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The Finite Volume Element Method for the Pollution In Groundwater Flow*

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Abstract
In this paper, we use the finite volume element method to solve the mathematic model of one kind of ion reactions for the problem of groundwater in 2-space variables. Piecewise quadratic trial functions and piecewise constant test functions are used to finally obtain error estimate in L^2 norms and H^1 norms. Numerical results that confirm the efficiency of our methods are presented.
Keywords-finite volume element method, error estimate, quadratic element, ion reaction, problem of groundwater

1. INTRODUCTION

The finite volume element method (FVEM) [1][2][3][4][5][6][7][8][9][10][11] is a discretization technique for partial differential equations. FVEM uses a volume integral formulation of the differential equation with a finite partitioning set of volume to discretize the equation. As far as the method is concerned, it is identical to the special case of the generalized difference method (GDM)[12][13][14] proposed by professor Ronghua Li that is, linear or bilinear finite element space is used as trial or admissible finite element space and piecewise constant space is used as test function space. The advantages of FVEM are as follows: first, the grid is flexible and the natural boundary conditions are easy to deal with; second, the computational effort is greater than in finite difference method (FDM) and less than finite element method (FEM) [15] while the accuracy is higher than with FDM and nearly the same as with FEM. Because the method keep conservation law of mass or energy, they are widely used in fluid and underground fluid computations. There are many results about finite volume element methods for elliptic problems and parabolic problems (see [4][6][7][8] and their references). It is a class of important numerical method of much current interest.

Groundwater [16][17] is one of the most important sources of drinking, irrigation and industrial process water. However, groundwater supplies are threatened by organic, inorganic and radioactive contaminant[1] introduced by improper disposal or accidental release. Therefore,

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protection of the quality groundwater supplies and their remediation is a problem with both economical and social significant. It is desirable that the discrete model inherits the conservation of mass in both a local and global sense.

The main goal of the paper is that we develop the finite volume element scheme for a class of groundwater problems which is characterized by nonlinear evolution systems, so that apply it to the real groundwater fluid flow problems in the future. In this paper we present the finite volume element method to solve a class of groundwater problem. Our approach is based on the piecewise quadratic trial functions and piecewise constant test functions and to obtain error estimate in L^2 and H^1.

The model of one kind of positive ion reactions for the problem of groundwater is:

$$mN_1 + r\overline{N_2} \xleftrightarrow[k_2]{k_1} rN_2 + m\overline{N_1}$$

We consider the following model of groundwater [18][19] in 2-space variables:

$$\frac{\partial s_1}{\partial t} - d\Delta s_1 = f_1, \quad (x,y)\in \Omega, t\in J \tag{1.1}$$

$$\frac{\partial s_2}{\partial t} - d\Delta s_2 = f_2, \quad (x,y)\in \Omega, t\in J \tag{1.2}$$

$$\frac{\partial c_1}{\partial t} + \rho\frac{\partial s_1}{\partial t} - D\Delta c_1 = 0, \quad (x,y)\in \Omega, t\in J \tag{1.3}$$

$$\frac{\partial c_2}{\partial t} + \rho\frac{\partial s_2}{\partial t} - D\Delta c_2 = 0, \quad (x,y)\in \Omega, t\in J \tag{1.4}$$

where $\Omega \subset R^2$ is a bounded domain with smoothly boundary. J=(0,T]. s_i and c_i denote by the concentration of $\overline{N_i}$ and N_i (i=1,2), respectively. D>d>0 is the diffusion coefficients, ρ >0 is the density of the solid, which are all constants.

$$f_1 = c_1 s_2^2 - 2c_2^2 s_1 \ , \tag{1.5}$$

$$f_2 = \frac{1}{2}c_2^2 s_1 - c_1 s_2^2 \ , \tag{1.6}$$

$$s_i(x,y,0) = s_i^0(x,y), \quad c_i(x,y,0) = c_i^0(x,y), \quad (x,y)\in \Omega \tag{1.7}$$

$$s_i\big|_{\partial\Omega} = 0, \quad c_i\big|_{\partial\Omega} = 0, \quad t\in J \tag{1.8}$$

The rest of this paper is organized as follows: In section 2, we give some preliminaries. In section 3, we present a full-discrete finite volume element scheme. The error estimates in L^2 and H^1 are presented in section 4 and 5, respectively. In section 6, we carry out numerical experiments to observe the performance of the proposed scheme.

2. PRELIMINARIES

In this paper, we use the standard notation as that in [20], for instance $\|\cdot\|_0$ and $\|\cdot\|_1$ which represent L^2 and H^1 norm, respectively.

Let T_h be a quasi-uniform triangulation of $\overline{\Omega}$. T_h consists of finite number of triangular elements K_Q, Q being the barycenter of the triangle, where h is the maximum length of all the sides.. The vertices of the triangles and the midpoints of the sides are taken as the nodes. $\overline{\Omega_h}$ denotes the set of the vertices of all the triangular elements, $\overline{M_h}$ the set of the midpoints of the sides of all the element, and $\dot{\Omega}_h = \overline{\Omega_h} \setminus \partial\Omega$, $\dot{M}_h = \overline{M_h} \setminus \partial\Omega$.

The dual decomposition of T_h is denoted by T_h^*, consisting of the polygons $K_{P_0}^*$ surrounding the nodes $P_0 \in \overline{\Omega_h}$ and K_M^* surrounding $M \in \overline{M_h}$. Their detailed construction is as follows:

1) Construction of $K_{P_0}^*$. Suppose that $P_0 \in \overline{\Omega_h}$, that $P_i (i = 1,2,\cdots,7)$ are its adjacent vertices, and that P_{0i} is a point on $\overline{P_0 P_i}$ such that $\overline{P_0 P_{0i}} = \frac{1}{3}\overline{P_0 P_i}$. Connect successively $P_{0i} (i = 1,\cdots,7)$ to obtain a polygon $K_{P_0}^*$ surrounding P_0 (see Figure1).

