Neural, Parallel, and Scientific Computations 16 (2008) 449 - 466

A Two-level Implicit Non-uniform Mesh Cubic Spline Method of $O(k^2h_l^{-1}+kh_l+h_l^3)$ for the Parabolic Equation $\varepsilon u_{xx}=\phi(x,t,u,u_x,u_t)$

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Abstract

We present a new two-level implicit high accuracy variable mesh cubic spline method for the non-linear parabolic partial differential equation $\varepsilon u_{xx} = \phi(x,t,u,u_x,u_t)$, $0<x<1$, $t>0$ subject to appropriate initial and Dirichlet boundary conditions prescribed, where $\varepsilon>0$ is a small real constant. The proposed variable mesh approximation produces at each time level a cubic spline function which can be used to obtain the solution at any point in the range 0<x<1. The presented variable mesh strategy is applicable to parabolic equations in polar coordinates. In all the cases, we require only 3-spatial variable grid points. The stability analysis for diffusion equations on a non-uniform mesh shows that the linear cubic spline scheme is unconditionally stable. The advantage of using this new variable mesh method is highlighted computationally especially for better stability with a relatively large time step.

Keywords - variable mesh, cubic spline, nonlinear parabolic equation, implicit method, diffusion equation, Burgers' equation, RMS errors.

1. INTRODUCTION

This paper primarily aims at developing a new highly accurate two-level implicit cubic spline method for the solution of nonlinear parabolic equations on a non-uniform mesh (see Fig.1). It is known that finite difference methods for obtaining approximate solutions of partial differential equations can vary considerably in terms of accuracy and efficiency. In the area of finite differences, the most familiar schemes are the central differences and upwind differences. It has been demonstrated that lower order

Received January 30, 2008

approximations suffer from computational instability. The upwind difference approximations are computationally stable, although only first order accurate (Roache, 1976; Segal, 1982; Brandt and Yavneh, 1991). In many cases the first order upwind and the second order central difference approximations may yield unreliable computational results. In the context of higher-order approximations, compact finite difference schemes have good numerical stability and efficiency, and offer two attractive features: higher-order accuracy and small stencil (Boisvert, 1981; Jain, 1984; Smith, 1996; Morton, 1996). In recent years high-order numerical methods have generated renewed interest and a variety of specialized techniques have been developed. (Mohanty, 2005 & 2006) has presented a class of three point variable mesh methods based on Numerov and arithmetic average discretizations for the solution of nonlinear two point boundary value problems. Recently, (Mohanty and Singh, 2005; Mohanty, 2007) have extended the technique and developed two-level implicit high accuracy variable mesh finite difference methods based on the same Numerov and arithmetic average discretizations for the solution of quasi-linear parabolic equations. These discretizations require only three spatial grid points. Using a cubic spline approach (Jain and Aziz, 1983; Mohanty et al., 2005) have discussed third order three point cubic spline methods for the solution of the nonlinear differential equation $y'' = f(x, y, y')$ on a non-uniform mesh. First, (Papamichael and Whiteman, 1973) have used a cubic spline technique to solve the 1-D heat conduction equation. Further, (Rubin and Graves, 1975) have solved viscous flow problems using a second order cubic spline approximation. (Jain and Lohar, 1979) have also used a second order cubic spline method for the solution of coupled nonlinear parabolic equations. A higher order cubic spline collocation method on a uniform mesh for parabolic equations has been studied by (Archer, 1977). Recently, using three spatial grid points on a uniform mesh, (Mohanty and Jain, 2009) have derived a new two level implicit cubic spline method of second order accuracy in time and fourth order accuracy in space for the solution of one space dimensional nonlinear parabolic equations. But to the author's knowledge no highly accurate variable mesh cubic spline methods for the solution of nonlinear parabolic equations have been discussed in the literature so far. In the present paper, we discuss a new two-level implicit highly accurate cubic spline method for the solution of nonlinear parabolic equations on a non-uniform mesh. However, in the case of a constant mesh, the proposed method reduces to the method derived by (Mohanty and Jain, 2009). Difficulties were experienced in the past for the cubic spline solution of parabolic equations in polar coordinates, especially on a variable mesh. The solution usually deteriorates in the vicinity of its singularity. We modify our method in such a way that the solutions retain their order and accuracy everywhere in the solution region even in the vicinity of its singularity.

This paper is arranged as follows. In the next section, we present the variable mesh cubic spline method for a 1-D nonlinear parabolic equation with significant first derivative terms. In section 3, we discuss the mathematical details of the derivation of the method. In section 4, we study the application of the proposed cubic spline method to a diffusion equation in polar coordinates and perform a stability analysis. Formation of grid points is discussed briefly and comparative numerical results are provided in section 5. Finally, concluding remarks are given in section 6.

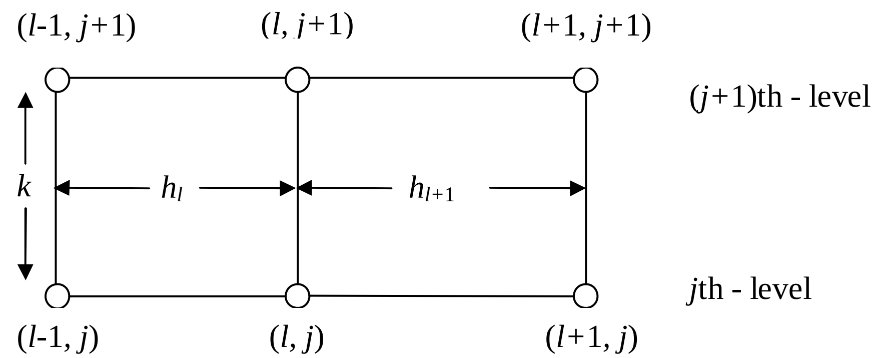

Fig. 1 (Non-uniform Grid Points)

2. VARIABLE MESH CUBIC SPLINE METHOD FOR THE PARABOLIC EQUATION

Consider the nonlinear parabolic differential equation of the form

$$\varepsilon \frac{\partial^2 u}{\partial x^2} = \phi(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}), \quad 0<x<1, t>0 \tag{1}$$

where $\varepsilon>0$ is a small real constant.

