate of x_p

σ = the prescribed tolerance bound.

3. FOURTH-ORDER FINITE DIFFERENCE SCHEME

A finite difference scheme is often a convenient choice for the numerical solution of two point boundary value problems. We used Chawla's [4] fourth- order finite difference method to solve the inner and outer region problems.

Outer region problem:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \ , \quad x_p \le x \le 1 \tag{9}$$

With $\quad qy(x_p) + ry'(x_p) = s$ (10a)

and $\ y(1) = \beta$; (10b)

Now let us rewrite equation (9) in the form:

$$\varepsilon y''(x) = f(x) - a(x)y'(x) - b(x)y(x) = g(x, y, y') \tag{16}$$

With $\quad qy(x_p) + ry'(x_p) = s$ (17a)

and $\ y(1) = \beta$; (17b)

Now we divide the interval $[x_p, 1]$ into N equal parts with constant mesh length h. Let $x_p = x_0, x_1, ..., x_N = 1$ be the mesh points. Then we have $x_i = x_p + ih$; i=0, 1, 2… N.

Let us denote the exact solution $y(x)$ at the grid points x_i by y_i; similarly, $y(x_p) = y_0$ and $y_i' = y'(x_i)$.

For i=1,2, ….., N-1, let

$$\bar{y}_i' = \frac{y_{i+1} - y_{i-1}}{2h} \tag{18a}$$

$$\bar{y}_{i+1}' = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} \tag{18b}$$

$$\bar{y}'_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h} \tag{18c}$$

$$\bar{\bar{y}}'_{i-1} = \bar{y}'_i - \frac{h}{20}(\bar{g}_{i+1} - \bar{g}_{i-1}) \tag{18d}$$

Then for each x_i , i=1, 2…N-1, (15) can be described as:

$$\frac{\varepsilon}{h^2}\delta^2 y_i = \frac{1}{12}(\bar{g}_{i+1} + 10\bar{\bar{g}}_i + \bar{g}_{i-1}) \tag{19}$$

Where $\bar{\bar{g}}_i = g(x_i y_i, \bar{\bar{y}}'_i)$ (20a)

And $\bar{g}_{i\pm1} = g(x_{i\pm1}, y_{i\pm1}, \bar{y}'_{i\pm1})$ (20b)

Using (18) and (20), terms of the right hand side expressions of (19) can be simplified:

$$\frac{1}{12}\bar{g}_{i+1} = \frac{\varepsilon f_{i+1}}{12} - (\frac{a_{i+1}}{8h} + \frac{b_{i+1}}{12})y_{i+1} + \frac{a_{i+1}}{6h}y_i - \frac{a_{i+1}}{24h}y_{i-1} \tag{21a}$$

$$\begin{aligned}\frac{10}{12}\bar{\bar{g}}_i &= (\frac{10a_i}{24h} - \frac{a_i a_{i+1}}{48} - \frac{a_i a_{i-1}}{16} + \frac{ha_i b_{i-1}}{24})y_{i-1} + (\frac{a_i(a_{i+1} + a_{i-1}) - 10b_i}{12})y_i \\ &+ (-\frac{10a_i}{24h} - \frac{a_i a_{i+1}}{16} - \frac{ha_i b_{i+1}}{24} - \frac{a_i a_{i-1}}{48}y_{i+1} + \frac{10}{12}f_i + \frac{ha_i}{24}f_{i+1} - \frac{ha_i}{24}f_{i-1}\end{aligned} \tag{21b}$$

$$\frac{1}{12}\bar{g}_{i-1} = \frac{f_{i-1}}{12} + \frac{a_{i-1}}{24h}y_{i+1} - \frac{a_{i-1}}{6h}y_i + (\frac{a_{i-1}}{8h} - \frac{b_{i-1}}{12})y_{i-1} \tag{21c}$$

Now substituting (21) in (19) we get:

$$\begin{aligned}\frac{\varepsilon}{h^2}(y_{i-1} - 2y_i + y_{i+1}) &= (\frac{-a_{i+1} + 10a_i}{24h} - \frac{a_i a_{i+1}}{48} - \frac{a_i a_{i-1}}{16} + \frac{ha_i b_{i-1}}{24} + \frac{a_{i-1}}{8h} - \frac{b_{i-1}}{12})y_{i-1} \\ &+ (\frac{a_{i+1}}{6h} + \frac{a_i(a_{i+1} + a_{i-1}) - 10b_i}{12} - \frac{a_{i-1}}{6h})y_i + (-\frac{a_{i+1}}{8h} - \frac{b_{i+1}}{12} - \frac{10a_i}{24h} - \frac{a_i a_{i+1}}{16} - \frac{ha_i b_{i+1}}{24} - \frac{a_i a_{i-1}}{48} \\ &+ \frac{a_{i-1}}{24h})y_{i+1} + (\frac{(f_{i+1} + 10f_i + f_{i-1})}{12} + \frac{ha_i(f_{i+1} - f_{i-1})}{24})\end{aligned} \tag{22}$$