2) Construction of K_M^*. Let $M \in \overline{M_h}$ be a midpoint of a common side of two adjacent triangular elements $K_{Q_1} = \Delta P_0 P_1 P_2$ and $K_{Q_2} = \Delta P_0 P_1 P_3$. Denote by Q_{12}, Q_{13}, Q_{02}, Q_{03} the midpoint of $\overline{P_{01}P_{02}}$, $\overline{P_{01}P_{03}}$, $\overline{P_{10}P_{12}}$, $\overline{P_{10}P_{13}}$ respectively. A polygon K_M^* surrounding M is obtain by connecting successively P_{10}, Q_{03}, Q_2, Q_{13}, P_{01}, Q_{12}, Q_1, Q_{02}, P_{10} (see Figure 2).

Select the trial function space $U_h \subset H_0^1(\Omega)$ as the quadratic element space of Lagrangian type with respect to T_h. The test function space $V_h \subset V \subset L^2(\Omega)$ corresponding to T_h^* is taken as the piecewise constant function space.

First define an interpolation projector $\Pi_h : H_0^1(\Omega) \to U_h$. Next, define an interpolation projector Π_h^*: $U_h \to V_h$ as

$$\Pi_h^* u_h = \sum_{P_0 \in \dot{\Omega}_h} u_h(P_0)\psi_{P_0}(x) + \sum_{M \in \dot{M}_h} u_h(M)\psi_M(x) \tag{2.1}$$

where $\psi_{P_0}(\psi_{p_0} \in \dot{\Omega}_h)$ and $\psi_M(\psi_M \in \dot{M}_h)$, the corresponding basis functions of V_h are the characteristic functions of $K_{P_0}^*$ and K_M^* respectively.

Define a(u,v) as follows:

(1) when (u,v)$\in H_0^1(\Omega)\times H_0^1(\Omega)$

$$a(u,v)=\int_{\Omega}\nabla u\cdot\nabla v dx \tag{2.2}$$

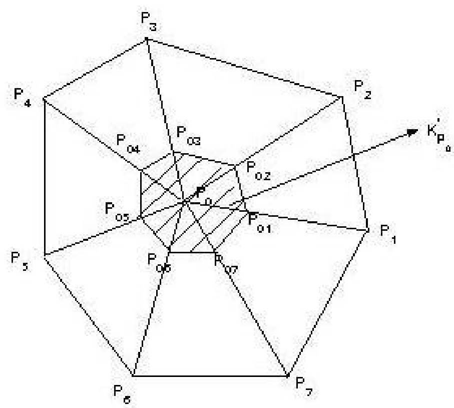

Figure 1

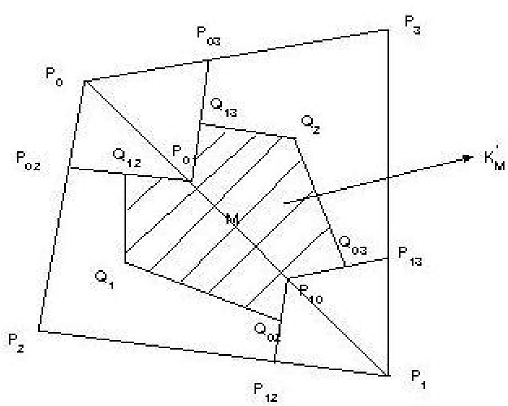

Figure 2

(2) when (u,v)∈ $H_0^1(\Omega)\times V_h$

$$a(u,v)=\sum_{P_0\in\dot{\Omega}_h} v_h(P_0)a(u,\psi_{P_0})+\sum_{M\in\dot{M}_h} v(M)a(u,\psi_M) \tag{2.3}$$

where

$$a(u,\psi_{P_0})=-\int_{\partial K_{P_0}^*}(\frac{\partial u}{\partial x}dy-\frac{\partial u}{\partial y}dx) \tag{2.4}$$

$$a(u,\psi_M)=-\int_{\partial K_M^*}(\frac{\partial u}{\partial x}dy-\frac{\partial u}{\partial y}dx) \tag{2.5}$$

For any $u_h\in U_h$, we introduce the following discrete norms

$$\|u_h\|_{0,h}=(\sum_{K\in T_h}|u_h|^2_{0,h,K})^{\frac{1}{2}} \tag{2.6}$$

$$|u_h|_{1,h}=(\sum_{K\in T_h}|u_h|^2_{1,h,K})^{\frac{1}{2}} \tag{2.7}$$

where

$$|u_h|_{0,h,K}=[(u_{P_i}^2+u_{P_j}^2+u_{P_k}^2+u_{M_i}^2+u_{M_j}^2+u_{M_k}^2)S_Q/6]^{\frac{1}{2}} \tag{2.8}$$

$$|u_h|_{1,h,K}=[(u_{P_i}-u_{M_i})^2+(u_{P_j}-u_{M_j})^2+(u_{P_k}-u_{M_k})^2+(u_{M_i}-u_{M_j})^2+(u_{M_i}-u_{M_k})^2]^{\frac{1}{2}} \tag{2.9}$$

where S_Q is the areas of the triangular element K_Q. P_l and M_l (l=i, j, k) are the vertices and midpoints of the triangular element K_Q.

The following lemmas, which can be found in[12], will be used in our analysis.

Lemma2.1 Within U_h, the norms $\|\cdot\|_{0,h}$ and $|\cdot|_{1,h}$ are equivalent to $\|\cdot\|_0$ and $|\cdot|_1$, namely, there exist positive constants M_1, M_2 independent of U_h, such that

$$M_1|u_h|_{1,h}\le|u_h|_1\le M_2|u_h|_{1,h} \qquad \forall u_h\in U_h \tag{2.10}$$

$$M_1\|u_h\|_{0,h}\le\|u_h\|_0\le M_2\|u_h\|_{0,h} \qquad \forall u_h\in U_h \tag{2.11}$$

Lemma2.2 There hold the following statements

(1) $(u_h,\Pi_h^*\overline{u_h})=(\overline{u_h},\Pi_h^*u_h) \qquad \forall u_h,\overline{u_h}\in U_h$

(2) set $|||u_h|||_0=(u_h,\Pi_h^*u_h)^{\frac{1}{2}}$, then $|||\cdot|||_0$ is equivalent to $\|\cdot\|_0$ on U_h, that is, there exist positive

constant M_3, M_4 such that

$$M_3\|u_h\|_0 \le |||u_h|||_0 \le M_4\|u_h\|_0 \qquad \forall u_h \in U_h \tag{2.12}$$

Lemma2.3 For any $u_h \in U_h$, there exists positive constant M such that

$$a(u_h, \Pi_h^* u_h) \ge M\|u_h\|_1^2$$

3. FULLY-DISCRETE FINITE VOLUME ELEMENT SCHEME

The variational problem related to {1.1}~ {1.4} is: Finding $s_1, s_2, c_1, c_2 \in H_0^1(\Omega)$ such that

$$(\frac{\partial s_i}{\partial t}, v) + da(s_i, v) = (f_i, v) \qquad \forall v \in V, t \in J \tag{3.1}$$

$$(\frac{\partial c_i}{\partial t}, z) + \rho(\frac{\partial s_i}{\partial t}, z) + Da(c_i, z) = 0 \quad \forall z \in V, t \in J \tag{3.2}$$

where $s_i(x,y,0) = s_i^0(x,y), c_i(x,y,0) = c_i^0(x,y), (i=1,2), (x,y) \in \Omega$.