The initial condition is prescribed by

$$u(x,0) = f(x), \ 0\le x\le 1 \tag{2}$$

and the Dirichlet boundary conditions are prescribed by

$$u(0,t) = g_0(t), \ u(1,t) = g_1(t), \ t\ge 0 . \tag{3}$$

We assume that the functions $\phi(x,t,u,u_x,u_t)$, $f(x)$, $g_0(t)$ and $g_1(t)$ are sufficiently smooth and their required high-order derivatives exist .

Let $\Omega = \{(x,t) \,|\, 0<x<1,\ t>0\}$ be the solution domain. We discretize the solution domain Ω such that $0 = x_0 < x_1 <\ldots< x_N < x_{N+1} = 1$. Let $h_{l+1} = x_{l+1} - x_l$, $l=0,1,\ldots,N$ be the mesh sizes in the x-direction and $k = t_{j+1} - t_j > 0$, $j=0,1,2,\ldots$ be the time step size in t-direction. The mesh ratio is given by $\sigma_l = (h_{l+1}/h_l) > 0$, $l=1(1)N$. Let $U_l^j = u(x_l,t_j)$ be the exact solution values of $u(x,t)$ and u_l^j be a discrete approximation to $u(x,t)$ at the grid point (x_l,t_j).

We let $S_j(x)$ denote the cubic spline polynomial interpolating the value u_l^j at the jth time level, and is given by

$$S_j(x) = \frac{(x_l - x)^3}{6h_l} M_{l-1}^j + \frac{(x - x_{l-1})^3}{6h_l} M_l^j + \left(u_{l-1}^j - \frac{h_l^2}{6} M_{l-1}^j \right)\left(\frac{x_l - x}{h_l} \right)$$
$$+ \left(u_l^j - \frac{h_l^2}{6} M_l^j \right)\left(\frac{x - x_{l-1}}{h_l} \right), \quad x_{l-1} \le x \le x_l, \quad l = 1(1)N+1, \ j > 0 \tag{4}$$

which satisfies at jth-level the following properties:

(i) $S_j(x)$ coincides with a polynomial of degree three on each $[x_{l-1}, x_l]$, $l = 1,2,\ldots,N+1$, $j>0$,

(ii) $S_j(x) \in C^2[0,1]$, and

(iii) $S_j(x_l) = u_l^j$, $l = 0,1,\ldots,N+1$, $j > 0$,

where

$$m_l^j = U_{xl}^j \text{ and } M_l^j = S_j''(x_l) = U_{xxl}^j = \frac{1}{\varepsilon}\phi(x_l, t_j, U_l^j, m_l^j, U_{tl}^j),\ l = 0,1,\ldots,N+1,\ j > 0.$$

At the grid point (x_l,t_j), we denote

$$P_l = \sigma_l^2 + \sigma_l - 1,\ Q_l = (1+\sigma_l)(1+3\sigma_l+\sigma_l^2),\ R_l = \sigma_l(1+\sigma_l-\sigma_l^2),\ S_l = \sigma_l(1+\sigma_l), \tag{5}$$

and

$$\phi_l^j = \phi(x_l, t_j, U_l^j, U_{xl}^j, U_{tl}^j) = \varepsilon U_{xxl}^j \quad , \tag{6a}$$

$$\phi_{l+1}^j = \phi(x_{l+1}, t_j, U_{l+1}^j, U_{xl+1}^j, U_{tl+1}^j) = \varepsilon U_{xxl+1}^j , \tag{6b}$$

$$\phi_{l-1}^j = \phi(x_{l-1}, t_j, U_{l-1}^j, U_{xl-1}^j, U_{tl-1}^j) = \varepsilon U_{xxl-1}^j . \tag{6c}$$

Now we need the following approximations

$$\bar{t}_j = t_j + \theta k , \tag{7}$$

$$\overline{U}_l^j = \theta U_l^{j+1} + (1-\theta) U_l^j , \tag{8a}$$

$$\overline{U}_{l+1}^j = \theta U_{l+1}^{j+1} + (1-\theta) U_{l+1}^j , \tag{8b}$$

$$\overline{U}_{l-1}^j = \theta U_{l-1}^{j+1} + (1-\theta) U_{l-1}^j . \tag{8c}$$

$$\overline{U}_{tl}^j = (U_l^{j+1} - U_l^j)/k , \tag{9a}$$

$$\overline{U}_{tl+1}^j = (U_{l+1}^{j+1} - U_{l+1}^j)/k , \tag{9b}$$

$$\overline{U}_{tl-1}^j = (U_{l-1}^{j+1} - U_{l-1}^j)/k , \tag{9c}$$

$$\overline{m}_l^j = \overline{U}_{xl}^j = (\overline{U}_{l+1}^j - (1-\sigma_l^2)\overline{U}_l^j - \sigma_l^2 \overline{U}_{l-1}^j)/(h_l S_l) , \tag{10a}$$

$$\overline{m}_{l+1}^j = \overline{U}_{xl+1}^j = ((1+2\sigma_l)\overline{U}_{l+1}^j - (1+\sigma_l)^2 \overline{U}_l^j + \sigma_l^2 \overline{U}_{l-1}^j)/(h_l S_l) , \tag{10b}$$

$$\overline{m}_{l-1}^j = \overline{U}_{xl-1}^j = (-\overline{U}_{l+1}^j + (1+\sigma_l)^2 \overline{U}_l^j - \sigma_l (2+\sigma_l)\overline{U}_{l-1}^j)/(h_l S_l) , \tag{10c}$$

