Neural, Parallel, and Scientific Computations 16 (2008) 505-516

CALCULATING ZEROS OF THE TWISTED EULER POLYNOMIALS

R. P. AGARWAL, YOUNG-HO KIM, AND C. S. RYOO

Department of Mathematical Sciences, Florida Institute of Technology,
Melbourne, Florida 32901-6975, USA
Department of Applied Mathematics, Changwon National University, Changwon,
Kyeong-Nam 641-773, Korea
Department of Mathematics, Hannam University, Daejeon 306-791, Korea

ABSTRACT: In this paper we introduce the twisted Euler numbers $E_{n,w}$ and polynomials $E_{n,w}(x)$. Finally, we investigate the zeros of the twisted Euler polynomials.

1. INTRODUCTION

Many mathematicians have studied Euler numbers and Euler polynomials. Euler polynomials posses many interesting properties and arising in many areas of mathematics and physics. We introduce the twisted Euler numbers and polynomials. In the 21st century, the computing environment would make more and more rapid progress. Using computer, a realistic study for new analogs of Euler numbers and polynomials is very interesting. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the twisted Euler polynomials $E_{n,w}(x)$. The outline of this paper is as follows. In Section 2, we study the the twisted Euler polynomials $E_{n,w}(x)$. In Section 3, we describe the beautiful zeros of the twisted Euler polynomials $E_{n,w}(x)$ using a numerical investigation. Also we display distribution and structure of the zeros of the the twisted Euler polynomials $E_{n,w}(x)$ by using computer. By using the results of our paper the readers can observe the regular behaviour of the roots of the twisted Euler polynomials $E_{n,w}(x)$. Finally, we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the twisted Euler polynomials $E_{n,w}(x)$. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$.

Received April 2, 2008

If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

$$[x]_q = [x:q] = \frac{1-q^x}{1-q}, \text{ cf. [1,4,5] }.$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$X = \varprojlim_{N} (\mathbb{Z}/dp^N\mathbb{Z}),$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. For any positive integer N,

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$$

is known to be a distribution on X, cf. [3,4,5]. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiaable function}\},$$

the p-adic q-integral was defined by [3,4,5]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{0 \leq x < p^N} g(x) q^x.$$

Note that

$$I_1(g) = \lim_{q\to 1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{0 \leq x < p^N} g(x)$$

(see [4,5,7]). Now, we consider the case $q \in (-1, 0)$ corresponding to q-deformed fermionic certain and annihilation operators and the literature given therein [4,5,7]. The expression for the $I_q(g)$ remains same, so it is tempting to consider the limit $q \to -1$. That is,

$$I_{-1}(g) = \lim_{q\to -1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0 \leq x < p^N} g(x)(-1)^x. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{1.2}$$

From (1.2), we obtain

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} g(l),$$

where $g_n(x) = g(x+n)$. First, we introduce the Euler numbers and Euler polynomials. The Euler numbers E_n are defined by the generating function:

$$F(t) = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \text{ cf. [1,4,5]} \tag{1.3}$$

where we use the technique method notation by replacing E^n by $E_n (n \geq 0)$ symbolically. For $x \in \mathbb{R}$ (= the field of real numbers), we consider the Euler polynomials $E_n(x)$ as follows:

$$F(x,t) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.4}$$

Note that $E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k x^{n-k}$. In the special case $x = 0$, we define $E_n(0) = E_n$.

2. THE TWISTED EULER NUMBERS AND POLYNOMIALS

In [7], we defined the twisted Euler numbers and polynomials. In this section, we introduce the twisted Euler numbers $E_{n,w}$ and polynomials $E_{n,w}(x)$ and investigate their properties. Let $T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{w | w^{p^N} = 1\}$ is the cyclic group of order p^N. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto w^x$. In (1.2), if we take $g(x) = \phi_w(x) e^{xt}$, then we easily see that

$$I_{-1}(\phi_w(x) e^{xt}) = \int_{\mathbb{Z}_p} \phi_w(x) e^{xt} d\mu_{-1}(x) = \frac{2}{we^t+1}.$$

Let us define the twisted Euler numbers $E_{n,w}$ and polynomials $E_{n,w}(x)$ as follows:

$$I_{-1}(\phi_w(y) e^{yt}) = \int_{\mathbb{Z}_p} \phi_w(y) e^{yt} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,w} \frac{t^n}{n!}, \tag{2.1}$$

$$I_{-1}(\phi_w(y) e^{(y+x)t}) = \int_{\mathbb{Z}_p} \phi_w(y) e^{(x+y)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,w}(x) \frac{t^n}{n!}. \tag{2.2}$$

By (2.1) and (2.2), we obtain the following Witt's formula.

