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Numerical Methods for Systems of Nonlinear Differential Functional Equations

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Abstract: *The paper deals with initial boundary value problems for nonlinear differential functional systems. We are interested in approximation of solutions of considered differential problems by solutions of suitable difference schemes. A complete convergence analysis for the methods is presented. The proof of the stability is based on a comparison technique with nonlinear estimates of the Perron type for given operators.*

Key Words Initial boundary value problems, difference functional equations, difference methods, stability and convergence, interpolating operators, error estimates.

1 INTRODUCTION

For any metric spaces X and Y we denote by $C(X, Y)$ the class of all continuous functions from X to Y. We denote by $M_{k\times n}$ the space of real $k \times n$ matrices. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

For each $x = (x_1, \dots, x_n) \in R^n$ we write $x = (x', x'')$ where $x' = (x_1, \dots, x_\kappa)$, $x'' = (x_{\kappa+1}, \dots, x_n)$, where $0 \le \kappa \le n$ is fixed. If $\kappa = n$ we have $x' = x$, if $\kappa = 0$ then $x'' = x$. We define the sets

$$E = [0, a] \times [-b', b') \times (-b'', b''], \quad D = [-d_0, 0] \times [0, d'] \times [-d'', 0]$$

where $a > 0$, $d_0 \in R_+$, $b = (b_1, \dots, b_n) \in R^n_+$ and $d = (d_1, \dots, d_n) \in R^n_+$ are given. Let $c = (c_1, \dots, c_n) = b + d$ and

$$E_0 = [-d_0, 0] \times [-b', c'] \times [-c'', b''],$$

$$\partial_0 E = ((0, a] \times [-b', c'] \times [-c'', b'']) \setminus E, \quad E^* = E_0 \cup E \cup \partial_0 E.$$

For $z : E^* \to R^k$, $z = (z_1, \dots, z_k)$, and $(t, x) \in [0, a] \times [-b, b]$ we define the function $z_{(t,x)} : D \to R^k$ as follows

$$z_{(t,x)}(\tau, y) = z(t + \tau, x + y), \quad (\tau, y) \in D.$$

The function $z_{(t,x)}$ is the restriction of z to the set $[t - d_0, t] \times [x', x' + d'] \times [x'' - d'', x'']$ and this restriction is shifted to the set D. Put $\Omega = E \times C(D, R^k) \times R^n$ and suppose that

$$f : \Omega \to R^k, \quad f = (f_1, \dots, f_k),$$

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$$\varphi : E_0 \cup \partial_0 E \to R^k, \quad \varphi = (\varphi_1, \dots, \varphi_k)$$

are given functions. We consider the system of differential functional equations

$$\partial_t z_i(t,x) = f_i(t,x,z_{(t,x)}, \partial_x z_i(t,x)), \quad 1 \le i \le k, \tag{1.1}$$

with the initial boundary condition

$$z(t,x) = \varphi(t,x) \ \text{ on } \ E_0 \cup \partial_0 E, \tag{1.2}$$

where $\partial_x z_i(t,x) = (\partial_{x_1} z_i(t,x), \dots, \partial_{x_n} z_i(t,x))$, $1 \le i \le k$. A function $v : E^* \to R^k$ is called a classical solution of problem (1.1), (1.2) if

(i) $v \in C(E^*, R^k)$ and v is of class C^1 on E,

(ii) $v = (v_1, \dots, v_k)$ satisfies system of equations (1.1) on E and condition (1.2) holds.

Systems of differential equations with deviated variables and differential integral problems can be derived from (1.1) by specializing the operator f.

2 DIFFERENCE FUNCTIONAL PROBLEMS

We are interested in the construction of a method for the approximation of classical solutions to problem (1.1), (1.2) with solutions of associated difference scheme and in the error estimate for the constructed method.