$$\overline{\phi}_l^j = \phi(x_l, \bar{t}_j, \overline{U}_l^j, \overline{m}_l^j, \overline{U}_{tl}^j) , \tag{11a}$$

$$\overline{\phi}_{l+1}^j = \phi(x_{l+1}, \bar{t}_j, \overline{U}_{l+1}^j, \overline{m}_{l+1}^j, \overline{U}_{tl+1}^j) , \tag{11b}$$

$$\overline{\phi}_{l-1}^j = \phi(x_{l-1}, \bar{t}_j, \overline{U}_{l-1}^j, \overline{m}_{l-1}^j, \overline{U}_{tl-1}^j) , \tag{11c}$$

$$\hat{m}_l^j = \hat{U}_{xl}^j = \overline{m}_l^j + a_l h_l (\overline{M}_{l+1}^j - \overline{M}_{l-1}^j) = \overline{U}_{xl}^j + \frac{a_l h_l}{\varepsilon}(\overline{\phi}_{l+1}^j - \overline{\phi}_{l-1}^j) , \tag{12a}$$

$$\hat{m}_{l+1}^j = \hat{U}_{xl+1}^j = \frac{\overline{U}_{l+1}^j - \overline{U}_l^j}{h_{l+1}} + \frac{h_{l+1}}{6}\left[\overline{M}_l^j + 2\overline{M}_{l+1}^j\right] = \frac{\overline{U}_{l+1}^j - \overline{U}_l^j}{h_{l+1}} + \frac{h_{l+1}}{6\varepsilon}\left[\overline{\phi}_l^j + 2\overline{\phi}_{l+1}^j\right], \tag{12b}$$

$$\hat{m}_{l-1}^j = \hat{U}_{xl-1}^j = \frac{\overline{U}_l^j - \overline{U}_{l-1}^j}{h_l} - \frac{h_l}{6}\left[\overline{M}_l^j + 2\overline{M}_{l-1}^j\right] = \frac{\overline{U}_l^j - \overline{U}_{l-1}^j}{h_l} - \frac{h_l}{6\varepsilon}\left[\overline{\phi}_l^j + 2\overline{\phi}_{l-1}^j\right], \tag{12c}$$

where

$$\overline{M}_l^j = \overline{\phi}_l^j / \varepsilon , \quad \overline{M}_{l+1}^j = \overline{\phi}_{l+1}^j / \varepsilon , \quad \overline{M}_{l-1}^j = \overline{\phi}_{l-1}^j / \varepsilon \quad \text{etc.}$$

Further, we define

$$\hat{\phi}_l^j = \phi(x_l, \bar{t}_j, \overline{U}_l^j, \hat{m}_l^j, \overline{U}_{t\,l}^{\,j}), \tag{13a}$$

$$\hat{\phi}_{l+1}^j = \phi(x_{l+1}, \bar{t}_j, \overline{U}_{l+1}^j, \hat{m}_{l+1}^j, \overline{U}_{t\,l+1}^{\,j}), \tag{13b}$$

$$\hat{\phi}_{l-1}^j = \phi(x_{l-1}, \bar{t}_j, \overline{U}_{l-1}^j, \hat{m}_{l-1}^j, \overline{U}_{t\,l-1}^{\,j}). \tag{13c}$$

Then at each grid point (x_l, t_j), a variable mesh cubic spline method with accuracy of $O(k^2 h_l^{-1} + k h_l + h_l^3)$ for the differential equation (1) may be written as

$$\varepsilon\left(\overline{U}_{l+1}^j - (1+\sigma_l)\overline{U}_l^j + \sigma_l \overline{U}_{l-1}^j\right) = \frac{h_l^2}{12}\left[P_l \hat{\phi}_{l+1}^j + Q_l \hat{\phi}_l^j + R_l \hat{\phi}_{l-1}^j\right] + \hat{T}_l^j, \; l = 1,2,\dots,N, \; j > 0 \tag{14}$$

where

$$\theta = \frac{1}{2}, \;\; a_l = \frac{-\sigma_l}{6(1+\sigma_l)} \text{ and } \hat{T}_l^j = O(k^2 h_l + k h_l^3 + h_l^5), \text{ provided } \sigma_l \neq 1.$$

Note that, the coefficients P_l, Q_l, R_l in (14) are positive if, $\frac{\sqrt{5}-1}{2} < \sigma_l < \frac{\sqrt{5}+1}{2}$ (Mohanty, 2005). For $\sigma_l = 1$ (constant mesh case), that is, for $h_{l+1} = h_l = h$, the method (14) reduces to the cubic spline method with accuracy of $O(k^2 + kh^2 + h^4)$ for the solution of the differential equation (1) (Mohanty and Jain, 2009).

Further note that the initial and boundary conditions are given by (2) and (3), respectively. Incorporating the initial and boundary conditions, we can easily express the cubic spline method (14) in tri-diagonal matrix form . If the differential equation is linear, we can solve the linear system using a tri-diagonal solver; in the non-linear case we can use the generalized Newton-Raphson method to solve the nonlinear system (Hageman and Young, 2004; Kelly, 1987; Evans, 1999).

3. MATHEMATICAL DETAILS AND DERIVATION PROCEDURE

For the derivation of the cubic spline method (14), we simply follow the approaches given by (Jain and Aziz, 1983; Mohanty and Jain, 2009).