From equation (22) we get the recurrence relation of the form:

$$E_i Y_{i-1} - F_i Y_i + G_i Y_{i+1} = H_i \quad ; \text{i=0, 1, 2, 3… N-1} \tag{23}$$

Where

$$E_i = \frac{\varepsilon}{h^2} + \frac{a_{i+1} - 10a_i}{24h} + \frac{a_i a_{i+1}}{48} + \frac{a_i a_{i-1}}{16} - \frac{ha_i b_{i-1}}{24} - \frac{a_{i-1}}{8h} + \frac{b_{i-1}}{12} \tag{24a}$$

$$F_i = \frac{2\varepsilon}{h^2} + \frac{a_{i+1} - a_{i-1}}{6h} + \frac{a_i(a_{i+1} + a_{i-1}) - 10b_i}{12} \tag{24b}$$

$$G_i = \frac{\varepsilon}{h^2} + \frac{a_{i+1}}{8h} + \frac{b_{i+1}}{12} + \frac{10a_i - a_{i-1}}{24h} + \frac{ha_i b_{i+1}}{24} + \frac{a_i a_{i+1}}{16} + \frac{a_i a_{i-1}}{48} \tag{24c}$$

$$H_i = \frac{(f_{i+1} + 10f_i + f_{i-1})}{12} + \frac{ha_i(f_{i+1} - f_{i-1})}{24} \tag{24d}$$

Equation (23) gives a system of N equations with N+1 unknowns y_{-1} to y_{N-1} .
To eliminate the unknowns y_{-1}, we make use of the equation (17a) given as boundary condition in implicit form.
By employing the second order central difference approximation in (17a), we get

$$y_{-1} = \frac{2hq}{r} y_0 + y_1 - \frac{2h_2 s}{r} \tag{25}$$

Where q, r and s are defined in (8).Making use of (25) in the first equation of the recurrence relation (23) at i=0, we get

$$-(F_0 - \frac{2hq}{r} E_0) y_0 + (E_0 + G_0) y_1 = H_0 + \frac{2hs}{r} E_0 \tag{26}$$

Now, equations (23) and (26) give an N by N tri-diagonal system which can be solved by using Thomas Algorithm.

The inner region Problem:

A similar approach to inner region problem

$$Y''(t) + A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon H(t),\ 0 \le t \le t_p \tag{13}$$

with $y(0) = \alpha$ and $Y(t_p) = y(x_p) = \gamma$ (14)

where $t_p = \frac{x_p}{\varepsilon}$ produces the recurrence relation

$$E_i Y_{i-1} - F_i Y_i + G_i Y_{i+1} = H_i \quad ;\ i=1, 2, 3\ldots N-1 \tag{27}$$

Where

$$E_i = \frac{1}{h^2} + \frac{A_{i+1} - 10A_i}{24h} + \frac{A_i A_{i+1}}{48} + \frac{A_i A_{i-1}}{16} - \frac{\varepsilon h A_i B_{i-1}}{24} - \frac{A_{i-1}}{8h} + \frac{\varepsilon B_{i-1}}{12} \tag{28a}$$

$$F_i = \frac{2}{h^2} + \frac{A_{i+1} - A_{i-1}}{6h} + \frac{A_i (A_{i+1} + A_{i-1}) - 10\varepsilon B_i}{12} \tag{28b}$$

$$G_i = \frac{1}{h^2} + \frac{A_{i+1}}{8h} + \frac{\varepsilon B_{i+1}}{12} + \frac{10A_i - A_{i-1}}{24h} + \frac{\varepsilon h A_i B_{i+1}}{24} + \frac{A_i A_{i+1}}{16} + \frac{A_i A_{i-1}}{48} \tag{28c}$$

$$H_i = \frac{\varepsilon (F_{i+1} + 10F_i + F_{i-1})}{12} + \frac{\varepsilon h A_i (F_{i+1} - F_{i-1})}{24} \tag{28d}$$

Where the interval $0 \le t \le t_p$ is subdivided in to N subintervals of equal mesh h=$\frac{t_p - 0}{N}$ with nodes $0 = t_0, t_1, ..., t_N = t_p$.To solve the tri diagonal system (27), we used Thomas Algorithm.