Let us denote by $F(X,Y)$ the class of all functions defined on X and taking values in Y, where X and Y are arbitrary sets. Let **N** and **Z** be the sets of natural numbers and integers, respectively. For $x = (x_1, \dots, x_n) \in R^n$, $p = (p_1, \dots, p_k) \in R^k$ and for the matrix

$$U \in M_{k\times n}, \quad U = [u_{ij}]_{i=1,\dots,k, j=1,\dots,n}$$

we write

$$\|x\| = \sum_{i=1}^n |x_i| \quad \text{and} \quad \|p\|_\infty = \max\{|p_i| : 1 \le i \le k\},$$

$$\|U\| = \max\Big\{ \sum_{j=1}^n |u_{ij}| : 1 \le i \le k \Big\}.$$

For a function $w \in C(D, R^k)$ we put

$$\|w\|_D = \max\{\|w(t,x)\|_\infty : (t,x) \in D\}.$$

We define a mesh on the set E^* and D in the following way. Let (h_0, h'), $h' = (h_1, \dots, h_n)$, stand for steps of the mesh. For $h = (h_0, h')$ and $(r,m) \in \mathbf{Z}^{1+r}$, where $m = (m_1, \dots, m_n)$, we define nodal points as follows

$$t^{(r)} = rh_0, \ \ x^{(m)} = (x_1^{(m_1)}, \dots, x_n^{(m_n)}) = (m_1 h_1, \dots, m_n h_n).$$

Let us denote by H the set of all $h = (h_0, h')$ such that there are $K_0 \in \mathbf{Z}$ and $N = (N_1, \ldots, N_n) \in \mathbf{N}^n$ with the properties $K_0 h_0 = d_0$ and $(N_1 h_1, \ldots, N_n h_n) = d$. Let $K \in \mathbf{N}$ be defined by the relations $K h_0 \le a < (K+1) h_0$. Write

$$R_h^{1+n} = \{(t^{(r)}, x^{(m)}) : (r, m) \in \mathbf{Z}^{1+n}\}$$

and

$$E_h = E \cap R_h^{1+n}, \quad E_{h.0} = E_0 \cap R_h^{1+n}, \quad D_h = D \cap R_h^{1+n},$$
$$\partial_0 E_h = \partial_0 E \cap R_h^{1+n}, \quad E_h^* = E_{h.0} \cup E_h \cup \partial_0 E_h.$$

For functions $z : E_h^* \to R^k$ and $w : D_h \to R^k$ we write

$$z^{(r,m)} = z(t^{(r)}, x^{(m)}) \ \text{ on } \ E_h^* \quad \text{and} \quad w^{(r,m)} = w(t^{(r)}, x^{(m)}) \ \text{ on } \ D_h.$$

For the above z and for a point $(t^{(r)}, x^{(m)}) \in E_h$ we define the function $z_{[r,m]} : D_h \to R^k$ by the formula

$$z_{[r,m]}(\tau, y) = z(t^{(r)} + \tau, x^{(m)} + y), \quad (\tau, y) \in D_h.$$

The function $z_{[r,m]}$ is the restriction of z to the set

$$([t^{(r)} - d_0] \times [x^{(m')}, x^{(m')} + d'] \times [x^{(m'')} - d'', x^{(m'')}]) \cap R_h^{1+n}$$

and the restriction is shifted to the set D_h. For a function $w : D_h \to R^k$ we write

$$\|w\|_{D_h} = \max\{\|w^{(r,m)}\|_\infty : (t^{(r)}, x^{(m)}) \in D_h\}.$$

Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n$, $1 \le j \le n$, where 1 is the j-th coordinate. We consider difference operators δ_0 and $\delta = (\delta_1, \ldots, \delta_n)$ defined in the following way

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\Big(z^{(r+1,m)} - z^{(r,m)}\Big) \tag{2.1}$$

and

$$\delta_j z^{(r,m)} = \frac{1}{h_j}\Big(z^{(r,m+e_j)} - z^{(r,m)}\Big), \quad 1 \le j \le \kappa, \tag{2.2}$$

$$\delta_j z^{(r,m)} = \frac{1}{h_j}\Big(z^{(r,m)} - z^{(r,m-e_j)}\Big), \quad \kappa + 1 \le j \le n. \tag{2.3}$$

Note that $\delta z^{(r,m)}$ is given by (2.3) if $\kappa = 0$ and $\delta z^{(r,m)}$ is defined by (2.2) for $\kappa = n$.

