Neural, Parallel, and Scientific Computations 16 (2008) 497-504

ESTIMATE FOR THE NUMBER OF ZEROS OF ABELIAN INTEGRALS FOR A KIND OF QUARTIC HAMILTONIANS

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ABSTRACT. In this paper, we give a lower upper bound of the number of zeros of part of the Abelian integral $I(h) = \int_{\delta(h)} P(x,y)dx + Q(x,y)dy$, $h \in \Sigma$, where $\delta(h)$ is an oval contained in the level set $\{H(x,y) = y^2 + x^4 - x^2 = h\}$, $P(x,y)$, $Q(x,y)$ are real polynomials of x and y with degree not greater than n, Σ is the maximal interval of the existence of the ovals $\{\delta(h)\}$. The corresponding vector space of the Abelian integral $I(h)$ defined on the open interval Σ obeys the Chebyshev property (the maximal number of isolated zeros of each function is less than the dimension of the space of functions).

Key Words: Abelian integral, Chebyshev property, Chebyshev accuracy

2000 Mathematics Subject Classification: 34C07

1. INTRODUCTION

It is well known that the second part of Hilbert's 16th problem is about the maximum number of limit cycles that a polynomial system may have. This a very difficult problem which has not been solved even for quadratic systems. But it has been known that: for sufficiently small ϵ, the limit cycles of the perturbed plane Hamiltonian system:

$$dH + \epsilon(Pdx + Qdy) = 0, \quad h \in \Sigma, \quad H, P, Q \in \mathbb{R}[x,y]$$

which tend to certain ovals from the continuous family when $\epsilon \longrightarrow 0$, are in one-to-one correspondence with the zeros of the complete Abelian integral

$$I(h) = \int_{\delta(h)} P(x,y)dx + Q(x,y)dy, \quad h \in \Sigma,$$

where $\delta(h) \subset \{(x,y) \in \mathbb{R}^2 : H(x,y) = h, h \in \Sigma\}$. So people usually consider to find the number of zero of the Abelian integral $I(h)$. The weakened Hilbert 16th problem called by Arnold (see [1], p. 313) is to find the number of zeros of the Abelian integral $I(h)$ in terms of the degrees of H, P, Q (compare with Hilbert [6]; see also [7], [8], [9]). The general results of solving the weakened Hilbert 16th problem are due to A. N. Varchenko and A. G. Khovansky (see [10]), who achieved independently the

Received February 3, 2008

existence of $Z(m,n)$, where $Z(m,n)$ denotes the upper bound of the number of zeros of the Abelian integrals $I(h)$ in terms of $deg\, H = m$, $deg\, P$, $deg\, Q \le n$. But the explicit expression of $Z(m,n)$ has not been obtained.

In general, the number of the isolated zeros of $I(h)$ is related to the Picard-Fuchsian equation:

$$(I_0(h), I_2(h))^T = A(h)(I_0'(h), I_2'(h))^T, \qquad (*)$$

where $I_0(h), I_2(h)$ satisfy:

$$I(h) = P_0(h)I_0(h) + P_2(h)I_2(h), h \in \Sigma,$$

and $A(h)$ is a first-degree polynomial matrix (here, $P_0(h)$ and $P_2(h)$ are real polynomials).

Recently, Lubomir Gavrilov and Iliya D. Iliev considered a two-parameter class of Fuchsian systems. They described that the corresponding vector space of Abelian integrals obeys the Chebyshev property if the above Picard-Fuchsian equation $(*)$ satisfies the following conditions (see [2]):

H(1). A' is a constant matrix with distinct real eigenvalues.

H(2). The equation $detA(h) = 0$ has real distinct roots h_0, h_1 (assume that $h_0 < h_1$) and the identity: $trace\, A(h) = (det\, A(h))'$ holds.

H(3). $I_0(h), I_2(h)$ are analytic in a neighborhood of h_0.

And they applied their results to some cases such as: (a) $H = y^2 + x^2 - x^3$, (b) $H = y^2 + x^2 - xy^2$, (c) $H = y^2 + x^2 - x^4$ and so on.

Let us adopt the denotations in [2]: For systems $(*)$ satisfying **H(1)** and **H(2)**, $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ denote the eigenvalues of the constant matrix A'; Define $\lambda^* = 2$ if λ is integer and $\lambda^* = \max\{|\lambda - 1|, 1 - |\lambda - 1|\}$ otherwise.