Theorem 1. *For $w \in T_p$, we have*

$$\int_{\mathbb{Z}_p} \phi_w(x) x^n d\mu_{-1}(x) = E_{n,w},$$

$$\int_{\mathbb{Z}_p} \phi_w(y)(x+y)^n d\mu_{-1}(y) = E_{n,w}(x).$$

Let q be a complex number with $|q| < 1$ and w be the p^N-th root of unity. By the meaning of (1.3) and (1.4), let us define the twisted Euler numbers $E_{n,w}$ and polynomials $E_{n,w}(x)$ as follows:

$$F_w(t) = \frac{2}{we^t+1} = \sum_{n=0}^{\infty} E_{n,w} \frac{t^n}{n!}, \tag{2.3}$$

$$F_w(x,t) = \frac{2}{we^t+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,w}(x)\frac{t^n}{n!}. \tag{2.4}$$

We have the following remark.

Remark. Note that

(1) $E_{n,w}(0) = E_{n,w}$,
(2) If $w=1$, then $E_{n,w}(x) = E_n(x), E_{n,w} = E_n$,
(3) If $w=1$, then $F_w(x,t) = F(x,t), F_w(t) = F(t)$.

By above definition, we obtain

$$\begin{aligned}\sum_{l=0}^{\infty} G_{l,q}(x)\frac{t^l}{l!} &= \frac{2}{we^t+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,w}\frac{t^n}{n!}\sum_{m=0}^{\infty} x^m\frac{t^m}{m!} \\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} E_{n,w}\frac{t^n}{n!}x^{l-n}\frac{t^{l-n}}{(l-n)!}\right) \\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n}E_{n,w}x^{l-n}\right)\frac{t^l}{l!}.\end{aligned}$$

By using comparing coefficients $\frac{t^l}{l!}$, we have the following theorem.

Theorem 2. *For any positive integer n, we have*

$$E_{n,w}(x) = \sum_{k=0}^{n}\binom{n}{k}E_{k,w}x^{n-k}.$$

Over five decades ago, Carlitz [1] defined q-extensions of the classical Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ and proved properties analogues to those satisfied by B_n and $B_n(x)$. Carlitz's q-Bernoulli numbers $\beta_n = \beta_{n,q}$ can be determined inductively by [1]

$$\beta_0 = 1, \quad q(q\beta+1)^k - \beta_k = \begin{cases} 1, & \text{if } k=1, \\ 0, & \text{if } k>1, \end{cases}$$

with the usual convention about replacing β^k by β_k. For the twisted Euler numbers, we obtain the following theorem.

Theorem 3. *The twisted Euler numbers $E_{n,w}$ are defined respectively by*

$$w(E_w+1)^n + E_{n,w} = \begin{cases} 2, & \text{if } n=0, \\ 0, & \text{if } n>0, \end{cases}$$

with the usual convention about replacing $(E_w)^n$ by $E_{n,w}$ in the binomial expansion.

Proof. From (2.3), we obtain

$$\frac{2}{we^t+1} = \sum_{n=0}^{\infty} E_{n,w}\frac{t^n}{n!} = \sum_{n=0}^{\infty}(E_w)^n\frac{t^n}{n!} = e^{E_w t}$$

which yields

$$2 = (we^t + 1)e^{E_w t} = we^{(E_w+1)t} + e^{GE_w t}.$$

Using Taylor expansion of exponential function, we have

$$\begin{aligned} 2 &= \sum_{n=0}^{\infty} \{w (E_w + 1)^n + (E_w)^n\} \frac{t^n}{n!} \\ &= w (E_w + 1)^0 + (E_w)^0 + \sum_{n=1}^{\infty} \{w (E_w + 1)^n + (E_w)^n\} \frac{t^n}{n!}. \end{aligned}$$

The result follows by comparing the coefficients.