Denote the partition of time interval (0,T] by $0 = t_0 < t_1 < \cdots < t_N = T$ and suppose $\Delta t = \frac{T}{N}$.

The fully-discrete finite volume element scheme is: Finding $S_i^{n+1} \in U_h, C_i^{n+1} \in U_h (i=1,2)$, such that

$$(\frac{S_i^{n+1} - S_i^n}{\Delta t}, v_h) + da(S_i^{n+1}, v_h) = (F_i^n, v_h) \qquad \forall v_h \in V_h \tag{3.3}$$

$$(\frac{C_i^{n+1} - C_i^n}{\Delta t}, z_h) + \rho(\frac{S_i^{n+1} - S_i^n}{\Delta t}, z_h) + Da(C_i^{n+1}, z_h) = 0 \ \forall z_h \in V_h \tag{3.4}$$

where initial data $S_i(0)$ and $C_i(0)$ are given approximation of s_i^0 and c_i^0 in U_h, respectively, (i=1,2). And where

$$F_1^n = C_1^n(S_2^n)^2 - 2(C_2^n)^2 S_1^n \tag{3.5}$$

$$F_2^n = \frac{1}{2} S_1^n (C_2^n)^2 - (S_2^n)^2 C_1^n \tag{3.6}$$

Next we turn to some error estimates for (3.3)~(3.4). For convenience, we denote by $\xi_i = S_i - \Pi_h s_i$, $\eta_i = s_i - \Pi_h s_i$, $\lambda_i = C_i - \Pi_h c_i$, $\theta_i = c_i - \Pi_h c_i$, (i=1,2). By the interpolation theory in Sobolev space [15], we have

$$\|\eta_i\|_0 + h\|\eta_i\|_1 \le Mh^3\|s_i\|_3 \qquad \text{(i=1,2)} \tag{3.7}$$

$$\|\theta_i\|_0 + h\|\theta_i\|_1 \le Mh^3\|c_i\|_3 \qquad \text{(i=1,2)} \tag{3.8}$$

4. ESTIMATES IN L^2 NORMS

From (3.1) ~(3.4), we have the error equations as follows: $(v_h, z_h \in V_h)$

$$(\frac{\xi_i^{n+1}-\xi_i^n}{\Delta t}, v_h) + da(\xi_i^{n+1}, v_h) = (\frac{\eta_i^{n+1}-\eta_i^n}{\Delta t}, v_h) + (\frac{\partial s_i^{n+1}}{\partial t} - \frac{s_i^{n+1}-s_i^n}{\Delta t}, v_h) + da(\eta_i^{n+1}, v_h)$$

$$+(F_i^n - f_i^{n+1}, v_h) \tag{4.1}$$

$$(\frac{\lambda_i^{n+1}-\lambda_i^n}{\Delta t}, z_h) + Da(\lambda_i^{n+1}, z_h) = (\frac{\theta_i^{n+1}-\theta_i^n}{\Delta t}, z_h) + (\frac{\partial c_i^{n+1}}{\partial t} - \frac{c_i^{n+1}-c_i^n}{\Delta t}, z_h) + Da(\theta_i^{n+1}, z_h)$$

$$-\rho(F_i^n - f_i^{n+1}, z_h) + \rho da(\xi_i^{n+1}, z_h) - \rho da(\eta_i^{n+1}, z_h) \tag{4.2}$$

First, we estimate ξ_i. Take i=1, $v_h = \Pi_h^*\xi_1^{n+1}$ in (4.1) to get

$$(\frac{\xi_1^{n+1}-\xi_1^n}{\Delta t}, \Pi_h^*\xi_1^{n+1}) \ge \frac{1}{2\Delta t}(\left\|\xi_1^{n+1}\right\|_0^2 - \left\|\xi_1^n\right\|_0^2) \tag{4.3}$$

$$\mathrm{da}(\xi_1^{n+1}, \Pi_h^*\xi_1^{n+1}) \ge Md\left\|\xi_1^{n+1}\right\|_1^2 \tag{4.4}$$

Using ε inequality, we obtain

$$(\frac{\eta_1^{n+1}-\eta_1^n}{\Delta t}, \Pi_h^*\xi_1^{n+1}) \le Mh^6 + \varepsilon\left\|\xi_1^{n+1}\right\|_0^2 \tag{4.5}$$

$$\frac{\partial s_1^{n+1}}{\partial t} - \frac{s_1^{n+1}-s_1^n}{\Delta t} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t-t_n) s_{1tt}\, dt$$

$$\left\|\frac{\partial s_1^{n+1}}{\partial t} - \frac{s_1^{n+1}-s_1^n}{\Delta t}\right\|_0 \le M\Delta t$$

$$(\frac{\partial s_1^{n+1}}{\partial t} - \frac{s_1^{n+1}-s_1^n}{\Delta t}, \Pi_h^*\xi_1^{n+1}) \le M(\Delta t)^2 + \varepsilon\left\|\xi_1^{n+1}\right\|_0^2 \tag{4.6}$$

Using the conclusion in [12], we have

$$\left|a(\eta_1^{n+1}, v_h)\right| \le M\left\|\eta_1^{n+1}\right\|_1\left\|v_h\right\|_1 \le Mh^2\left\|v_h\right\|_1 \qquad v_h \in V_h$$

So, we have

$$a(\eta_1^{n+1},\Pi_h^*\xi_1^{n+1}) \le Mh^4 + \varepsilon\left\|\xi_1^{n+1}\right\|_1^2 \tag{4.7}$$

Assume that the space step size and time step size satisfying the following condition:

$$\Delta t = o(h) \tag{4.8}$$

It follows from the assumption (4.8) that, we introduce another assumption by introduction:

$$\sup_{0\le n\le N-1}\left\|\xi_1^n\right\|_{0,\infty} \le K_1 \tag{4.9}$$

$$\sup_{0\le n\le N-1}\left\|\lambda_1^n\right\|_{0,\infty} \le K_2 \tag{4.10}$$

where K_1 and K_2 are constants.