On the basis of the work in [2], we consider a specific Hamiltonian:

$$H = y^2 + x^4 - x^2 = h, \qquad (1)$$

(where $l \in \mathbb{R}$) and the corresponding perturbed Hamiltonian system:

$$\begin{cases} \dot{x} = -2y - \epsilon Q(x,y) \\ \dot{y} = 4x^3 - 2x + \epsilon P(x,y) \end{cases} \qquad (2)$$

The Hamiltonian (1) has been considered in [4]. We improve on the results in [4] partly. In the present paper, we denote

$$V_n = \{I(h) |\, I(h) = \int_{\delta(h)} P(x,y)dx + Q(x,y)dy$$
$$P,\, Q \in \mathbb{R}[x,y],\, deg\, P,\, deg\, Q\, \le n,\, h \in \Sigma\},$$

where $\mathbb{R}[x,y]$ is the set of real polynomials about x, y. Σ is the maximal interval on which a continuous family of ovals $\{\delta(h)\}$ exists.

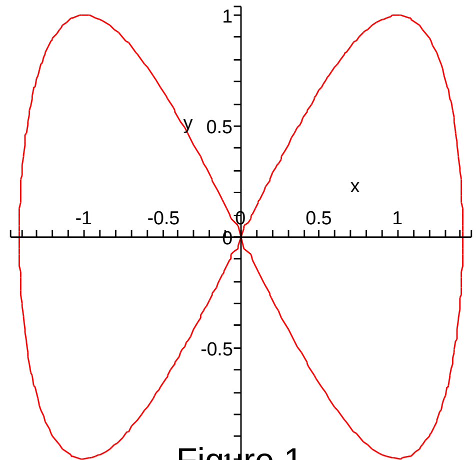


Figure 1

It's evident that the unperturbed system (2) has three critical points: $(0,0)$ and $(\pm\frac{\sqrt{2}}{2},0)$. And the critical values of H at these points are: $H(0,0) = 0 = h_{\max}$ and $H(\pm\frac{\sqrt{2}}{2},0) = -\frac{1}{4} = h_{\min}$ respectively. When $h \in \Sigma = (-\frac{1}{4},0)$, $\{\delta(h)\}$ are surrounding the centers $(\pm\frac{\sqrt{2}}{2},0)$ with two components. When $h \in \Sigma = (0,\infty)$, $\{\delta(h)\}$ are surrounding the three critical points $(0,0)$ and $(\pm\frac{\sqrt{2}}{2},0)$ consisting of one component. See Figure 1.

This paper is organized as follows: In section 2, we introduce some results which have been known. In section 3, $I(h)$ is expressed as a linear combination of $I_0(h), I_1(h), I_2(h)$. Then we derive a Picard-Fuchsian equation of $I_0(h), I_2(h)$. Finally we prove that when the coefficient of $I_1(h)$ is identical to zero, the spaces of Abelian integrals corresponding to systems (2) obey the Chebyshev property.

2. PRELIMINARIES

Definition 2.1. The space V of functions defined on the domain D is called Chebyshev with accuracy k (in D), if every non-zero function in V has at most $dim\,V+k-1$ zeros in D. V is said to be Chebyshev (in D) if every non-zero function has at most $dim\,V-1$ zeros in D.

Definition 2.2. Let $I(h)$, $h \in \mathbb{C}$ be a function of locally analytic in a neighborhood of ∞, and $S \in \mathbb{R}$. We shall write $I(h) \lesssim h^S$, if there exists a non-zero constant C_S such that $I(h) \leqslant C_S|h|^S$ for all sufficiently big $|h|$, $h \in \mathbf{S}$, where $\mathbf{S}$ is any sector centered at ∞.

Denote $V_S = \{I(h) \mid I(h) = P_0(h)I_0(h) + P_2(h)I_2(h),\ P_0,\ P_2 \in \mathbb{R}[h],\ I(h) \lesssim h^S\}$. We need the following Lemmas.