Here is the list of the first the twisted Euler numbers $E_{n,w}$.

$$\begin{aligned} E_{0,w} &= \frac{2}{1+w}, \quad E_{2,w} = -\frac{2w}{(1+w)^2}, \\ E_{3,w} &= \frac{2(-1+w)w}{(1+w)^3}, \\ E_{3,w} &= -\frac{2w(1-4w+w^2)}{(1+w)^4}, \\ E_{4,w} &= \frac{2(-1+w)w(1-10w+w^2)}{(1+w)^5}, \cdots , \end{aligned}$$

Because

$$\frac{\partial}{\partial x} F_w(x,t) = tF_w(x,t) = \sum_{n=0}^{\infty} \frac{d}{dx} F_{n,w}(x) \frac{t^n}{n!},$$

it follows the important relation

$$\frac{d}{dx} E_{n,w}(x) = nE_{n-1,w}(x).$$

Here is the list of the first the twisted Euler Polynomials $G_{n,w}(x)$.

$$\begin{aligned} E_{0,w}(x) &= \frac{2}{1+w}, \\ E_{1,w}(x) &= \frac{2(-w+x+wx)}{(1+w)^2}, \\ E_{2,w}(x) &= \frac{2(-w+w^2-2wx-2w^2x+x^2+2wx^2+w^2x^2)}{(1+w)^3}, \\ E_{3,w}(x) &= \frac{2(-w+4w^2-w^3-3wx+3w^3x-3wx^2-6w^2x^2-3w^3x^2)}{(1+w)^4} \\ &\quad + \frac{2(x^3+3wx^3+3w^2x^3+w^3x^3)}{(1+w)^4}, \cdots , \end{aligned}$$

Since

$$\begin{aligned}\sum_{l=0}^{\infty} E_{l,w}(x+y)\frac{t^l}{l!} &= \frac{2}{we^t+1}e^{(x+y)t}\\ &= \sum_{n=0}^{\infty} E_{n,w}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty} y^m\frac{t^m}{m!}\\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} E_{n,w}(x)\frac{t^n}{n!}y^{l-n}\frac{t^{l-n}}{(l-n)!}\right)\\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n}E_{n,w}(x)y^{l-n}\right)\frac{t^l}{l!},\end{aligned}$$

we have the following theorem.

Theorem 4. *The twisted Euler polynomials $E_{n,w}(x)$ satisfies the following relation:*

$$E_{l,w}(x+y) = \sum_{n=0}^{l}\binom{l}{n}E_{n,w}(x)y^{l-n}.$$

It is easy to see that

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,w}(x)\frac{t^n}{n!} &= \frac{2}{we^t+1}e^{xt}\\ &= \frac{2t}{w^n e^{mt}+1}e^{xt}\sum_{a=0}^{m-1}(-1)^a w^a e^{at}\\ &= \frac{1}{m}\sum_{a=0}^{m-1}(-1)^a w^a\frac{2m}{w^m e^{mt}+1}e^{\left(\frac{a+x}{m}\right)(mt)}\\ &= \frac{1}{m}\sum_{a=0}^{m-1}(-1)^a w^a\sum_{n=0}^{\infty}E_{n,w^m}\left(\frac{a+x}{m}\right)\frac{(mt)^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(m^{n-1}\sum_{a=0}^{m-1}(-1)^a w^a E_{n,w^m}\left(\frac{a+x}{m}\right)\right)\frac{t^n}{n!}.\end{aligned}$$

Hence we have the below theorem.