Notice that

$$C_1^n(S_2^n)^2 - c_1^{n+1}(s_2^{n+1})^2 = -c_1^{n+1}[(s_2^{n+1})^2-(s_2^n)^2] - (s_2^n)^2(c_1^{n+1}-C_1^n) - \lambda_1^n[(s_2^n)^2-(S_2^n)^2]$$

$$-\Pi_h c_1^n[(s_2^n)^2-(S_2^n)^2] \tag{4.11}$$

So we obtain

$$\left|(C_1^n(S_2^n)^2 - c_1^{n+1}(s_2^{n+1})^2, \Pi_h^*\xi_1^{n+1})\right| \le M[\left\|\xi_2^n\right\|_0^2 + \left\|\lambda_1^n\right\|_0^2 + h^6 + (\Delta t)^2] + \varepsilon\left\|\xi_1^{n+1}\right\|_0^2 \tag{4.12}$$

Similarly, we have

$$\left|(C_2^n(S_1^n)^2 - c_2^{n+1}(s_1^{n+1})^2, \Pi_h^*\xi_1^{n+1})\right| \le M[\left\|\xi_1^n\right\|_0^2 + \left\|\lambda_2^n\right\|_0^2 + h^6 + (\Delta t)^2] + \varepsilon\left\|\xi_1^{n+1}\right\|_0^2 \tag{4.13}$$

Combining (4.3)~(4.7) and (4.12},(4.13} lead to

$$\frac{1}{2\Delta t}(\left\|\xi_1^{n+1}\right\|_0^2 - \left\|\xi_1^n\right\|_0^2) + Md\left\|\xi_1^{n+1}\right\|_1^2$$

$$\le M[\left\|\xi_1^n\right\|_0^2 + \left\|\xi_2^n\right\|_0^2 + \left\|\lambda_1^n\right\|_0^2 + \left\|\lambda_2^n\right\|_0^2 + h^4 + (\Delta t)^2] + \varepsilon\left\|\xi_1^{n+1}\right\|_1^2 \tag{4.14}$$

Choose ε properly and sum from n=0 to N-1. Then we multiply Δt from both sides. Using Lemma 2.2 we have

$$\left\|\xi_1^N\right\|_0^2 + M\Delta t\sum_{n=0}^{N-1}\left\|\xi_1^{n+1}\right\|_1^2 - \left\|\xi_1^0\right\|_0^2$$

$$\le M\{(\Delta t)\sum_{n=0}^{N-1}(\left\|\xi_1^n\right\|_0^2 + \left\|\xi_2^n\right\|_0^2 + \left\|\lambda_1^n\right\|_0^2 + \left\|\lambda_2^n\right\|_0^2) + h^4 + (\Delta t)^2\} \tag{4.15}$$

Similarly, we obtain

$$\left\|\xi_2^N\right\|_0^2 + M\Delta t\sum_{n=0}^{N-1}\left\|\xi_2^{n+1}\right\|_1^2 - \left\|\xi_2^0\right\|_0^2$$

$$\le M\{(\Delta t)\sum_{n=0}^{N-1}(\left\|\xi_1^n\right\|_0^2 + \left\|\xi_2^n\right\|_0^2 + \left\|\lambda_1^n\right\|_0^2 + \left\|\lambda_2^n\right\|_0^2) + h^4 + (\Delta t)^2\} \tag{4.16}$$

Next, we estimate λ_1. Take i=1, $z_h = \Pi_h^* \lambda_1^{n+1}$ in (4.2). Using the same methods employed for proving (4.3)~(4.7) and (4.12), (4.13).

Notice that

$$\rho da(\xi_1^{n+1}, \Pi_h^* \lambda_1^{n+1}) \le \frac{M}{2}\left\|\xi_1^{n+1}\right\|_1^2 + \varepsilon\left\|\lambda_1^{n+1}\right\|_1^2 \tag{4.17}$$

$$\rho da(\eta_1^{n+1}, \Pi_h^* \lambda_1^{n+1}) \le Mh^4 + \varepsilon\left\|\lambda_1^{n+1}\right\|_1^2 \tag{4.18}$$

So, we have

$$\begin{aligned}&\frac{\left\|\lambda_1^{n+1}\right\|_0^2 - \left\|\lambda_1^n\right\|_0^2}{2\Delta t} + M\left\|\lambda_1^{n+1}\right\|_1^2\\ &\le M[\left\|\xi_1^n\right\|_0^2 + \left\|\xi_2^n\right\|_0^2 + \left\|\lambda_1^n\right\|_0^2 + \left\|\lambda_2^n\right\|_0^2 + h^4 + (\Delta t)^2] + \frac{M}{2}\left\|\xi_1^{n+1}\right\|_1^2 + \varepsilon\left\|\lambda_1^{n+1}\right\|_1^2 + \varepsilon\left\|\lambda_1^{n+1}\right\|_0^2\end{aligned} \tag{4.19}$$

Choose ε properly and sum from n=0 to N-1. Then we multiply Δt from both sides. Using Lemma 2.2 we have

$$\begin{aligned}&\left\|\lambda_1^N\right\|_0^2 + \frac{M}{2}\Delta t\sum_{n=0}^{N-1}\left\|\lambda_1^{n+1}\right\|_1^2 - \left\|\lambda_1^0\right\|_0^2 - \frac{M}{2}\sum_{n=0}^{N-1}\Delta t\left\|\xi_1^{n+1}\right\|_1^2\\ &\le M\{(\Delta t)\sum_{n=0}^{N-1}(\left\|\xi_1^n\right\|_0^2 + \left\|\xi_2^n\right\|_0^2 + \left\|\lambda_1^n\right\|_0^2 + \left\|\lambda_2^n\right\|_0^2) + h^4 + (\Delta t)^2\}\end{aligned} \tag{4.20}$$