Lemma 2.1 (2, Proposition 4)**.** *Let $S \geq \lambda^*$, and λ, μ be not integer. Then:*

$$dim\, V_S = \begin{cases} 2S-1 & \text{if } \lambda-\mu \text{ and } S-\frac{1}{2} \text{ are interger,} \\ [S-\lambda]+[S-\mu]+2 & \text{otherwise.} \end{cases}$$

Lemma 2.2 (2, Theorem 1)**.** *Assume that conditions **H(1)**–**H(3)** hold. If $\lambda \notin \mathbb{Z}$, then V_S is a Chebyshev vector space with accuracy $1+[\lambda^*]$ in the complex domain $\mathbb{D} = \mathbb{C}\backslash[h_1, \infty)$. If $\lambda \in \mathbb{Z}$, then V_S coincides with the space of real polynomials of degree at most $[S]$, which vanish at h_0 and h_1.*

Lemma 2.3. *If $I_0(h)$ and $I_2(h)$ satisfy **H(1)-H(3)**, and h is near infinity, then $I_0(h), I_2(h)$ have the following form:*

$$\begin{pmatrix} I_0 \\ I_2 \end{pmatrix} = a \begin{pmatrix} h^{\lambda} - \frac{\lambda}{2}h^{\lambda-1} + \cdots \\ \alpha h^{\lambda-1} + \cdots \end{pmatrix} + b \begin{pmatrix} \beta h^{\mu-1} + \cdots \\ h^{\mu} - \frac{\mu}{2}h^{\mu-1} + \cdots \end{pmatrix},$$

where $\alpha = \frac{\lambda}{2\omega(\mu-\lambda+1)}$, $\beta = \frac{\mu\omega}{2(\lambda-\mu+1)}$ and ω is a free parameter.

Proof: See the proof of Proposition 4 in [2]. Denote

$$I_{ij}(h) = \int_{\delta(h)} x^i y^j dx$$

and

$$I_0 = I_{01}, \qquad I_1(h) = I_{11}(h), \qquad I_2(h) = I_{21}(h),\ h \in \Sigma.$$

Lemma 2.4 (4, Lemma 3)**.** *For the system (2), any $I(h) \in V_n$ can be expressed as:*

$$I(h) = p(h)I_0(h) + r(h)I_1(h) + q(h)I_2(h), \tag{3}$$

where $p(h), r(h), q(h) \in \mathbb{R}[h]$, and $deg\, p(h) \leq [\frac{n-1}{2}] = a$, $deg\, r(h) \leq [\frac{n-2}{2}] = b$, $deg\, q(h) \leq [\frac{n-3}{2}] = c$. And a, b, c are the lowest upper bounds of $deg\, p(h)$, $deg\, r(h)$, $deg\, q(h)$.

Remark 1: By Lemma 2.4,

$$V_n = \{I(h) = p(h)I_0(h) + r(h)I_1(h) + q(h)I_2(h)\},$$

where $p(h)$, $r(h)$, $q(h) \in \mathbb{R}[h]$, $deg\, p(h) \leq [\frac{n-1}{2}]$, $deg\, r(h) \leq [\frac{n-2}{2}]$, $deg\, q(h) \leq [\frac{n-3}{2}]$.

3. MAIN RESULTS

Proposition 3.1. *I_0, I_2 satisfy the following Picard-Fuchsian equation:*

$$\begin{pmatrix} I_0 \\ I_2 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}h & \frac{2}{3} \\ \frac{2h}{15} & \frac{4}{5}h + \frac{4}{15} \end{pmatrix} \begin{pmatrix} I_0' \\ I_2' \end{pmatrix}. \tag{4}$$

Proof: Denote $I_k = \int_{\delta(h)} x^k y dx$. From (1):

$$ydy = xdx - 2x^3dx. \tag{5}$$

Differentiate (1) with respect to h, it yields $\frac{dy}{dh} = \frac{1}{2y}$. By partial integral and (5), we have

$$\begin{aligned} I_k &= -\frac{1}{k+1}\int_{\delta(h)} \frac{x^{k+1}(x-2x^3)}{y}\,dx \\ &= -\frac{2}{k+1}\,(I'_{k+2} - 2I'_{k+4}). \end{aligned} \tag{6}$$

On the other hand,

$$I_k = 2\int_{\delta(h)} \frac{x^k y^2}{2y}\,dx = 2\int_{\delta(h)} \frac{x^k(h+x^2-x^4)}{2y}\,dx = 2hI'_k + 2I'_{k+2} - 2I'_{k+4}. \tag{7}$$

Following from (6), (7), we have:

$$\frac{-2I'_{k+2}}{k+1} + \frac{4I'_{k+4}}{k+1} = 2hI'_k + 2I_{k+2'} - 2I'_{k+4}.$$