Theorem 5. *For any positive integer m(=odd), we have*

$$E_{n,w}(x) = m^{n-1}\sum_{i=0}^{m-1}(-1)^i w^i E_{n,w^m}\left(\frac{i+x}{m}\right),\ \textit{for } n \geq 0.$$

3. DISTRIBUTION AND STRUCTURE OF THE ZEROS

In this section, we investigate the zeros of the twisted Euler polynomials $E_{n,w}(x)$ by using computer. Let $w = e^{\frac{2\pi i}{N}}$ in $\mathbb{C}$. We plot the zeros of $E_{n,w}(x), x \in \mathbb{C}$ for $N = 1, 3, 5, 7$. (Figures 1, 2, 3, and 4). Next, we plot the zeros of $E_{n,w}(x), x \in \mathbb{C}$ for $n = 12, 13, 14, 15, N = 7$. (Figures 5, 6, 7, and 8). Next, we plot the zeros of

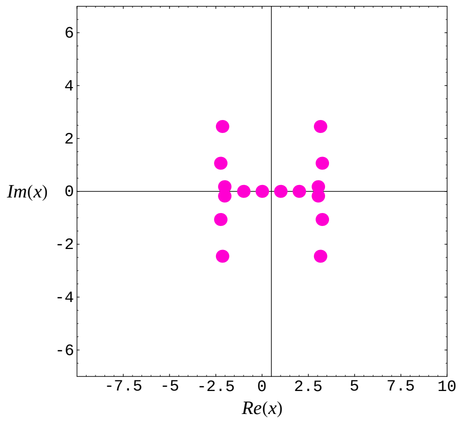

FIGURE 1. Zeros of $E_{16,w}(x)$

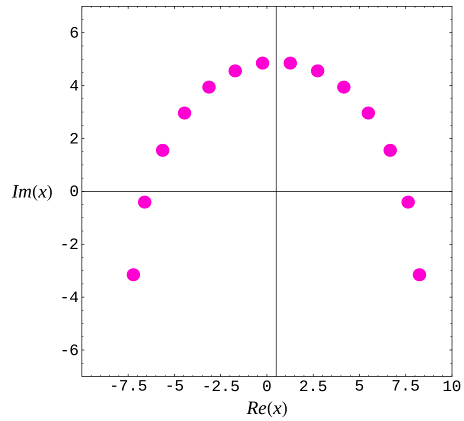

FIGURE 2. Zeros of $E_{16,w}(x)$

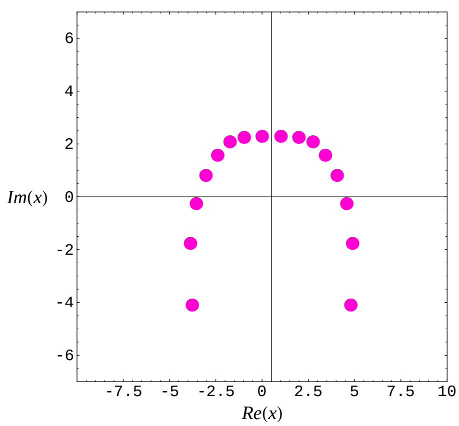

FIGURE 3. Zeros of $E_{16,w}(x)$

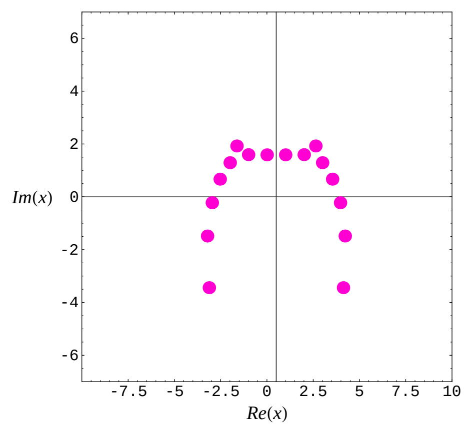