Right-hand sides of equations (1.1) contain the functional variable $z_{(t,x)}$ which is the element of the space $C(D, R^k)$. Therefore we need an interpolating operator $T_h : F(D_h, R^k) \to F(D, R^k)$ and the following assumptions on the operator T_h.

Assumption $H[T_h]$. Suppose that the operator $T_h : F(D_h, R^k) \to F(D, R^k)$ satisfies the conditions

1) if $w, \widetilde{w} \in F(D_h, R^k)$ then $T_h[w], T_h[\widetilde{w}] \in C(D, R^k)$ and

$$\|T_h[w] - T_h[\widetilde{w}]\|_D \le \|w - \widetilde{w}\|_{D_h}, \tag{2.4}$$

2) if $w : D \to R^k$ is of class C^1 then there is $\gamma : H \to R_+$ such that

$$\| T_h[w_h] - w \|_D \leq \gamma(h) \text{ and } \lim_{h \to 0} \gamma(h) = 0, \tag{2.5}$$

where w_h is the restriction of w to the set D_h.

Remark 2.1 The condition 1) of Assumption $H[T_h]$ states that the operator T_h satisfies the Lipschitz condition with the coefficient equal to 1.

Assumption (2.5) implies that the function w is approximated by $T_h[w_h]$ and the error of this approximation is estimated by $\gamma(h)$.

We formulate a difference problem corresponding to (1.1), (1.2). Write

$$\delta_0 z = (\delta_0 z_1, \ldots, \delta_0 z_k),$$
$$F[z]^{(r,m)} = (F_1[z]^{(r,m)}, \ldots, F_k[z]^{(r,m)})$$

and

$$F_i[z]^{(r,m)} = f_i(t^{(r)}, x^{(m)}, T_h z_{[r,m]}, s_i \delta z_i^{(r,m)} + (1 - s_i)\delta z_i^{(r+1,m)}), \quad 1 \leq i \leq k,$$

where

$$s_i \delta z_i^{(r,m)} = (s_{i1}\delta_1 z_i^{(r,m)}, \ldots, s_{in}\delta_n z_i^{(r,m)}),$$
$$(1 - s_i)\delta z_i^{(r+1,m)} = ((1 - s_{i1})\delta_1 z_i^{(r+1,m)}, \ldots, (1 - s_{in})\delta_n z_i^{(r+1,m)})$$

and where $0 \leq s_{ij} \leq 1$, $i = 1, \ldots, k$, $j = 1, \ldots, n$, are given constants. We consider the difference functional system

$$\delta_0 z^{(r,m)} = F[z]^{(r,m)} \tag{2.6}$$

with initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \text{ on } E_{h.0} \cup \partial_0 E_h \tag{2.7}$$

where $\varphi_h : E_{h.0} \cup \partial_0 E_h \to R^k$, $\varphi_h = (\varphi_{h.1}, \ldots, \varphi_{h.k})$, are given function.

Classical difference methods for partial differential or functional differential equations consist in replacing partial derivatives by difference expressions. Then the original problem is transformed into difference or difference functional equations.

In recent years, a number of papers concerning numerical methods for functional partial differential equations have been published.

Difference methods for nonlinear first order partial functional equations were studied in [1], [2], [3], [4], [6]. The main question in this investigations is to construct a difference functional equation which satisfies consistency conditions with respect to the original problem and to find sufficient conditions for the stability of the difference schemes.

Our difference functional problems have the following property: each equation in system (2.6) contains the parameters $s_i = (s_{i1}, \ldots, s_{in})$, $1 \leq i \leq k$. If $s_i = (0, \ldots, 0) \in R^n$ for

$1 \le i \le k$ then (2.6), (2.7) reduces to the explicit difference scheme. It is clear that there exists exactly one solution of problem (2.6), (2.7) in this case. Sufficient conditions for the convergence of the explicit difference methods for first order partial differential equations can be found in the monograph [5] (Chapter V).