Simplifying the above equation, we obtain:

$$\left(\frac{-1}{k+1} - 1\right) I'_{k+2} + \left(\frac{2}{k+1} + 1\right) I'_{k+4} = hI'_k. \tag{8}$$

Using $k = 0$ in (8), we get:

$$I'_4 = \frac{2I'_2}{3} + \frac{hI'_0}{3}. \tag{9}$$

Using (6), (7) again (getting rid of I'_{k+4}), we have:

$$\frac{k+3}{4} I_k = hI'_k + \frac{1}{2}\,I'_{k+2}. \tag{10}$$

Let $k = 0, 2$ respectively in (10), so we have:

$$I_0 = \frac{4h}{3}\,I'_0 + \frac{2}{3}\,I'_2, \tag{11}$$

$$\frac{5}{4}\,I_2 = hI'_2 + \frac{1}{2}\,I'_4. \tag{12}$$

Displacing I'_4 with (9), we obtain:

$$I_2 = \frac{2h}{15}\,I'_0 + \left(\frac{4}{5}\,h + \frac{4}{15}\right) I'_2. \tag{13}$$

Combining (11) and (13), we can get the Picard-Fuchsian equation (4). The proof is completed.

Theorem 3.1. *For the perturbed Hamiltonian systems* (2), *the linear space of integrals* V_n *is Chebyshev in* Σ *if* $r(h) \equiv 0$ *in* (3), *where* Σ *is the maximal open interval on which a continuous family of ovals* $\{\delta(h)\}$ *exists. So* $B(n) \le 2[\frac{n-1}{2}]$ ($B(n)$ *is the upper bound of the number of zeros of the Abelian integrals* $I(h)$ *on the open interval* Σ).

Proof: We will proof the theorem in two cases.

Case 1: $\Sigma = (-\frac{1}{4}, 0)$. By Proposition 3.1, the coefficient matrix of (4) satisfies the conditions **H(1)**–**H(2)**. Then we get $\lambda = \frac{3}{4}$, $\mu = \frac{5}{4}$. So $[\lambda^*] = 0$. We choose $S = [\frac{n+1}{2}]$. From Lemma 2.1, $dimV_S = [S-\lambda] + [S-\mu] + 2$. So, $dim\, V_S = 2[\frac{n-1}{2}] + 1$. We suppose that $r(h) \equiv 0$ for any $I(h) \in V_n$ in the following. By Lemma 2.4, $dim\, V_n = 2[\frac{n-1}{2}] + 1$ when $r(h) \equiv 0$. Evidently, $dim\, V_n = dim\, V_s$. Next, we would like to prove that $V_n \subseteq V_s$. Under the condition $r(h) \equiv 0$, $V_n = \{I(h) = P_0(h)I_0(h) + P_2(h)I_2(h),\, deg\, P_0 \leq [\frac{n-1}{2}],\, deg\, P_2 \leq [\frac{n-1}{2}] - 1\}$ by Lemma 2.4.

By Lemma 2.3, when h is near infinity, it's obvious that $deg\, P_0(h) + deg\, I_0(h) = deg\, P_0(h) + \lambda = deg\, P_0(h) + \frac{3}{4} < [\frac{n+1}{2}] = S$. Similarly, $deg\, P_2(h) + deg\, I_2(h) < S$. Then by the definition of V_S, $I(h) \in V_S$ for any $I(h) \in V_n$. So $V_n \subseteq V_s$. Thus we have: $V_n = V_s$.

It's easy to check that the coefficient matrix of (4) and $I_0(h)$, $I_2(h)$ satisfy the three conditions **H(1)**–**H(3)**. So V_n is Chebyshev with accuracy 1 in $\mathbb{D} = \mathbb{C} \backslash (0, \infty)$ by Lemma 2.2 (because $[\lambda^*] = 0$). And V_n is Chebyshev in Σ, because we take into account that $I(h)$ has always a zero at $h_0 = -\frac{1}{4}$, which completes the proof.

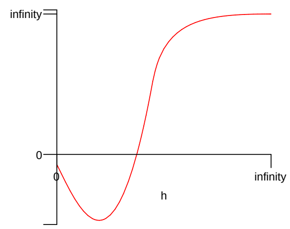
Case 2: $\Sigma = (0, \infty)$. In this case, the condition **H(3)** isn't satisfied. But see [4 Theorem 1] has proved that the space of integrals $I(h)$ of real polynomial forms of degree at most n is Chebyshev in the domain $\mathcal{D}^+/0$ where $\mathcal{D}^+$ is the plane with a cut along the ray $\{h \leq -\frac{1}{4}\}$; And its dimension is $2[\frac{n-1}{2}] + 1$. So $B(n) \leq 2[\frac{n-1}{2}]$ on the interval $(0, \infty)$.