FIGURE 4. Zeros of $E_{16,w}(x)$

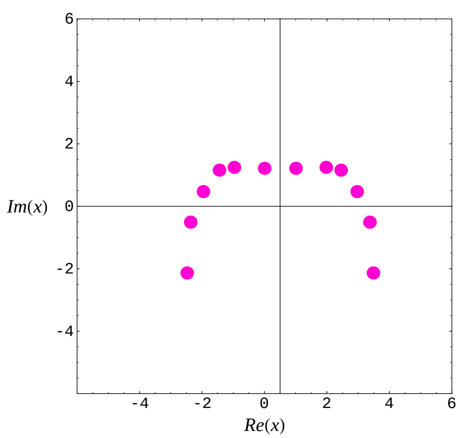

FIGURE 5. Zeros of $E_{12,w}(x)$

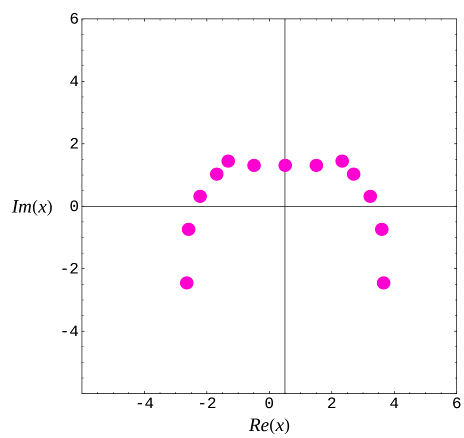

FIGURE 6. Zeros of $E_{13,w}(x)$

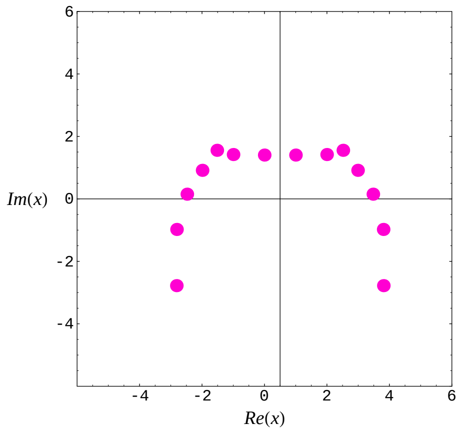

FIGURE 7. Zeros of $E_{14,w}(x)$

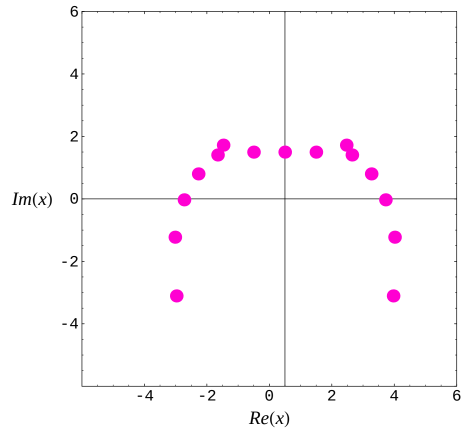

FIGURE 8. Zeros of $E_{15,w}(x)$

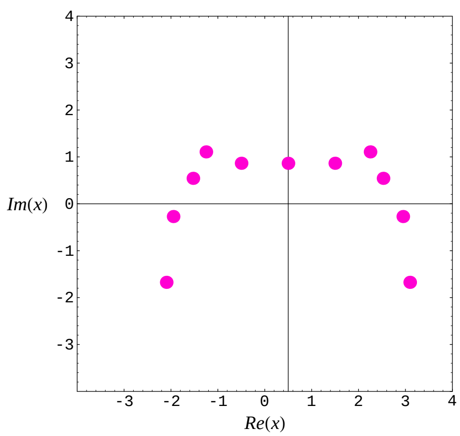

FIGURE 9. Zeros of $E_{11,w}(x)$

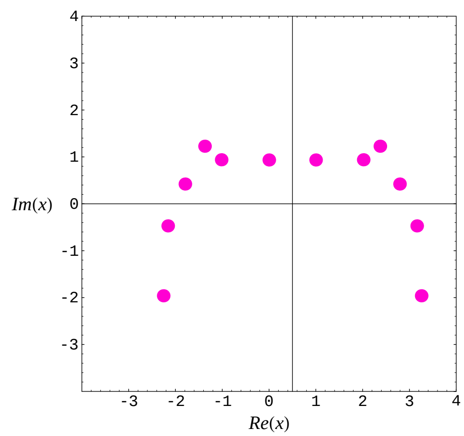

FIGURE 10. Zeros of $E_{12,w}(x)$

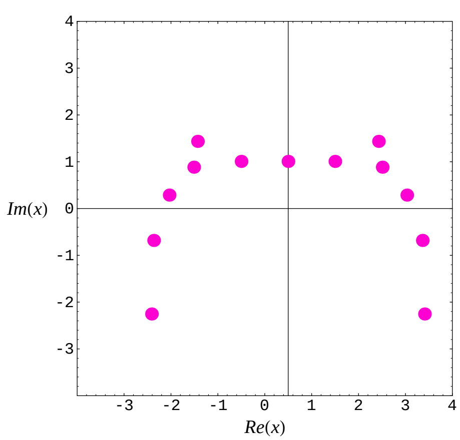

FIGURE 11. Zeros of $E_{13,w}(x)$

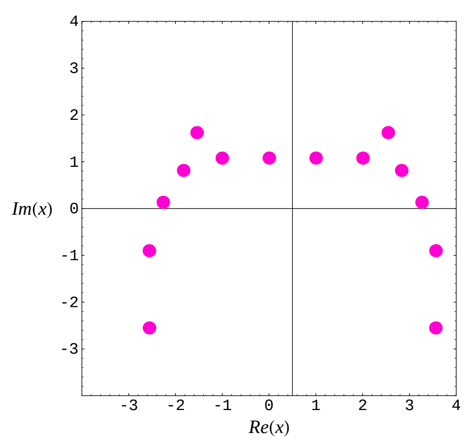

FIGURE 12. Zeros of $E_{14,w}(x)$

$E_{n,w}(x), x \in \mathbb{C}$ for $n = 11, 12, 13, 14, N = 9$. (Figures 9, 10, 11, and 12). Finally, we plot the zeros of $E_{n,w}(x), x \in \mathbb{C}$ for $n = 16, N = 11, 13, 15, 17$. (Figures 13, 14, 15, and 16).