Note that if $k = 1$ and $s = (s_1, \ldots, s_n) = (1, \ldots, 1) \in R^n$ then (2.6), (2.7) reduces to the implicit difference scheme considered in [7].

Numerical methods for nonlinear parabolic problems were investigated in [8]-[10]. Difference schemes considered in the above papers depend on two parameters $s, \widetilde{s} \in [0, 1]$. Right hand sides of difference equations corresponding to parabolic equations contain the expressions

$$s\delta z^{(r,m)} + (1-s)\delta z^{(r+1,m)} \quad \text{and} \quad \widetilde{s}\delta^{(2)} z^{(r,m)} + (1-\widetilde{s})\delta^{(2)} z^{(r+1,m)},$$

where $\delta = (\delta_1, \ldots, \delta_n)$ and $\delta^{(2)} = [\delta_{ij}]_{i,j=1,\ldots,n}$ are difference operators corresponding to the partial derivatives $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ and $\partial_{xx} = [\partial_{x_i x_j}]_{i,j=1\ldots,n}$ and z is a scalar unknown function. Our results are motivated by papers [8]-[10].

In the paper we start the investigations of difference schemes for nonlinear mixed problems. We prove that under natural assumptions on given functions and on the mesh there is a class of difference schemes for a mixed problem which is convergent.

The paper is organized as follows. In Section 2 we construct a class of difference schemes for (1.1), (1.2). The convergence theorem and an error estimate for considering difference methods are presented in Section 3.

3 SOLVABILITY AND CONVERGENCE OF DIFFERENCE METHODS

We first prove that there exists exactly one solution $u_h : E_h^* \to R^k$ of problem (2.6), (2.7). For each $x^{(m)} \in B_h$ we put

$$\Delta^{(m)} = \{x^{(m+e_j)} : 1 \le j \le \kappa\} \cup \{x^{(m-e_j)} : \kappa + 1 \le j \le n\}.$$

Assumption $H[f]$. Suppose that the function $f : \Omega \to R^k$, $f = (f_1, \ldots, f_k)$, of the variables (t, x, w, q) is such that

1) for each $P = (t, x, w, q) \in \Omega$ there exist partial derivatives

$$\partial_q f(P) = [\partial_{q_j} f_i(P)]_{i=1,\ldots,k, j=1,\ldots,n}$$

and $\partial_q f_h(t, x, w, \cdot) \in C(R^n, M_{k\times n})$,

2) for each $P \in \Omega$ and for $i = 1, \ldots, k$ the estimates

$$\partial_{q_j} f_i(P) \ge 0 \quad \text{for} \quad 1 \le j \le \kappa, \qquad \partial_{q_j} f_i(P) \le 0 \quad \text{for} \quad \kappa + 1 \le j \le n$$

are satisfied.

Lemma 3.1 *If Assumption $H[f]$ is satisfied and $\varphi_h : E_{h.0} \cup \partial_0 E_h \to R^k$ then there exists exactly one solution $u_h : E_h^* \to R^k$ of* (2.6), (2.7).

Proof. Is follows from (2.7) that u_h is defined on $E_{h.0}$. Suppose that $0 \le r < K$ is fixed and that $u_{h.i}$ is defined on $E_h^* \cap ([-d_0, t^r] \times R^n)$ for $1 \le i \le k$. Assume now that i is fixed, $1 \le i \le k$. Consider the problem

$$z_i^{(r+1,m)} = u_{h.i}^{(r,m)} + h_0 f_{h.i}(t^{(r)}, x^{(m)}, (u_h)_{[r,m]}, s_i \delta u_i^{(r,m)} + (1 - s_i)\delta z_i^{(r+1,m)}) \tag{3.1}$$

$$u_{h.i}^{(r+1,m)} = \varphi_{h.i}^{(r+1,m)} \ \text{ for } \ x^{(m)} \in \partial_0 B_h. \tag{3.2}$$