By the above two steps, the proof is finished.

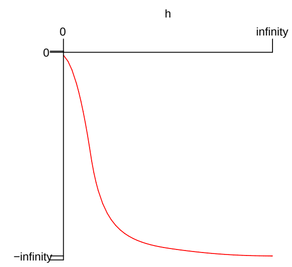
4. **SIMULATION**

In the proof of Theorem 3.1, we extend $I(h)$ from real interval Σ to complex domain, and then give the conclusion of Theorem 3.1. Expanding $I_0(h)$ and $I_2(h)$ in the interval Σ, where Σ is the maximal interval on which the continuous ovals $\delta(h)$ exist, we give some examples to simulate the result of Theorem 3.1. See Figure 2 and Figure 3.

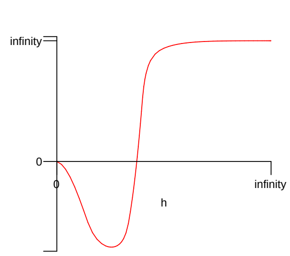
Figure 2 shows that: in the interval $(0, \infty)$, the number of zeros of $I(h)$ with $deg\, p(h) = 1$ or 2, $deg\, q(h) = 0$ or 1 and $r(h) = 0$, is not great than 2. The zeros of $I(h)$ in the interval $(-\frac{1}{4}, 0)$ in Figure 3 is also less than the bound we give in Theorem 3.1. All these numerical results and the conclusion of Theorem 3.1 are consistent.



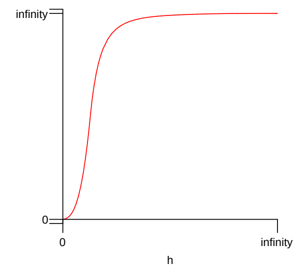
(a) $p(h) = h$, $q(h) = -5$



(b) $p(h) = -h$, $q(h) = 1$

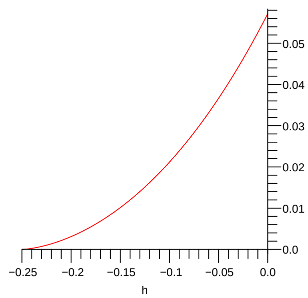


(c) $p(h) = h^2$, $q(h) = -5h$

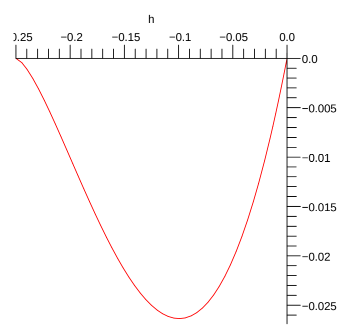


(d) $p(h) = h^2$, $q(h) = h$

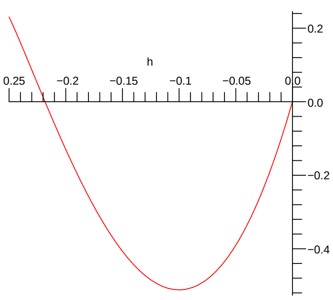
Figure 2: Diagram for $I(h)$ with $r(h) = 0$ in (3), and $\Sigma = (0, \infty)$.



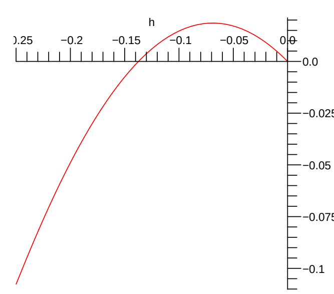
(e) $p(h) = h$, $q(h) = 1$



(f) $p(h) = h^2$, $q(h) = 10h$



(g) $p(h) = -h^2 - 2.5h$, $q(h) = 100h$



(h) $p(h) = h^4 - h^3 + h^2 - 2h$, $q(h) = 10h^3$

Figure 3: Diagram for $I(h)$ with $r(h) = 0$ in (3), and $\Sigma = (-\frac{1}{4}, 0)$.

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