At the grid point (x_l, t_j), we denote

$$U_{pq} = \frac{\partial^{p+q} U_l^j}{\partial x^p \partial t^q}, \quad \alpha_l^j = \frac{\partial \phi}{\partial t_j}, \quad \beta_l^j = \frac{\partial \phi}{\partial U_l^j}, \quad \gamma_l^j = \frac{\partial \phi}{\partial U_{x\,l}^{\;j}}, \quad \Psi_l^j = \frac{\partial \phi}{\partial U_{t\,l}^{\;j}}. \tag{15}$$

Differentiating the differential equation (1) partially with respect to 't' at the grid point (x_l, t_j), we obtain the relation

$$-\Psi_l^j U_{02} = \alpha_l^j + \beta_l^j U_{01} + \gamma_l^j U_{11} - \varepsilon U_{21}. \tag{16}$$

By the help of a Taylor expansion, we first obtain

$$\varepsilon\left[U_{l+1}^j - (1+\sigma_l)U_l^j + \sigma_l U_{l-1}^j\right] = \frac{h_l^{\,2}}{12}\left[P_l \phi_{l+1}^j + Q_l \phi_l^j + R_l \phi_{l-1}^j\right] + T_l^j \tag{17}$$

where

$T_i^j = O\left(h_l^{\,5}\right)$, provided $\sigma_l \neq 1$.

Using equations (7)- (11c) and (15), we obtain

$$\overline{U}_l^j = U_l^j + \theta k U_{01} + O\left(k^2\right), \tag{18a}$$

$$\overline{U}_{l+1}^j = U_{l+1}^j + \theta\left[k U_{01} + k\sigma_l h_l U_{11}\right] + O\left(k^2\right), \tag{18b}$$

$$\overline{U}_{l-1}^j = U_{l-1}^j + \theta\left[k U_{01} - k h_l U_{11}\right] + O\left(k^2\right), \tag{18c}$$

$$\overline{U}_{t\,l}^{\;j} = U_{t\,l}^{\;j} + \frac{k}{2} U_{02} + O\left(k^2\right), \tag{19a}$$

$$\overline{U}_{t\,l+1}^{\;j} = U_{t\,l+1}^{\;j} + \frac{k}{2} U_{02} + \frac{k\sigma_l h_l}{2} U_{12} + O\left(k^2\right), \tag{19b}$$

$$\overline{U}_{t\,l-1}^{\;j} = U_{t\,l-1}^{\;j} + \frac{k}{2} U_{02} - \frac{k h_l}{2} U_{12} + O\left(k^2\right), \tag{19c}$$

$$\overline{m}_l^j = m_l^j + \frac{\sigma_l h_l^{\,2}}{6} U_{30} + \theta k U_{11} + O\left(k^2 h_l^{-1} + k h_l + h_l^{\,3}\right), \tag{20a}$$

$$\overline{m}_{l+1}^j = m_{l+1}^j - \frac{\sigma_l (1+\sigma_l) h_l^{\,2}}{6} U_{30} + \theta k U_{11} + O\left(k^2 h_l^{-1} + k h_l + h_l^{\,3}\right), \tag{20b}$$

$$\overline{m}_{l-1}^j = m_{l-1}^j - \frac{(1+\sigma_l) h_l^{\,2}}{6} U_{30} + \theta k U_{11} + O\left(k^2 h_l^{-1} + k h_l + h_l^{\,3}\right), \tag{20c}$$

$$\overline{\phi}_l^j = \phi_l^j + \theta k\left[\alpha_l^j + U_{01}\beta_l^j + U_{11}\gamma_l^j\right] + \frac{k}{2} U_{02} \Psi_l^j$$

$$+ \frac{\sigma_l h_l^{\,2}}{6} U_{30} \gamma_l^j + O(k h_l + h_l^{\,3}), \tag{21a}$$

$$\bar{\phi}_{l+1}^{j} = \phi_{l+1}^{j} + \theta k\left[\alpha_l^j + U_{01}\beta_l^j + U_{11}\gamma_l^j\right] + \frac{k}{2}U_{02}\Psi_l^j$$

$$-\frac{\sigma_l(1+\sigma_l)h_l^{\,2}}{6}U_{30}\gamma_l^j + O(k^2 h_l^{-1} + kh_l + h_l^{\,3})\,, \tag{21b}$$

$$\bar{\phi}_{l-1}^{j} = \phi_{l-1}^{j} + \theta k\left[\alpha_l^j + U_{01}\beta_l^j + U_{11}\gamma_l^j\right] + \frac{k}{2}U_{02}\Psi_l^j$$

$$-\frac{(1+\sigma_l)h_l^{\,2}}{6}U_{30}\gamma_l^j + O(k^2 h_l^{-1} + kh_l + h_l^{\,3})\,. \tag{21c}$$

Using equations (12a), (20a) and (21b) - (21c), we obtain

$$\hat{m}_l^j = m_l^j + \theta k U_{11} + \frac{h_l^{\,2}}{6}[\sigma_l + 6a_l(1+\sigma_l)]U_{30} + O(k^2 h_l^{-1} + kh_l + h_l^{\,3}), \sigma_l \neq 1\,. \tag{22}$$

Equating the coefficient of $h_l^{\,2}$ to zero in equation (22), we obtain $a_l = \dfrac{-\sigma_l}{6(1+\sigma_l)}$ and the equation (22) reduces to

$$\hat{m}_l^j = m_l^j + \theta k U_{11} + O(k^2 h_l^{-1} + kh_l + h_l^{\,3})\,. \tag{23a}$$

Similarly, simplifying (12b) and (12c) by the help of the approximations (18a)-(18c) and (21a)-(21c), we obtain

$$\hat{m}_{l+1}^j = m_{l+1}^j + \theta k U_{11} + O(k^2 h_l^{-1} + kh_l + h_l^{\,3})\,, \tag{23b}$$

$$\hat{m}_{l-1}^j = m_{l-1}^j + \theta k U_{11} - O(k^2 h_l^{-1} + kh_l + h_l^{\,3})\,. \tag{23c}$$