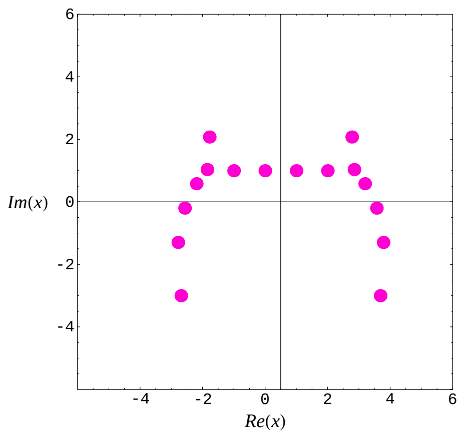

FIGURE 13. $E_{16,w}(x)$, $w = e^{\frac{2\pi i}{11}}$

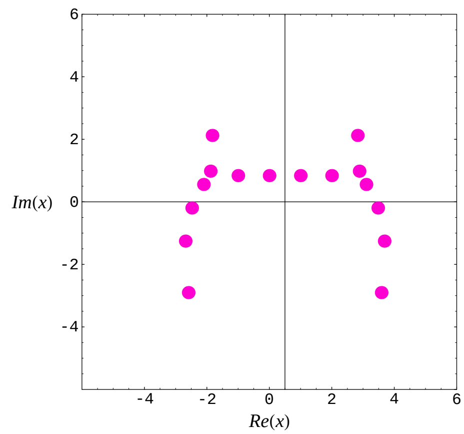

FIGURE 14. $E_{16,w}(x)$, $w = e^{\frac{2\pi i}{13}}$

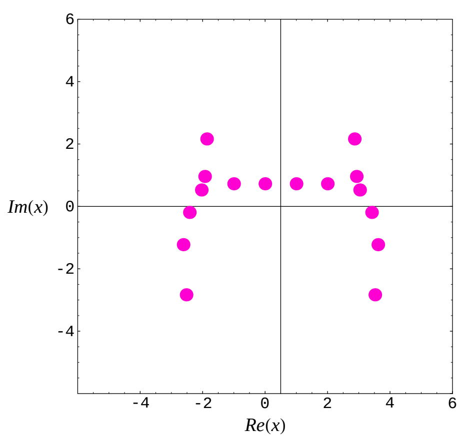

FIGURE 15. $E_{16,w}(x)$, $w = e^{\frac{2\pi i}{15}}$

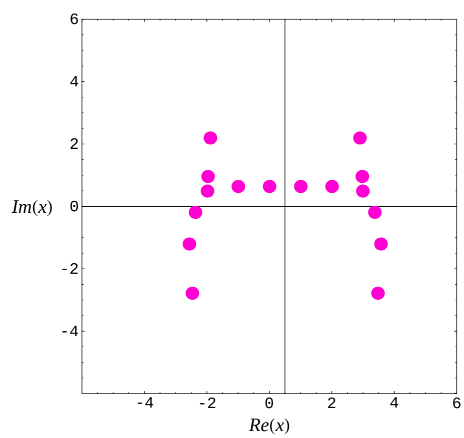

FIGURE 16. $E_{16,w}(x)$, $w = e^{\frac{2\pi i}{17}}$

In Figures 1-16, $E_{n,w}(x), x \in \mathbb{C}$, has $Re(x) = 1/2$ reflection symmetry. This translates to the following open problem: Prove or disprove: $E_{n,w}(x), x \in \mathbb{C}$, has $Re(x) = 1/2$ reflection symmetry.

Our numerical results for numbers of real and complex zeros of $E_{n,w}(x)$ are displayed in Table 1.

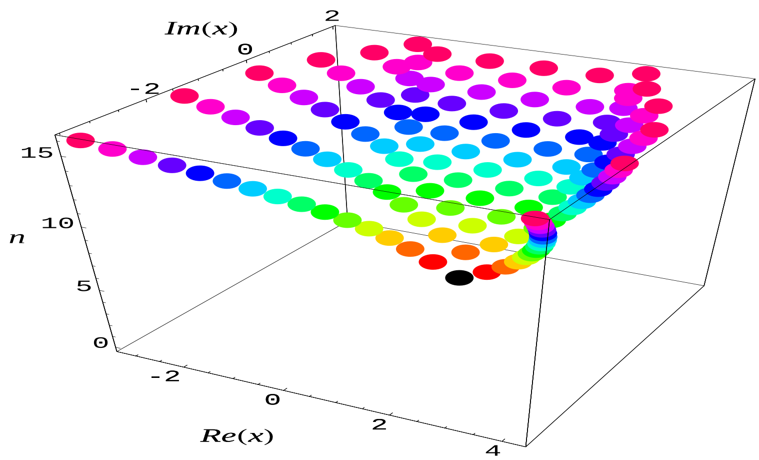

FIGURE 17. Stacks of zeros $E_{n,w}(x), N = 7,$ for $1 \leq n \leq 16$

Table 1. Numbers of real and complex zeros of $E_{n,w}(x)$

degree n	$w = e^{\frac{2\pi i}{3}}$		$w = e^{\frac{2\pi i}{4}}$	
	real zeros	complex zeros	real zeros	complex zeros
1	0	1	0	1
2	0	2	0	2
3	0	3	0	3
4	0	4	0	4
5	0	5	0	5
6	0	6	0	6
7	0	7	0	7
8	0	8	0	8
9	0	9	0	9
10	0	10	0	10

We shall consider the more general open problem. In general,how many roots does $E_{n,w}(x)$ have ? Prove or disprove: $E_{n,q}(x)$ has n distinct solutions. Find the numbers of complex zeros $C_{E_{n,w}(x)}$ of $E_{n,w}(x), Im(x) \neq 0$. Prove or give a counterexample: *Conjecture*: Since n is the degree of the polynomial $E_{n,w}(x)$, the number of real zeros $R_{E_{n,w}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{E_{n,w}(x)} = n - C_{E_{n,w}(x)}$, where $C_{E_{n,w}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,w}(x)}$ and $C_{E_{n,w}(x)}$. Find the equation of envelope curves bounding the real zeros lying on the plane, and the equation of a trajectory curve running through the complex zeros on any one of the arcs. We plot the $E_{n,w}(x)$, respectively (Figures 1–16). These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the $E_{n,w}(x)$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The authors have no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the twisted Euler polynomials $E_{n,w}(x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [6], [7], [8], [9]. We calculated an approximate solution satisfying $E_{n,w}(x), N = 3, 4, x \in \mathbb{C}$. The results are given in Table 2 and Table 3.