Similarly, we obtain

$$\begin{aligned}&\left\|\lambda_2^N\right\|_0^2 + \frac{M}{2}\Delta t\sum_{n=0}^{N-1}\left\|\lambda_2^{n+1}\right\|_1^2 - \left\|\lambda_2^0\right\|_0^2 - \frac{M}{2}\sum_{n=0}^{N-1}\Delta t\left\|\xi_2^{n+1}\right\|_1^2\\ &\le M\{(\Delta t)\sum_{n=0}^{N-1}(\left\|\xi_1^n\right\|_0^2 + \left\|\xi_2^n\right\|_0^2 + \left\|\lambda_1^n\right\|_0^2 + \left\|\lambda_2^n\right\|_0^2) + h^4 + (\Delta t)^2\}\end{aligned} \tag{4.21}$$

From (4.15),(4.20), we have

$$\begin{aligned}&\left\|\xi_1^N\right\|_0^2 + \left\|\lambda_1^N\right\|_0^2 + \Delta t\sum_{n=0}^{N-1}(\left\|\xi_1^{n+1}\right\|_1^2 + \left\|\lambda_1^{n+1}\right\|_1^2) - \left\|\xi_1^0\right\|_0^2 - \left\|\lambda_1^0\right\|_0^2\\ &\le M\{(\Delta t)\sum_{n=0}^{N-1}(\left\|\xi_1^n\right\|_0^2 + \left\|\xi_2^n\right\|_0^2 + \left\|\lambda_1^n\right\|_0^2 + \left\|\lambda_2^n\right\|_0^2) + h^4 + (\Delta t)^2\}\end{aligned} \tag{4.22}$$

Similarly, combining (4.16),(4.21), we have

$$\begin{aligned}&\left\|\xi_2^N\right\|_0^2 + \left\|\lambda_2^N\right\|_0^2 + \Delta t\sum_{n=0}^{N-1}(\left\|\xi_2^{n+1}\right\|_1^2 + \left\|\lambda_2^{n+1}\right\|_1^2) - \left\|\xi_2^0\right\|_0^2 - \left\|\lambda_2^0\right\|_0^2\\ &\le M\{(\Delta t)\sum_{n=0}^{N-1}(\left\|\xi_1^n\right\|_0^2 + \left\|\xi_2^n\right\|_0^2 + \left\|\lambda_1^n\right\|_0^2 + \left\|\lambda_2^n\right\|_0^2) + h^4 + (\Delta t)^2\}\end{aligned} \tag{4.23}$$

So, we obtain

$$\left\|\xi_1^N\right\|_0^2+\left\|\xi_2^N\right\|_0^2+\left\|\lambda_1^N\right\|_0^2+\left\|\lambda_2^N\right\|_0^2+\Delta t\sum_{n=0}^{N-1}(\left\|\xi_1^{n+1}\right\|_1^2+\left\|\xi_2^{n+1}\right\|_1^2+\left\|\lambda_1^{n+1}\right\|_1^2+\left\|\lambda_2^{n+1}\right\|_1^2)$$

$$\le M\{(\Delta t)\sum_{n=0}^{N-1}(\left\|\xi_1^n\right\|_0^2+\left\|\xi_2^n\right\|_0^2+\left\|\lambda_1^n\right\|_0^2+\left\|\lambda_2^n\right\|_0^2)+h^4+(\Delta t)^2\}+\left\|\xi_1^0\right\|_0^2+\left\|\xi_2^0\right\|_0^2+\left\|\lambda_1^0\right\|_0^2+\left\|\lambda_2^0\right\|_0^2 \quad (4.24)$$

Initial data $S_i(0)$ and $C_i(0)$ are chosen the interpolation projector $\Pi_h s_i(0)$ and

$\Pi_h c_i(0)$ of s_i^0 and c_i^0,

respectively. So, we get $\xi_i^0=0$ and $\lambda_i^0=0$,(i=1,2).

Using Gronwall inequality yields

$$\left\|\xi_1^N\right\|_0^2+\left\|\xi_2^N\right\|_0^2+\left\|\lambda_1^N\right\|_0^2+\left\|\lambda_2^N\right\|_0^2+\Delta t\sum_{n=0}^{N-1}(\left\|\xi_1^{n+1}\right\|_1^2+\left\|\xi_2^{n+1}\right\|_1^2+\left\|\lambda_1^{n+1}\right\|_1^2+\left\|\lambda_2^{n+1}\right\|_1^2)\le M[h^4+(\Delta t)^2] \quad (4.25)$$

where M depends on $s_i,\dfrac{\partial s_i}{\partial t},c_i,\dfrac{\partial c_i}{\partial t}$.

Next,we turn to proof (4.9), (4.10). When n=0, conclusion holds. If the condition(4.8) holds, using inverse
properly, we have

$$\sup_{0\le n\le N-1}\left\|\xi_1^n\right\|_{0,\infty}\le Mh^{-1}(h^2+\Delta t)\le M(h+\frac{\Delta t}{h})\le K_1$$

Similarly, we have

$$\sup_{0\le n\le N-1}\left\|\lambda_1^n\right\|_{0,\infty} K_2$$

Then (4.9), (4.10) hold.
Combining (3.7),(3.8), (4.25), we have the following theorem.