4. NUMERICAL EXAMPLES

Example 4.1: Consider the following singular perturbation problem from fluid dynamics for fluid of small viscosity, Reinhardt [[12], Example 2].

$\varepsilon y''(x) + y'(x) = 1 + 2x$; $0 \le x \le 1$, with y(0)=0 and y(1)=1.

Outer region problem:

$$\varepsilon y''(x) + y'(x) = 1 + 2x\,,\ x_p \le x \le 1,\ \text{with } qy(x_p) + ry'(x_p) = s \text{ and } y(1) = 1$$

Using the transformation t=x/ε

Inner region problem:

$$Y''(t) + Y'(t) = 1 + 2\varepsilon t\,,\ 0 \le t \le t_p,\ \text{with } Y(0) = 0 \text{ and } Y(t_p) = y(x_p) = \gamma$$

The exact solution is given by: $y(x) = x(x+1-2\varepsilon) + (2\varepsilon - 1)\left(\frac{1-\exp(-x/\varepsilon)}{1-\exp(-1/\varepsilon)}\right)$

Numerical maximum errors are presented in table 1 for $\varepsilon=10^{-3}$ and $\varepsilon=10^{-4}$ respectively.

Example 4.2: Consider the following singular perturbation problem from Kevorkian and Cole [[8] Page 33 equations 2.3.26 and 2.3.27 with α=-1/2]

$$\varepsilon y''(x) + (1 - \frac{x}{2})y'(x) - \frac{1}{2}y(x) = 0;\ 0 \le x \le 1,\ \text{with y(0)=0 and y(1)=1.}$$

Outer region problem:

$$\varepsilon y''(x) + (1 - \frac{x}{2})y'(x) - \frac{1}{2}y(x) = 0;\ x_p \le x \le 1,\ \text{with } qy(x_p) + ry'(x_p) = s \text{ and } y(1) = 1$$

Using the transformation t=x/ε

Inner region problem:

$$Y''(t) + (1 - \frac{\varepsilon t}{2})Y'(t) - \frac{\varepsilon}{2}Y(t) = 0\,,\ 0 \le t \le t_p,\ \text{with } Y(0) = 0 \text{ and } Y(t_p) = y(x_p) = \gamma$$

The exact solution is given by: $y(x) = \frac{1}{2-x} - \frac{1}{2}e^{-(x-x^2/4)/\varepsilon}$

Numerical maximum errors are presented in table 2 for $\varepsilon=10^{-3}$ and $\varepsilon=10^{-4}$ respectively.

5. NON-LINEAR PROBLEM

To solve non-linear singular perturbation problems we used the method of quasilinearization.

Example 5.1: Consider the following singular perturbation problem from Bender and Orszag [[2], page 463; equations: 9.7.1]

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0;\ 0 \le x \le 1,\ \text{with y(0)=0 and y(1)=0.}$$

The linear problem concerned to this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1}y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right]$$

Outer region problem: $\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1}y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right];\ x_p \le x \le 1$

with $qy(x_p) + ry'(x_p) = s$ and $y(1) = 0$

Using the transformation t=x/ε

Inner region problem: $Y''(t)+Y'(t)+\dfrac{2\varepsilon}{\varepsilon t+1}Y(t)=\varepsilon\left(\dfrac{2}{\varepsilon t+1}\right)\left[\log\left(\dfrac{2}{\varepsilon t+1}\right)-1\right], 0\le t\le t_p$,

with $Y(0)=0$ and $Y(t_p)=y(x_p)=\gamma$

We have chosen to use Bender and Orszag's uniformly valid approximation [[2], page 463; equation: 9.7.6] for comparison,

$$y(x)=\log_e\left(\frac{2}{x+1}\right)-(\log_e 2)e^{-2x/\varepsilon}$$

Numerical maximum errors are presented in table 3 for $\varepsilon=10^{-3}$ and $\varepsilon=10^{-4}$ respectively.

6. RIGHT BOUNDARY LAYER PROBLEMS

Now let us discuss our present method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form:

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x)=f(x)\ ,\ 0\le x\le 1 \tag{29}$$

With $y(0)=\alpha$ (30a)

and $y(1)=\beta$ (30b)

where ε is a small positive parameter (0<ε<<1) and α, β are known constants. We assume that $a(x), b(x)$ and f(x) are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that $a(x)\le M<0$ throughout the interval [0, 1], where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x=1$.