Table 2. Approximate solutions of $E_{n,w}(x) = 0, w = e^{\frac{2\pi i}{3}}$

degree n	x
1	$0.50000 + 0.86603i$
2	$-0.50000 + 0.86603i$, $1.50000 + 0.86603i$
3	$-1.29144 + 0.60191i$, $0.5000 + 1.3942i$, $2.2914 + 0.6019i$
4	$-1.9630 + 0.2084i$, $-0.3936 + 1.5236i$, $1.3936 + 1.5236i$, $2.9630 + 0.2084i$
5	$-2.5580 - 0.2656i$, $-1.1897 + 1.4655i$, $0.5000 + 1.9303i$, $2.1897 + 1.4655i$, $3.5580 - 0.2656i$
6	$-3.0992 - 0.7946i$, $-1.9102 + 1.2875i$, $-0.3411 + 2.1052i$, $1.341 + 2.105i$, $2.910 + 1.287i$, $4.0992 - 0.7946i$

Table 3. Approximate solutions of $E_{n,w}(x) = 0, w = e^{\frac{2\pi i}{4}}$

degree n	x
1	$0.50000 + 0.50000i$
2	$-0.20711 + 0.50000i$, $1.20711 + 0.50000i$
3	$-0.75437 + 0.34355i$, $0.50000 + 0.81291i$, $1.75437 + 0.34355i$
4	$-1.20587 + 0.09999i$, $-0.14033 + 0.90001i$, $1.14033 + 0.90001i$ $2.20587 + 0.09999i$
5	$-1.59983 - 0.20378i$, $-0.69955 + 0.88218i$, $0.5000 + 1.1432i$, $1.6995 + 0.8822i$, $2.5998 - 0.2038i$
6	$-1.95697 - 0.54893i$, $-1.19062 + 0.78778i$, $-0.1063 + 1.2611i$, $1.1063 + 1.2611i$, $2.1906 + 0.7878i$, $2.9570 - 0.5489i$

REFERENCES

[1] Carlitz, L. (1948) *q-Bernoulli numbers and polynomials*, *Duke Math. J.*, v. 15, pp. 987–1000.

[2] Cenkci, M., Can, M., Kurt, V.(2004). *q-adic interpolation functions and kummer type congruence for q-twisted and q-generalized twisted Euler numbers*, *Advan. Stud. Contemp. Math.*, v. 9, pp. 203–216.

[3] Kim, T.(2007). *A note on the q-Genocchi numbers and polynomials* , *to appear in Journal of Inequalities and Applications*

[4] Kim, T.(2007). *On the q-extension of Euler and Genocchi numbers*, *J. Math. Anal. Appl.*, v.326, pp. 1458–1465.

[5] Kim, T., Jang, L.C., Pak, H-K.(2001). *A note on q-Euler and Genocchi numbers* , *Proc. Japan Acad.* , v.77 A, pp. 139–141.

[6] Kim, T., Ryoo, C.S., Jang, L.C., Rim, S.H.,(2005). *Exploring the q-Riemann Zeta function and q-Bernoulli polynomials* , *Discrete Dynamics in Nature and Society*, v.2005(2), pp. 171–181.

[7] Ryoo, C.S., Seo, J.J., Kim, T.(2007). *A note on generalized q-Euler numbers and polynomials* , *Journal of Computational Analysis and Applications*, to appear.

[8] Ryoo, C.S., Kim, T., Agarwal, R.P.(2005). *Exploring the multiple Changhee q-Bernoulli polynomials*, *Inter. J. Comput. Math.*, v.82(4), pp. 483–493.

[9] Ryoo,C.S., Song, H., Agarwal, R.P. (2004). *On the roots of the q-analogue of Euler-Barnes' polynomials* , *Advan. Stud. Contemp. Math.*, v. 9, pp. 153–163.

[10] Ryoo,C.S., Kim, T., Agarwal, R.P. (2005). *The structure of the zeros of the generalized Bernoulli polynomials*, *Neural Parallel Sci. Comput.*, v.13, pp. 371–379.

[11] Santos, J.P. (2001). *On a new combinatorial interpolation for a theorem of Euler* , *Advan. Stud. Contemp. Math.*, v. 3, pp. 31–38.