Theorem 4.1 Let $\{S_1^n,S_2^n,C_1^n,C_2^n\}$ be the solutions to the fully-discrete scheme. If the condition (4.8) holds, then the following error estimate holds

$$\sup_{n\Delta t\le T}\sum_{i=1}^{2}\{\left\|s_i^n-S_i^n\right\|_0^2+\left\|c_i^n-C_i^n\right\|_0^2\}\le M(h^4+(\Delta t)^2) \quad (4.26)$$

5. ESTIMATES IN H^1 NORMS

Also, we have the error equations as follows: $(v_h,z_h\in V_h)$

$$(\frac{\xi_i^{n+1}-\xi_i^n}{\Delta t},v_h)+da(\xi_i^{n+1},v_h)=(\frac{\eta_i^{n+1}-\eta_i^n}{\Delta t},v_h)+(\frac{\partial s_i^{n+1}}{\partial t}-\frac{s_i^{n+1}-s_i^n}{\Delta t},v_h)+da(\eta_i^{n+1},v_h)$$

$$+(F_i^n-f_i^{n+1},v_h) \quad (5.1)$$

$$(\frac{\lambda_i^{n+1}-\lambda_i^n}{\Delta t},z_h)+Da(\lambda_i^{n+1},z_h)=(\frac{\theta_i^{n+1}-\theta_i^n}{\Delta t},z_h)+(\frac{\partial c_i^{n+1}}{\partial t}-\frac{c_i^{n+1}-c_i^n}{\Delta t},z_h)-\rho(\frac{\xi_i^{n+1}-\xi_i^n}{\Delta t},z_h)$$

$$+Da(\theta_i^{n+1},z_h)+\rho(\frac{\eta_i^{n+1}-\eta_i^n}{\Delta t},z_h)+\rho(\frac{\partial s_i^{n+1}}{\partial t}-\frac{s_i^{n+1}-s_i^n}{\Delta t},z_h) \quad (5.2)$$

First, we estimate ξ_i. Take i=1, $v_h=\frac{\Pi_h^*\xi_1^{n+1}-\Pi_h^*\xi_1^n}{\Delta t}$ in (5.1) to get

$$(\frac{\xi_1^{n+1}-\xi_1^n}{\Delta t},\frac{\Pi_h^*\xi_1^{n+1}-\Pi_h^*\xi_1^n}{\Delta t})=\left\|d_t\xi_1^n\right\|_0^2 \quad (5.3)$$

where denote by $d_t\xi_1^n=\frac{\xi_1^{n+1}-\xi_1^n}{\Delta t}$.

$$da(\xi_1^{n+1},\frac{\Pi_h^*\xi_1^{n+1}-\Pi_h^*\xi_1^n}{\Delta t})\geq\frac{d}{2\Delta t}(\left\|\xi_1^{n+1}\right\|_1^2-\left\|\xi_1^n\right\|_1^2) \quad (5.4)$$

$$(\frac{\eta_1^{n+1}-\eta_1^n}{\Delta t},\frac{\Pi_h^*\xi_1^{n+1}-\Pi_h^*\xi_1^n}{\Delta t})\leq Mh^4+\varepsilon\left\|d_t\xi_1^n\right\|_0^2 \quad (5.5)$$

$$(\frac{\partial s_1^{n+1}}{\partial t}-\frac{s_1^{n+1}-s_1^n}{\Delta t},\frac{\Pi_h^*\xi_1^{n+1}-\Pi_h^*\xi_1^n}{\Delta t})\leq M(\Delta t)^2+\varepsilon\left\|d_t\xi_1^n\right\|_0^2 \quad (5.6)$$

Using inverse properly, we have

$$da(\eta_1^{n+1},\frac{\Pi_h^*\xi_1^{n+1}-\Pi_h^*\xi_1^n}{\Delta t})\leq Mh^2\left\|d_t\xi_1^n\right\|_1\leq Mh\left\|d_t\xi_1^n\right\|_0\leq Mh^2+\varepsilon\left\|d_t\xi_1^n\right\|_0^2 \quad (5.7)$$

$$(F_1^n-f_1^{n+1},\frac{\Pi_h^*\xi_1^{n+1}-\Pi_h^*\xi_1^n}{\Delta t})\leq M[\left\|\xi_1^n\right\|_0^2+\left\|\xi_2^n\right\|_0^2+\left\|\lambda_1^n\right\|_0^2+\left\|\lambda_2^n\right\|_0^2+h^4+(\Delta t)^2]+\varepsilon\left\|d_t\xi_1^n\right\|_0^2 \quad (5.8)$$

We combine (5.3)~(5.8), choose ε properly and sum from n=0 to N-1. Using Lemma 2.2 we obtain

$$\left\|\xi_1^N\right\|_1^2+M\Delta t\sum_{n=0}^{N-1}\left\|d_t\xi_1^n\right\|_0^2-\left\|\xi_1^0\right\|_1^2\leq M\{(\Delta t)\sum_{n=0}^{N-1}(\left\|\xi_1^n\right\|_0^2+\left\|\xi_2^n\right\|_0^2+\left\|\lambda_1^n\right\|_0^2+\left\|\lambda_2^n\right\|_0^2)+h^2+(\Delta t)^2\} \quad (5.9)$$

Similarly, we have

$$\left\|\xi_2^N\right\|_1^2+M\Delta t\sum_{n=0}^{N-1}\left\|d_t\xi_2^n\right\|_0^2-\left\|\xi_2^0\right\|_1^2\leq M\{(\Delta t)\sum_{n=0}^{N-1}(\left\|\xi_1^n\right\|_0^2+\left\|\xi_2^n\right\|_0^2+\left\|\lambda_1^n\right\|_0^2+\left\|\lambda_2^n\right\|_0^2)+h^2+(\Delta t)^2\} \quad (5.10)$$

Next, we estimate λ_1. Take i=1, $z_h=\frac{\Pi_h^*\lambda_1^{n+1}-\Pi_h^*\lambda_1^n}{\Delta t}$ in (5.2). Using the same methods employed for proving (5.3)~(5.7).

Notice that

$$\rho\left(\frac{\xi_1^{n+1}-\xi_1^n}{\Delta t},\frac{\Pi_h^*\lambda_1^{n+1}-\Pi_h^*\lambda_1^n}{\Delta t}\right)\le\frac{M}{2}\left\|d_t\xi_1^n\right\|_0^2+\varepsilon\left\|d_t\lambda_1^n\right\|_0^2 \tag{5.11}$$

$$\rho\left(\frac{\eta_1^{n+1}-\eta_1^n}{\Delta t},\frac{\Pi_h^*\lambda_1^{n+1}-\Pi_h^*\lambda_1^n}{\Delta t}\right)\le Mh^4+\varepsilon\left\|d_t\lambda_1^n\right\|_0^2 \tag{5.12}$$

$$\rho\left(\frac{\partial s_1^{n+1}}{\partial t}-\frac{s_1^{n+1}-s_1^n}{\Delta t},\frac{\Pi_h^*\lambda_1^{n+1}-\Pi_h^*\lambda_1^n}{\Delta t}\right)\le M(\Delta t)^2+\varepsilon\left\|d_t\lambda_1^n\right\|_0^2 \tag{5.13}$$