Consider x_p be the cutting point or thickness of the boundary layer (inner region). Now we divide the original problem into two problems, an inner region problem and an outer region problem. The outer region problem is defined in the interval $0\le x\le(1-x_p)$ and the inner region problem is defined in the interval $(1-x_p)\le x\le 1$.

Terminal boundary condition at the cutting point:

By using Taylor's expansion, we have

$$y'(x+x_p)\approx y'(x)+x_p y''(x) \tag{31}$$

Using (31) in to (29), we get

$$\varepsilon y'(x+x_p)-\varepsilon y'(x)+x_p a(x)y'(x)+x_p b(x)y(x)=x_p f(x) \tag{32}$$

Again, we approximate

$$y'(x+x_p)=\frac{y(x+x_p)-y(x)}{x_p} \tag{33}$$

Substituting (33) in (32), we get

$$\varepsilon y(x+x_p)-\varepsilon y(x)-\varepsilon x_p y'(x)+x_p^2 a(x)y'(x)+x_p^2 b(x)y(x)=x_p^2 f(x) \tag{34}$$

Evaluating (34) at $x = 1 - x_p$, we get:

$$qy(1 - x_p) + ry'(1 - x_p) = s \tag{35}$$

where $q = x_p^2 b(1 - x_p) - \varepsilon$ (36a)

$$r = x_p(x_p a(1 - x_p) - \varepsilon) \tag{36b}$$

$$s = x_p^2 f(1 - x_p) - \varepsilon y(1) \tag{36c}$$

Equation (35) which is in explicit form is taken as the terminal boundary condition at $x = 1 - x_p$ ***(the terminal point).***

The **outer region** problem

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x)\ , \quad 0 \le x \le (1 - x_p) \tag{37}$$

With $y(0) = \alpha$ and $qy(1 - x_p) + ry'(1 - x_p) = s$; (38)

From the solution of the outer region problem we get the value of $y(1 - x_p)$ .Let us denote it by $y(1 - x_p) = \gamma$

The **inner region** problem:

To solve the inner region problem, we take the transformation

$$t = \frac{1 - x}{\varepsilon} \tag{39}$$

By using (39), we transform equation (37) with

$$y(x) = y(1 - t\varepsilon) = Y(t) \tag{40a}$$

$$y'(x) = -\frac{y'(1 - t\varepsilon)}{\varepsilon} = -\frac{Y'(t)}{\varepsilon} \tag{40b}$$

$$y''(x) = \frac{y''(1 - t\varepsilon)}{\varepsilon^2} = \frac{Y''(t)}{\varepsilon^2} \tag{40c}$$

$$a(x) = a(1 - t\varepsilon) = A(t) \tag{40d}$$

$$b(x) = b(1 - t\varepsilon) = B(t) \tag{40e}$$

$$f(x) = f(1 - t\varepsilon) = F(t) \tag{40f}$$

to obtain the **new inner region** problem of the form:

$$Y''(t) + A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon H(t)\, ,\ t_p \le t \le 1 \tag{41}$$

with $Y(t_p) = y(x_p) = \gamma$ and $y(1) = \beta$ where $t_p = \dfrac{1 - x_p}{\varepsilon}$ (42)

Solution of the original problem

To solve the two-point boundary value problems given in equations (37)-(38) [outer region problem] and (41)-(42) [inner region problem), we used Chawla's fourth- order finite difference method. In fact, any standard analytic or numerical method can be used. Finally, we combine the solutions of both the inner region defined on $(1 - x_p) \le x \le 1$ and outer region defined on $0 \le x \le (1 - x_p)$ problems to get the approximate solution of the original problem. We repeat the process (numerical scheme) for various choices of

x_p (the terminal point), until the solution profiles do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criterion, namely

$$\left| y^{m+1}(x) - y^{m}(x) \right| \le \sigma \quad 0 \le x \le x_p \tag{43}$$

Where $y^{m}(x)$ = the solution for the mth iterate of x_p,σ = the prescribed tolerance bound.

7. EXAMPLES WITH RIGHT -END BOUNDARY LAYER

Example 7.1 Consider the following singular perturbation problem

$\varepsilon y''(x) - y'(x) = 0$; $0 \le x \le 1$, with y(0)=1 and y(1)=0.

Outer region problem:

$\varepsilon y''(x) - y'(x) = 0$; $0 \le x \le (1 - x_p)$, with $y(0) = \alpha$ and $qy(1 - x_p) + ry'(1 - x_p) = s$;

For this example the stretching transformation is $t = \dfrac{1 - x}{\varepsilon}$

Inner region problem: $Y''(t) + Y'(t) = 0$; with, $Y(0) = y(1) = 0$ and $Y(t_p) = y(1 - x_p)$

The exact solution is given by: $y(x) = \dfrac{\left(e^{(x-1)/\varepsilon} - 1\right)}{\left(e^{-1/\varepsilon} - 1\right)}$

Numerical maximum errors are presented in table 4 for ε=10^{-3} and ε=10^{-4} respectively.