Finally, from equations (7), (18a)-(19c), (23a)-(23c) and (13a)-(13c), we obtain

$$\hat{\phi}_l^j = \phi_l^j + \theta k\left[\alpha_l^j + U_{01}\beta_l^j + U_{11}\gamma_l^j\right] + \frac{k}{2}U_{02}\Psi_l^j + O(k^2 h_l^{-1} + kh_l + h_l^{\,3})\,, \tag{24a}$$

$$\hat{\phi}_{l+1}^j = \phi_{l+1}^j + \theta k\left[\alpha_l^j + U_{01}\beta_l^j + U_{11}\gamma_l^j\right] + \frac{k}{2}U_{02}\Psi_l^j + O(k^2 h_l^{-1} + kh_l + h_l^{\,3})\,, \tag{24b}$$

$$\hat{\phi}_{l-1}^j = \phi_{l-1}^j + \theta k\left[\alpha_l^j + U_{01}\beta_l^j + U_{11}\gamma_l^j\right] + \frac{k}{2}U_{02}\Psi_l^j - O(k^2 h_l^{-1} + kh_l + h_l^{\,3})\,. \tag{24c}$$

Using equations (18a)-(18c), (24a)-(24c) and (14), we obtain

$$\varepsilon\left[U_{l+1}^{j}-(1+\sigma_l)U_l^{j}+\sigma_l U_{l-1}^{j}+\theta S_l \frac{kh_l^{2}}{2}U_{21}\right]$$
$$=\frac{h_l^{2}}{12}\left[P_l\phi_{l+1}^{j}+Q_l\phi_l^{j}+R_l\phi_{l-1}^{j}+(P_l+Q_l+R_l)\left\{\theta k\left(\alpha_l^{j}+U_{01}\beta_l^{j}+U_{11}\gamma_l^{j}\right)+\frac{k}{2}U_{02}\Psi_l^{j}\right\}\right]+\hat{T}_l^{j} \tag{25}$$

where

$$P_l+Q_l+R_l=6S_l.$$

Now, with the help of equations (16), (17) and (25) we get the following expression for the local truncation error:

$$\hat{T}_l^{j}=S_l\frac{kh_l^{2}}{2}\left(\theta-\frac{1}{2}\right)U_{02}\psi_l^{j}+O(k^2h_l+kh_l^{3}+h_l^{5}),\ \ \sigma_l\neq 1. \tag{26}$$

In order for the proposed cubic spline method (14) to be of $O(k^2h_l^{-1}+kh_l+h_l^{3})$, the coefficient of kh_l^{2} in (26) must be zero.

Thus we obtain the value of the parameter $\theta=\frac{1}{2}$ and the local truncation error $\hat{T}_l^{j}$ reduces to

$$\hat{T}_l^{j}=O(k^2h_l+kh_l^{3}+h_l^{5}),\quad \sigma_l\neq 1. \tag{27}$$

4. APPLICATION TO PARABOLIC EQUATIONS WITH SINGULAR COEFFICIENTS

Consider the linear parabolic equation

$$\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\alpha}{x}\frac{\partial u}{\partial x}\right)=\frac{\partial u}{\partial t}+f(x,t),\ \ 0<x<1,\ t>0 \tag{28}$$

subject to appropriate initial and Dirichlet boundary conditions given by (2) and (3), respectively, where $0<\nu<<1$ is called diffusivity, α is a real constant and $\alpha\in(0,1)$ or it may take values 0,1 or 2. For $\alpha=0$, the equation above represents a time dependent diffusion equation. If we replace x by r in equation (28) then we obtain an unsteady diffusion equation either in cylindrical coordinates, if $\alpha=1$, or in spherical coordinates, if $\alpha=2$.

Applying the difference method (14) to the differential equation (28), and neglecting the local truncation error, we obtain the scheme

$$\nu\left[\bar{u}_{l+1}^{j}-(1+\sigma_l)\bar{u}_{l}^{j}+\sigma_l\bar{u}_{l-1}^{j}\right]=\frac{h_l^2}{12}\Big[P_l\bar{u}_{t\,l+1}^{j}+Q_l\bar{u}_{t\,l}^{j}+R_l\bar{u}_{t\,l-1}^{j}$$
$$-\frac{\alpha\nu P_l}{x_{l+1}}\left(\frac{\bar{u}_{l+1}^{j}-\bar{u}_{l}^{j}}{\sigma_l h_l}+\frac{\sigma_l h_l}{6\nu}\left(\bar{u}_{t\,l}^{j}+2\bar{u}_{t\,l+1}^{j}-\frac{\alpha\nu}{x_l}\bar{u}_{x\,l}^{j}-\frac{2\alpha\nu}{x_{l+1}}\bar{u}_{x\,l+1}^{j}+\bar{f}_{l}^{j}+2\bar{f}_{l+1}^{j}\right)\right)$$
$$-\frac{\alpha\nu Q_l}{x_l}\left(\bar{u}_{x\,l}^{j}-\frac{\sigma_l h_l}{6\nu(1+\sigma_l)}\left(\bar{u}_{t\,l+1}^{j}-\bar{u}_{t\,l-1}^{j}-\frac{\alpha\nu}{x_{l+1}}\bar{u}_{x\,l+1}^{j}+\frac{\alpha\nu}{x_{l-1}}\bar{u}_{x\,l-1}^{j}+\bar{f}_{l+1}^{j}-\bar{f}_{l-1}^{j}\right)\right)$$
$$-\frac{\alpha\nu R_l}{x_{l-1}}\left(\frac{\bar{u}_{l}^{j}-\bar{u}_{l-1}^{j}}{h_l}-\frac{h_l}{6\nu}\left(\bar{u}_{t\,l}^{j}+2\bar{u}_{t\,l-1}^{j}-\frac{\alpha\nu}{x_l}\bar{u}_{x\,l}^{j}-\frac{2\alpha\nu}{x_{l-1}}\bar{u}_{x\,l-1}^{j}+\bar{f}_{l}^{j}+2\bar{f}_{l-1}^{j}\right)\right)$$
$$+P_l\bar{f}_{l+1}^{j}+Q_l\bar{f}_{l}^{j}+R_l\bar{f}_{l-1}^{j}\Big],\quad l=1,2,\ldots,N,\ j=0,1,2,\ldots \tag{29}$$

where

$$\bar{f}_{l}^{j}=f(x_l,\bar{t}_j),\ \bar{f}_{l\pm1}^{j}=f(x_{l\pm1},\bar{t}_j)\ \text{etc.}$$