So, we have

$$\left\|\lambda_1^N\right\|_1^2+\frac{M}{2}\Delta t\sum_{n=0}^{N-1}\left\|d_t\lambda_1^n\right\|_0^2\le M\left(\frac{\Delta t}{2}\sum_{n=0}^{N-1}\left\|d_t\xi_1^n\right\|_0^2+h^2+(\Delta t)^2\right)+\left\|\lambda_1^0\right\|_1^2 \tag{5.14}$$

Similarly, we obtain

$$\left\|\lambda_2^N\right\|_1^2+\frac{M}{2}\Delta t\sum_{n=0}^{N-1}\left\|d_t\lambda_2^n\right\|_0^2\le M\left(\frac{\Delta t}{2}\sum_{n=0}^{N-1}\left\|d_t\xi_2^n\right\|_0^2+h^2+(\Delta t)^2\right)+\left\|\lambda_2^0\right\|_1^2 \tag{5.15}$$

From (5.9), (5.14), we have

$$\left\|\xi_1^N\right\|_1^2+\left\|\lambda_1^N\right\|_1^2+\Delta t\sum_{n=0}^{N-1}\left(\left\|d_t\xi_1^n\right\|_0^2+\left\|d_t\lambda_1^n\right\|_0^2\right)-\left\|\xi_1^0\right\|_1^2-\left\|\lambda_1^0\right\|_1^2$$

$$\le M\{(\Delta t)\sum_{n=0}^{N-1}\left(\left\|\xi_1^n\right\|_1^2+\left\|\xi_2^n\right\|_1^2+\left\|\lambda_1^n\right\|_1^2+\left\|\lambda_2^n\right\|_1^2\right)+h^2+(\Delta t)^2\} \tag{5.16}$$

From (5.10), (5.15), we obtain

$$\left\|\xi_2^N\right\|_1^2+\left\|\lambda_2^N\right\|_1^2+\Delta t\sum_{n=0}^{N-1}\left(\left\|d_t\xi_2^n\right\|_0^2+\left\|d_t\lambda_2^n\right\|_0^2\right)-\left\|\xi_2^0\right\|_1^2-\left\|\lambda_2^0\right\|_1^2$$

$$\le M\{(\Delta t)\sum_{n=0}^{N-1}\left(\left\|\xi_1^n\right\|_1^2+\left\|\xi_2^n\right\|_1^2+\left\|\lambda_1^n\right\|_1^2+\left\|\lambda_2^n\right\|_1^2\right)+h^2+(\Delta t)^2\} \tag{5.17}$$

So, we have

$$\left\|\xi_1^N\right\|_1^2+\left\|\xi_2^N\right\|_1^2+\left\|\lambda_1^N\right\|_1^2+\left\|\lambda_2^N\right\|_1^2+\Delta t\sum_{n=0}^{N-1}\left(\left\|d_t\xi_1^n\right\|_0^2+\left\|d_t\xi_2^n\right\|_0^2+\left\|d_t\lambda_1^n\right\|_0^2+\left\|d_t\lambda_2^n\right\|_0^2\right)$$

$$\le M\{(\Delta t)\sum_{n=0}^{N-1}\left(\left\|\xi_1^n\right\|_1^2+\left\|\xi_2^n\right\|_1^2+\left\|\lambda_1^n\right\|_1^2+\left\|\lambda_2^n\right\|_1^2\right)+h^2+(\Delta t)^2\}+\left\|\xi_1^0\right\|_1^2+\left\|\xi_2^0\right\|_1^2+\left\|\lambda_1^0\right\|_1^2+\left\|\lambda_2^0\right\|_1^2 \tag{5.18}$$

Initial data $S_i(0)$ and $C_i(0)$ are chosen the interpolation projector $\Pi_h s_i(0)$ and $\Pi_h c_i(0)$ of s_i^0 and c_i^0, respectively. So, we get $\xi_i^0=0$ and $\lambda_i^0=0$,(i=1,2).

Using Gronwall inequality yields

$$\left\|\xi_1^N\right\|_1^2+\left\|\xi_2^N\right\|_1^2+\left\|\lambda_1^N\right\|_1^2+\left\|\lambda_2^N\right\|_1^2+\Delta t\sum_{n=0}^{N-1}(\left\|d_t\xi_1^n\right\|_0^2+\left\|d_t\xi_2^n\right\|_0^2+\left\|d_t\lambda_1^n\right\|_0^2+\left\|d_t\lambda_2^n\right\|_0^2)$$

$$\leq M\,[h^2+(\Delta t)^2] \tag{5.19}$$

where M depends on $s_i, \dfrac{\partial s_i}{\partial t}, c_i, \dfrac{\partial c_i}{\partial t}$.

Combining (3.7),(3.8), (5.19), we have the following theorem.

Theorem 5.1 Let $\{S_1^n, S_2^n, C_1^n, C_2^n\}$ be the solutions to the fully-discrete scheme. If the condition (4.8) holds, then the following error estimate holds

$$\sup_{n\Delta t\leq T}\sum_{i=1}^{2}\{\left\|s_i^n-S_i^n\right\|_0^2+\left\|c_i^n-C_i^n\right\|_0^2\}\leq M(h^2+(\Delta t)^2) \tag{5.20}$$

6. NUMERICAL EXPERIMENT

In this section, we provide a numerical example to illustrate the effectiveness of scheme (3.3), (3.4).

Consider the following problem:

$$\frac{\partial s}{\partial t}-\frac{5}{2}\Delta s=\frac{1}{2}c-s \qquad (x,y)\in\Omega \quad t\in J \tag{6.1}$$

$$\frac{\partial c}{\partial t}+\frac{\partial s}{\partial t}-12\Delta c=0 \qquad (x,y)\in\Omega \quad t\in J \tag{6.2}$$

$$s|_{\partial\Omega}=0 \qquad t\in J \tag{6.3}$$

$$c|_{\partial\Omega}=0 \qquad t\in J \tag{6.4}$$

where $\Omega=(0,2\pi)\times(0,2\pi)$, J=(0,1]. Let space step size be h and time step size be τ. We consider the case of the exact solution $c(x,y,t)=e^{-2t}\sin\frac{1}{2}x\sin\frac{1}{2}y$ and $s(x,y,t)=2e^{-2t}\sin\frac{1}{2}x\sin\frac{1}{2}y$, respectively.