Suppose now that the numbers $u_{h.i}(t^{(r+1)}, y)$ where $y \in \Delta^{(m)}$ are known. Write

$$\psi_i(\tau) = u_{h.i}^{(r,m)} + h_0 f_{h.i}(t^{(r)}, x^{(m)}, (u_h)_{[r,m]}, Q_i^{(r,m)}(\tau)),$$

where

$$\begin{aligned} Q_i^{(r,m)}(\tau) = \Bigg(& \frac{1}{h_1}\Big(s_{i1}(u_{h.i}^{(r,m+e_1)} - u_{h.i}^{(r,m)}) + (1 - s_{i1})(u_{h.i}^{(r+1,m+e_1)} - \tau)\Big), \dots, \\ & \frac{1}{h_\kappa}\Big(s_{i\kappa}(u_{h.i}^{(r,m+e_\kappa)} - u_{h.i}^{(r,m)}) + (1 - s_{i\kappa})(u_{h.i}^{(r+1,m+e_\kappa)} - \tau)\Big), \\ & \frac{1}{h_{\kappa+1}}\Big(s_{i\kappa+1}(u_{h.i}^{(r,m)} - u_{h.i}^{(r,m-e_{\kappa+1})}) - (1 - s_{i\kappa+1})(\tau - u_{h.i}^{(r+1,m-e_{\kappa+1})})\Big), \dots, \\ & \frac{1}{h_n}\Big(s_{in}(u_{h.i}^{(r,m)} - u_{h.i}^{(r,m-e_n)}) - (1 - s_{in})(\tau - u_{h.i}^{(r+1,m-e_n)})\Big)\Bigg), \end{aligned}$$

Then $\psi = (\psi_1, \dots, \psi_k) : R \to R^k$ is of class C^1 and

$$\psi_i'(\tau) = -h_0 \sum_{j=1}^{n} \frac{1}{h_j}(1 - s_{ij})|\partial_{q_j} f_{h.i}(t^{(r)}, x^{(m)}, (u_h)_{[r,m]}, Q_i^{(r,m)}(\tau))| \le 0$$

for $\tau \in R$. Then equation $\tau = \psi_i(\tau)$ has exactly one solution and consequently the number $u_{h.i}^{(r+1,m)}$ can be calculated. Since $u_{h.i}^{(r+1,m)}$ is given for $x^{(m)} \in \partial_0 B_h$ it follows that there exists exactly one solution $u_{h.i}^{(r+1,m)}$ of (3.3), (3.4) for $x^{(m)} \in B_h$. Then $u_{h.i}$ is defined on $E_{h.r+1}$. Then by induction the solution exists and it is unique on E_h^*. □

We proof now the convergence of the difference method (2.6), (2.7).

Assumption $H[f, \sigma]$. Suppose that

1) the function $\sigma : [0, a] \times R_+ \to R_+$ satisfies the following conditions:

(i) $\sigma(t, \cdot) : R_+ \to R_+$ is continuous and nondecreasing for each $t \in [0, a]$,

(ii) the maximal solution of the Cauchy problem

$$w'(t) = \sigma(t, w(t)), \quad w(0) = 0, \tag{3.3}$$

is $\widetilde{w}(t) = 0$ for $t \in [0, a]$,

2) the estimate

$$\|f(t,x,w,q) - f(t,x,\bar{w},q)\|_\infty \le \sigma(t, \|w-\bar{w}\|_D) \tag{3.4}$$

is satisfied on Ω.