Note that the linear cubic spline variable mesh scheme (29) is of $O(k^2h_l^{-1}+kh_l+h_l^3)$ accuracy for the solution of the parabolic differential equation (28), however, the scheme fails to compute when the solution is to be determined at $l=1$ due to zero division. We overcome this difficulty by using the following approximations.

Let

$$\frac{1}{x_l}\equiv X_0, \tag{30a}$$

$$\frac{1}{x_{l+1}}=\frac{1}{x_l}-\frac{\sigma_l h_l}{x_l^2}+\frac{\sigma_l^2 h_l^2}{x_l^3}+O(h_l^3)\equiv X_1, \tag{30b}$$

$$\frac{1}{x_{l-1}}=\frac{1}{x_l}+\frac{h_l}{x_l^2}+\frac{h_l^2}{x_l^3}+O(h_l^3)\equiv X_2, \tag{30c}$$

$$\bar{f}_{l}^{j}=f(x_l,\bar{t}_j)\equiv F_0, \tag{31a}$$

$$\bar{f}_{l+1}^{j}=\bar{f}_{l}^{j}+\sigma_l h_l\bar{f}_{x\,l}^{j}+\frac{\sigma_l^2h_l^2}{2}\bar{f}_{xx\,l}^{j}+O(h_l^3)\equiv F_1, \tag{31b}$$

$$\bar{f}_{l-1}^{j}=\bar{f}_{l}^{j}-h_l\bar{f}_{x\,l}^{j}+\frac{h_l^2}{2}\bar{f}_{xx\,l}^{j}+O(h_l^3)\equiv F_2, \tag{31c}$$

where

$$\bar{f}_{x\,l}^{\,j}=\frac{\partial f\,(x_l,\bar{t}_j)}{\partial x},\quad \bar{f}_{xx\,l}^{\,j}=\frac{\partial^2 f\,(x_l,\bar{t}_j)}{\partial x^2}\ \text{etc.}$$

Now substituting the approximations (30a)-(31c) in (29) and neglecting high order terms, we obtain

$$\begin{aligned}
\nu\left[\bar{u}_{l+1}^{\,j}-(1+\sigma_l)\bar{u}_l^{\,j}+\sigma_l\bar{u}_{l-1}^{\,j}\right]&=\frac{h_l^{\,2}}{12}\Big[P_l\bar{u}_{t\,l+1}^{\,j}+Q_l\bar{u}_{t\,l}^{\,j}+R_l\bar{u}_{t\,l-1}^{\,j}\\
&-\alpha\nu P_l X_1\left(\frac{\bar{u}_{l+1}^{\,j}-\bar{u}_l^{\,j}}{\sigma_l h_l}+\frac{\sigma_l h_l}{6\nu}\left(\bar{u}_{t\,l}^{\,j}+2\bar{u}_{t\,l+1}^{\,j}-\alpha\nu X_0\bar{u}_{x\,l}^{\,j}-2\alpha\nu X_1\bar{u}_{x\,l+1}^{\,j}+F_0+2F_1\right)\right)\\
&-\alpha\nu Q_l X_0\left(\bar{u}_{x\,l}^{\,j}-\frac{\sigma_l h_l}{6\nu(1+\sigma_l)}\left(\bar{u}_{t\,l+1}^{\,j}-\bar{u}_{t\,l-1}^{\,j}-\alpha\nu X_1\bar{u}_{x\,l+1}^{\,j}+\alpha\nu X_2\bar{u}_{x\,l-1}^{\,j}+F_1-F_2\right)\right)\\
&-\alpha\nu R_l X_2\left(\frac{\bar{u}_l^{\,j}-\bar{u}_{l-1}^{\,j}}{h_l}-\frac{h_l}{6\nu}\left(\bar{u}_{t\,l}^{\,j}+2\bar{u}_{t\,l-1}^{\,j}-\alpha\nu X_0\bar{u}_{x\,l}^{\,j}-2\alpha\nu X_2\bar{u}_{x\,l-1}^{\,j}+F_0+2F_2\right)\right)\\
&+(P_l\times F_1)+(Q_l\times F_0)+(R_l\times F_2)\Big]\ ,\quad l=1,2,\ldots,N,\ j=0,1,2,\ldots
\end{aligned}\tag{32}$$

Note that the cubic spline method (32) is of $O(k^2h_l^{-1}+kh_l+h_l^{\,3})$ accuracy and free from the term $1/(x_{l\pm1})$, hence very easily solved for l=1,2,…,N in the solution region 0<x<1, t>0. This technique shows that the proposed cubic spline method is applicable to singular problems and we do not require the presence of any fictitious points outside the solution region to handle the numerical scheme near the boundary.