The initial function $s(x,y,0)=2\sin\frac{1}{2}x\sin\frac{1}{2}y$ and $c(x,y,0)=\sin\frac{1}{2}x\sin\frac{1}{2}y$, respectively. In our numerical experiment, we place a right triangular decomposition on Ω as shown in Figure 3.

We choose $h=\dfrac{\pi}{5}$ and $\tau=\dfrac{1}{5}$. The numerical results are shown in Table 1. The error line of the finite volume element solution is shown in Figure 4.

Table 1. Comparison of approximation errors of s and c

Point	Variable	True Solution	Finite Volume Element Solution
$(\frac{\pi}{10},\frac{\pi}{10})$	S	0.006624	0.007218
	C	0.003312	0.004364
$(\frac{\pi}{5},\frac{\pi}{5})$	S	0.025847	0.028251
	C	0.012923	0.017063
$(\frac{2\pi}{5},\frac{2\pi}{5})$	S	0.093514	0.102388
	C	0.046757	0.061736
$(\frac{3\pi}{5},\frac{3\pi}{5})$	S	0.177156	0.194122
	C	0.088578	0.116964
$(\frac{4\pi}{5},\frac{4\pi}{5})$	S	0.244824	0.268371
	c	0.122412	0.161648

We specially evaluate the maximum absolute errors (MAE) and the average absolute errors (ABE) of finite volume element method of quadratic element (FVEM2) and linear element (FVEM1) with different space and time step, when $t_n = 1$, which is shown in Table 2.

It is observed from Table 2 that the accuracy of FVEM2 is higher than FVEM1, and the numerical results support our theory.

Table 2. Comparison of FVEM with different space and time step

Space step and time step	Method	MAE	ABE
$h=\frac{2\pi}{5}, t=\frac{1}{5}$	FVEM1	0.071687	0.012683
	FVEM2	0.043237	0.010988
$h=\frac{\pi}{5}, t=\frac{1}{10}$	FVEM1	0.043215	0.008806
	FVEM2	0.020377	0.005970
$h=\frac{\pi}{6}, t=\frac{1}{12}$	FVEM1	0.036654	0.007739
	FVEM2	0.016772	0.005024

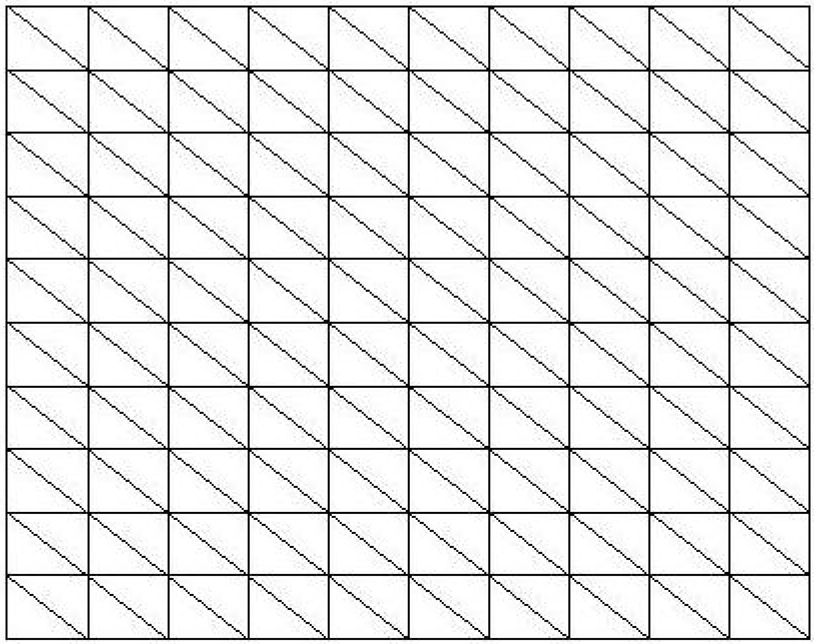

Figure 3

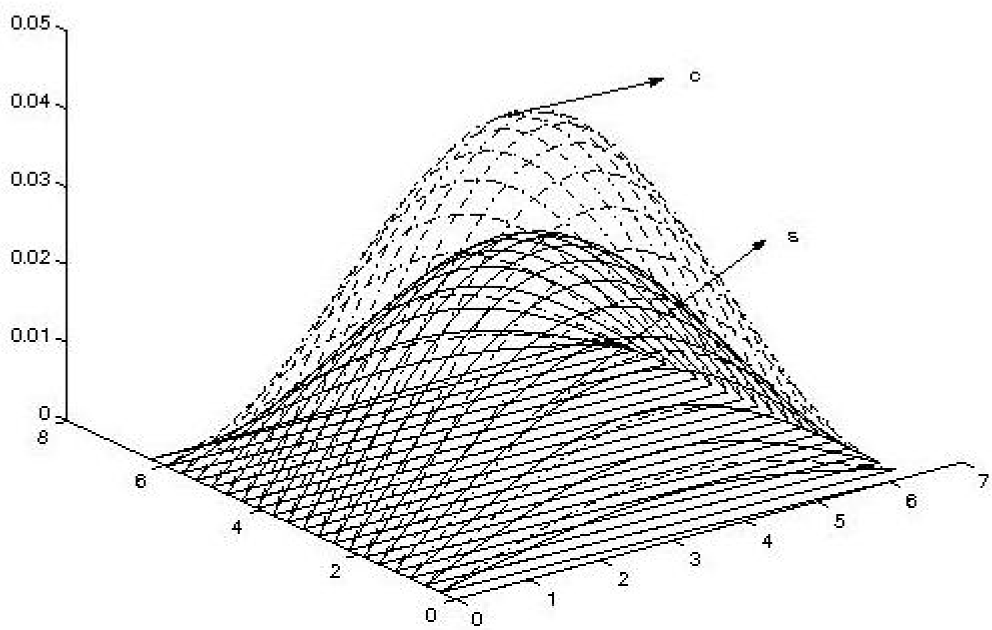

Figure 4

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