Theorem 3.2 *Suppose that Assumptions $H[f]$, $H[f,\sigma]$ are satisfied and*

1) the function $v : E^* \to R^k$ is a solution of (1.1), (1.2) *and is of class C^1,*

2) $h \in H$ and the function $u_h : E_h^* \to R^k$ is a solution of (2.6), (2.7) *and there is $\alpha_0 : H \to R_+$ such that*

$$\|\varphi^{(r,m)} - \varphi_h^{(r,m)}\|_\infty \le \alpha_0(h) \quad \textit{on} \quad E_{h.0} \cup \partial_0 E_h \quad \textit{and} \quad \lim_{h\to 0} \alpha_0(h) = 0,$$

3) the operator $T_h : F(D_h, R^k) \to C(D, R^k)$ satisfies the Assumption $H[T_h]$,

4) for $P \in \Omega$ we have

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} s_{ij} |\partial_{q_j} f_i(P)| \ge 0, \quad 1 \le i \le k. \tag{3.5}$$

Then there exists a function $\alpha : H \to R_+$ such that

$$\|v_h^{(r,m)} - u_h^{(r,m)}\|_\infty \le \alpha(h) \quad \textit{on} \quad E_h \quad \textit{and} \quad \lim_{h\to 0} \alpha(h) = 0. \tag{3.6}$$

Proof. Let the function $\Gamma_h : E_h' \to R^k$ be defined by

$$\delta_0 v_h^{(r,m)} = F[v_h]^{(r,m)} + \Gamma_h^{(r,m)} \quad \text{on} \quad E_h'. \tag{3.7}$$

It follows from the assumptions of theorem that there exists a function $\widetilde{\gamma} : H \to R_+$ such that

$$\|\Gamma_h^{(r,m)}\|_\infty \le \widetilde{\gamma}(h) \quad \text{on} \quad E_h' \quad \text{and} \quad \lim_{h\to 0} \widetilde{\gamma}(h) = 0.$$

We write

$$\varepsilon_h^{(r)} = \max\{\|(v_h - u_h)^{(i,m)}\|_\infty : (t^{(i)}, x^{(m)}) \in E_h^* \cap ([-d_0, t^{(r)}] \times R^n)\}.$$

It follows from the definitions of difference operator δ_0 and δ that

$$(v_{h.i} - u_{h.i})^{(r+1,m)} \Big[1 + h_0 \sum_{j=1}^{n} \frac{1}{h_j}(1-s_{ij})|\partial_{q_j} f_i(P_{ij})|\Big]$$

$$= (v_{h.i} - u_{h.i})^{(r,m)} \Big[1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} s_{ij} |\partial_{q_j} f_i(P_{ij})|\Big]$$

$$+h_0 \sum_{j=1}^{\kappa} \partial_{q_j} f_i(P_{ij}) \Big[s_{ij} \frac{1}{h_j}(v_{h.i} - u_{h.i})^{(r,m+e_j)} + (1-s_{ij}) \frac{1}{h_j}(v_{h.i} - u_{h.i})^{(r+1,m+e_j)}\Big]$$

$$-h_0 \sum_{j=\kappa+1}^{n} \partial_{q_j} f_i(P_{ij}) \Big[s_{ij} \frac{1}{h_j}(v_{h.i} - u_{h.i})^{(r,m-e_j)} + (1-s_{ij}) \frac{1}{h_j}(v_{h.i} - u_{h.i})^{(r+1,m-e_j)}\Big]$$

$$+h_0\Big[f_i(t^{(r)},x^{(m)},(v_h)_{(t^{(r)},x^{(m)})},s_i\delta v_{h.i}^{(r,m)}+(1-s_i)\delta v_{h.i}^{(r+1,m)})$$
$$-f_i(t^{(r)},x^{(m)},(T_hu_h)_{[r,m]},s_i\delta v_{h.i}^{(r,m)}+(1-s_i)\delta v_{h.i}^{(r+1,m)})\Big]+h_0\Gamma_{h.i}^{(r,m)},\quad 1\le i\le k,$$

where $P_{ij}\in\Omega$ are intermediate points. According to Assumption $H[T_h]$ we have the estimate

$$\|(v_h)_{(t^{(r)},x^{(m)})}-(T_hu_h)_{[r,m]}\|_D\le\gamma(h)+\varepsilon_h^{(r)}.$$