For stability we consider the variable mesh scheme for the diffusion equation

$$\nu\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}\ ,\quad 0<x<1,\ t>0. \tag{33}$$

Substituting D = 0 and f = 0 in the numerical method (32), we obtain the following linear cubic spline scheme for the solution of the above differential equation

$$a_{-1}u_{l-1}^{j+1}+a_0u_l^{j+1}+a_1u_{l+1}^{j+1}=b_{-1}u_{l-1}^{j}+b_0u_l^{j}+b_1u_{l+1}^{j} \tag{34}$$

where

$$\lambda_l=\left(k/h_l^{\,2}\right),$$

$$a_{-1} = -6\nu\lambda_l\sigma_l + R_l,\;\; a_0 = 6\nu\lambda_l(1+\sigma_l) + Q_l,\;\; a_1 = -6\nu\lambda_l + P_l, \tag{35a}$$

$$b_{-1} = 6\nu\lambda_l\sigma_l + R_l,\;\; b_0 = -6\nu\lambda_l(1+\sigma_l) + Q_l,\;\; b_1 = 6\nu\lambda_l + P_l. \tag{35b}$$

To study the stability of the linear cubic spline scheme (34), we use the Von Neumann linear stability analysis. Assume that there exists an error at each grid point (x_l, t_j) of the form $\varepsilon_l^j = \xi^j e^{i\beta l}$, where $i=\sqrt{-1}$, ξ is the amplitude, which may be complex and the phase angle β is real. The amplification factor is found to be

$$\xi = \frac{1+6\nu\lambda_l L(\beta)}{1-6\nu\lambda_l L(\beta)} \tag{36}$$

where

$$L(\beta) = \frac{(1+\sigma_l)(\cos\beta - 1) + i(1-\sigma_l)\sin\beta}{(1+\sigma_l)[(1+3\sigma_l+\sigma_l^2) - (1-3\sigma_l+\sigma_l^2)\cos\beta] - i(1-\sigma_l)(1+\sigma_l+\sigma_l^2)\sin\beta} \tag{37}$$

We may re-write equation (36) as

$$\xi = \frac{(C_l + 6\nu\lambda_l A_l) + i(D_l + 6\nu\lambda_l B_l)}{(C_l - 6\nu\lambda_l A_l) + i(D_l - 6\nu\lambda_l B_l)} \tag{38}$$

where

$$\begin{aligned}
A_l &= (1+\sigma_l)(\cos\beta - 1), \\
B_l &= (1-\sigma_l)\sin\beta, \\
C_l &= (1+\sigma_l)[(1+3\sigma_l+\sigma_l^2) - (1-3\sigma_l+\sigma_l^2)\cos\beta], \\
D_l &= (\sigma_l - 1)(1+\sigma_l+\sigma_l^2)\sin\beta.
\end{aligned}$$

For stability, it is required that $|\xi|^2 \le 1$ for all values of β in [-π, π]. Imposing this condition on the characteristic equation (38) yields $\nu\lambda_l(A_l C_l + B_l D_l) \le 0$. Since $\nu > 0$ and $\lambda_l > 0$, hence from (37), we obtain $\mathrm{Re}[L(\beta)] \equiv \dfrac{A_l C_l + B_l D_l}{C_l^2 + D_l^2} \le 0$ as a necessary and sufficient condition for linear stability. This condition is satisfied for all choices of $-\pi \le \beta \le \pi$. Thus we conclude that the scheme (34) is unconditionally stable.

5. COMPARATIVE RESULTS

We have solved three benchmark problems using the method described by equation (14) and compared our results with those obtained by the variable mesh method of $O(k^2 h_l^{-1} + kh_l + h_l^3)$ accuracy based on finite difference discretization, discussed by (Mohanty, 2007) for the solution of 1D nonlinear parabolic equations. The exact solutions are provided in each case. The right hand side homogeneous function, initial and boundary conditions may be obtained using the exact solution as a test procedure. The linear difference equation has been solved using a tri-diagonal solver, whereas non-linear difference equations have been solved using the Newton-Raphson method. While using the Newton-Raphson method, the iterations were stopped when absolute error tolerance $\leq 10^{-15}$ was achieved. All computations were carried out using double precision arithmetic.

The unit interval [0,1] in the space-direction is divided into $(N+1)$ points with

$$0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1,$$

where

$$h_l = x_l - x_{l-1} \text{ and } \sigma_l = (h_{l+1} / h_l) > 0, \; l=1,2,\ldots,N.$$

We may write

$$\begin{aligned} 1 &= x_{N+1} - x_0 = (x_{N+1} - x_N) + (x_N - x_{N-1}) + \ldots.. + (x_1 - x_0) \\ &= h_{N+1} + h_N + \ldots.. + h_1 \\ &= (\sigma_1 + \sigma_1 \sigma_2 + \ldots\ldots. + \sigma_1 \sigma_2 \ldots\ldots. \sigma_N)\, h_1. \end{aligned} \tag{39}$$

For simplicity, we consider $\sigma_l = \sigma$ (a constant), $l=1,2,\ldots,N$, then from (39) we have

$$h_1 = \frac{1-\sigma}{1-\sigma^{N+1}}, \quad \sigma \neq 1 \tag{40}$$

By prescribing the total number of mesh points to be $(N+2)$, we can compute the value of h_1 from (40). This is the first mesh spacing on the left of the boundary and the remaining mesh is determined by $h_{l+1} = \sigma h_l$, $l=1,2,\ldots,N$. Throughout our computation we choose the values of $\sigma \in \left(\frac{1}{2}(\sqrt{5}-1), \frac{1}{2}(\sqrt{5}+1)\right)$, which have already been reported in the section 2. We have considered $N+1 = 8, 16, 32, 64$ as the total number of grid points in x-

direction. In order to obtain numerical solution at t = 1.0, it is required to choose the time step $k = 0.2/(N+1)^2$, $0.4/(N+1)^2$, $0.8/(N+1)^2$, $1.6/(N+1)^2$, $3.2/(N+1)^2$,…etc. Throughout our computation we use the large time step $k = 1.6/(N+1)^2$. For N+1 = 8, 16, 32, and 64, the values of $k = 1.6/(N+1)^2$ become $\frac{1}{40}$, $\frac{1}{160}$, $\frac{1}{640}$, and $\frac{1}{2560}$ respectively, that is, we require 40, 160, 640 and 2560 time steps respectively, to obtain numerical solution at t = 1.0.