Example 7.2 Now we consider the following singular perturbation problem

$\varepsilon y''(x) - y'(x) - (1 + \varepsilon)y(x) = 0$; $0 \le x \le 1$ with $y(0) = 1 + \exp(-(1 + \varepsilon)/\varepsilon)$ and $y(1) = 1 + 1/e$.

Outer region problem: $\varepsilon y''(x) - y'(x) - (1 + \varepsilon)y(x) = 0$; $0 \le x \le (1 - x_p)$

with $y(0) = 1 + \exp(-(1 + \varepsilon)/\varepsilon)$ and $qy(1 - x_p) + ry'(1 - x_p) = s$.

For this example the stretching transformation is: $t = \dfrac{1 - x}{\varepsilon}$

and the inner region problem is given by: $Y''(t) + Y'(t) - \varepsilon(1 + \varepsilon)Y(t) = 0$;

with $Y(0) = y(1) = 1 + 1/e$ and $Y(t_p) = y(1 - x_p)$

The exact solution is given by $y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$

Numerical maximum errors are presented in table 5 for ε=10^{-3} and ε=10^{-4} respectively.

8. DISCUSSION AND CONCLUSION

We have described a terminal boundary condition method for the numerical solution of singularly perturbed two-point boundary value problems. As mentioned the method is iterative on the terminal point x_p and the process is to be repeated for different values of x_p (the terminal point which is not unique), until the solution profile stabilizes in both the inner and outer region. We have implemented the present method first on two linear problems with left-end boundary layer, one non-linear problem and two problems with right-end boundary layer, by taking different values of ε.We have tabulated the maximum errors occurred in the approximation by taking the mesh size $h = \varepsilon$ in the outer region and $h = 10^{-2}$ in the inner regions in all examples cindered. It is observed that the present approach approximates the exact solution very well.

Table 1 Maximum Errors for Example 4.1

	$t_p = 5$		$t_p = 10$		$t_p = 20$	
	Inner	Outer	Inner	Outer	Inner	Outer
$\varepsilon = 10^{-3}$	5.62E-03	4.00E-07	9.04E-04	4.00E-07	4.33E-03	4.00E-07
$\varepsilon = 10^{-4}$	6.56E-03	1.00E-07	4.72E-04	1.00E-07	3.42E-03	1.00E-07

Table 2 Maximum Errors for Example 4.2

	$t_p = 5$		$t_p = 10$		$t_p = 20$	
	Inner	Outer	Inner	Outer	Inner	Outer
$\varepsilon = 10^{-3}$	6.45E-03	2.60E-03	2.28E-03	2.70E-03	3.77E-03	2.70E-03
$\varepsilon = 10^{-4}$	3.53E-03	1.43E-04	1.39E-03	1.43E-04	4.16E-03	1.43E-04

Table 3 Maximum Errors for Example 5.1

	$t_p = 5$		$t_p = 10$		$t_p = 20$	
	Inner	Outer	Inner	Outer	Inner	Outer
$\varepsilon = 10^{-3}$	9.03E-04	7.67E-04	9.07E-04	7.67E-04	2.34E-03	7.67E-04
$\varepsilon = 10^{-4}$	1.95E-04	2.63E-05	9.90E-04	2.63E-05	2.69E-03	2.63E-05

Table 4 Maximum Errors for Example 7.1

	$t_p = 5$		$t_p = 10$		$t_p = 20$	
	Inner	Outer	Inner	Outer	Inner	Outer
$\varepsilon = 10^{-3}$	2.92E-02	3.48E-03	3.48E-03	1.30E-02	9.39E-03	6.15E-03
$\varepsilon = 10^{-4}$	1.85E-02	8.04E-04	7.92E-03	8.04E-04	5.92E-03	8.04E-04

Table 5 Maximum Errors for Example 7.2

	$t_p = 5$		$t_p = 10$		$t_p = 20$	
	Inner	Outer	Inner	Outer	Inner	Outer
$\varepsilon = 10^{-3}$	2.47E-02	7.84E-04	1.00E-02	7.84E-04	6.41E-03	7.84E-04
$\varepsilon = 10^{-4}$	1.77E-02	1.83E-05	6.76E-03	1.83E-05	1.80E-03	1.83E-05

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