Problem 1: $\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\alpha}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial u}{\partial t}+f(r,t)$, $0<r<1$, $t>0$. (41),

with $f(r,t)$ chosen appropriately so that the exact solution is given by $u = e^{-\nu t}\cosh r$. The root mean square (RMS) errors for u at t = 1.0 are tabulated in Table1 for a fixed $\sigma = 1.2 \in \left(\frac{1}{2}(\sqrt{5}-1), \frac{1}{2}(\sqrt{5}+1)\right)$ and α =1, 2 and various values of ν.

Table 1: The RMS errors

(N+1)	$O(k^2h_l^{-1}+kh_l+h_l^3)$-Cubic Spline Method				$O(k^2h_l^{-1}+kh_l+h_l^3)$-Finite Difference Method			
	α =1		α =2		α =1		α =2	
	ν = 0.01	ν = 0.001	ν = 0.01	ν = 0.001	ν = 0.01	ν = 0.001	ν = 0.01	ν = 0.001
8	.2905(-03)	.6624(-04)	.5898(-03)	.1724(-03)	.3344(-03)	.8602(-04)	.6998(-03)	.2329(-03)
16	.3314(-04)	.1162(-04)	.5676(-04)	.2384(-04)	.3370(-04)	.1276(-04)	.6012(-04)	.2440(-04)
32	.6282(-05)	.7744(-06)	.1576(-04)	.1812(-05)	.6333(-05)	.7859(-06)	.1634(-04)	.1848(-05)
64	.4989(-05)	.5299(-06)	.1439(-04)	.1443(-05)	.5054(-05)	.5364(-06)	.1518(-04)	.1521(-05)

Problem 2: $\nu\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}$, $0<x<1$, $t>0$. (*Burgers' Equation*) (42)

The exact solution is given by $u(x,t) = \frac{2\nu\pi\sin(\pi x)e^{-\nu\pi^2 t}}{2+\cos(\pi x)e^{-\nu\pi^2 t}}$, where $R_e = \nu^{-1} > 0$ is called the Reynolds number. The RMS errors for u at t = 1.0 are tabulated in Table 2 for a fixed $\sigma = 0.75 \in \left(\frac{1}{2}(\sqrt{5}-1), \frac{1}{2}(\sqrt{5}+1)\right)$ and various values of R_e.

Table 2: The RMS errors

(N+1)	$O(k^2h_l^{-1}+kh_l+h_l^3)$-Cubic Spline Method				$O(k^2h_l^{-1}+kh_l+h_l^3)$-Finite Difference Method			
	R_e=1.0	R_e=10	R_e=10^2	R_e=10^3	R_e=1.0	R_e=10	R_e=10^2	R_e=10^3
8	.3344(-05)	.1513(-03)	.1699(-04)	.2261(-06)	.2952(-05)	.2810(-03)	.1720(-04)	.2310(-06)
16	.8463(-06)	.4479(-04)	.5762(-05)	.7468(-07)	.1060(-05)	.1069(-03)	.5783(-05)	.7523(-07)
32	.7439(-06)	.2859(-04)	.3663(-05)	.4800(-07)	.8897(-06)	.6995(-04)	.3726(-05)	.4831(-07)
64	.5421(-06)	.2001(-04)	.2583(-05)	.3306(-07)	.7822(-06)	.4905(-04)	.2612(-05)	.3386(-07)

Problem 3: $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + f(x,t)$, 0<$x$<1, t>0. (*Non-linear Equation*) (43),

with $f(x,t)$ chosen appropriately so that the exact solution is given by $u(x,t) = e^{-t}\sin(\pi x)$. The RMS errors for u at t = 1.0 are tabulated in Table 3 for a fixed σ = 1.35 $\in \left(\frac{1}{2}(\sqrt{5}-1), \frac{1}{2}(\sqrt{5}+1)\right)$ and various values of α.

Table 3: The RMS errors

(N+1)	$O(k^2h_l^{-1}+kh_l+h_l^3)$-Cubic Spline Method			$O(k^2h_l^{-1}+kh_l+h_l^3)$-Finite Difference Method		
	α = 10	α = 50	α = 100	α = 10	α = 50	α =100
8	0.4118(-03)	0.1366(-03)	0.1883(-02)	0.4277(-03)	0.1397(-03)	0.1870(-02)
16	0.2252(-03)	0.4400(-04)	0.1002(-02)	0.2372(-03)	0.4405(-04)	0.1005(-02)
32	0.1577(-03)	0.3021(-04)	0.6800(-03)	0.1617(-03)	0.3071(-04)	0.6820(-03)
64	0.1002(-03)	0.2012(-04)	0.4705(-03)	0.1134(-03)	0.2155(-04)	0.4783(-03)

6. CONCLUDING REMARKS

Using three spatial variable mesh points (see Fig. 1), we have developed a new two level implicit method of $O(k^2h_l^{-1}+kh_l+h_l^3)$ based on cubic spline polynomial approximation for the solution of the non-linear parabolic partial differential equation (1). Although the proposed variable mesh cubic spline method involves more algebra, we do not require any fictitious points near the boundaries to solve singular parabolic equations. The proposed method when applied to a linear equation is shown to be unconditionally stable with respect to initial values. The numerical results indicate that the proposed variable mesh cubic spline method is computationally slightly better than the corresponding variable mesh finite difference method of $O(k^2 h_l^{-1} + kh_l + h_l^3)$, but numerical oscillation do not appear for large values of R_e or α.

ACKNOWLEDGEMENTS

This research was supported by 'The University of Delhi' under Research Grant No. Dean(R)/R&D/2007/Ph-III/386.

The author thanks the reviewers for their valuable suggestions, which substantially improved the standard of the paper.

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