Then $\varepsilon_h^{(r)}$ satisfies the difference inequality

$$\varepsilon_h^{(r+1)}\le\varepsilon_h^{(r)}+h_0\sigma(t^{(r)},\gamma(h)+\varepsilon_h^{(r)})+h_0\widetilde{\gamma}(h),\quad 0\le r\le K-1. \tag{3.8}$$

Let us consider the Cauchy problem

$$w'(t)=\sigma(t,\gamma(h)+w(t))+\widetilde{\gamma}(h), \tag{3.9}$$

$$w(0)=\alpha_0(h). \tag{3.10}$$

It follows from condition 1)-(ii) of Assumption $H[f,\sigma]$ that there exists the maximal solution $\widetilde{w}_h$ of the problem (3.9), (3.10) and $\widetilde{w}_h$ is defined on $[0,a]$. Moreover

$$\lim_{h\to 0}\widetilde{w}_h(t)=0\quad\text{uniformly on}\quad[0,a].$$

It is easily seen that $\widetilde{w}_h$ satisfies the recurrent inequality

$$\widetilde{w}_h^{(r+1)}\ge\widetilde{w}_h^{(r)}+h_0\sigma(t^{(r)},\gamma(h)+\widetilde{w}_h^{(r)})+h_0\widetilde{\gamma}(h),\quad 0\le r\le K-1$$

and it follows from (3.10) that the inequality

$$\widetilde{w}_h^{(r)}\le\alpha_0(h),\quad -K_0\le r\le 0$$

holds. By the above relations and (3.8) we have

$$\varepsilon_h^{(r)}\le\widetilde{w}_h^{(r)},\quad 0\le r\le K.$$

Thus we get (3.6) for $\alpha(h)=\widetilde{w}_h(a)$. This completes the proof. □

Now we give an example of the operator T_h satisfying Assumption $H[T_h]$ and the error estimate for the difference method (2.6), (2.7).

Put

$$S_*=\{(j,s):j\in\{0,1\},s=(s_1,\ldots,s_n),s_i\in\{0,1\}\quad\text{for}\quad 1\le i\le n\}.$$

Let $w\in F(D_h,R^k)$ and $(t,x)\in D$. There exists $(t^{(r)},x^{(m)})\in D_h$ such that

$$t^{(r)}\le t\le t^{(r+1)},\quad x^{(m)}\le x\le x^{(m+1)},\qquad (t^{(r+1)},x^{(m+1)})\in D_h.$$

We define

$$(T_hw)(t,x)=\sum_{(j,s)\in S_*}w^{(r+j,m+s)}\left(\frac{Y-Y^{(r,m)}}{h}\right)^{(j,s)}\left(1-\frac{Y-Y^{(r,m)}}{h}\right)^{1-(j,s)}$$

where

$$\left(\frac{Y-Y^{(r,m)}}{h}\right)^{(j,s)} = \left(\frac{t-t^{(r)}}{h_0}\right)^{j} \prod_{k=1}^{n} \left(\frac{x_k-x_k^{(m_k)}}{h_k}\right)^{s_k}$$

and

$$\left(1-\frac{Y-Y^{(r,m)}}{h}\right)^{1-(j,s)} = \left(1-\frac{t-t^{(r)}}{h_0}\right)^{1-j} \prod_{k=1}^{n} \left(1-\frac{x_k-x_k^{(m_k)}}{h_k}\right)^{1-s_k}$$

and we take $0^0 = 1$ in the above formulas.

Lemma 3.3 *Suppose that*

1) *the solution* $v : E^* \to R^k$ *of differential problem* (1.1), (1.2) *is of class* C^2 *and the assumptions of Theorem* 3.1 *are satisfied with* $\sigma(t,p) = Lp$, $L > 0$,
2) *there exist* $\widetilde{M} \in R_+$ *and* $\widetilde{C} \in R_+$ *such that*

$$\|\partial_q f(t,x,z,q)\| \le \widetilde{M},$$

$$\|\partial_{x_j} v(t,x)\|_\infty, \quad \|\partial_{tt} v(t,x)\|_\infty, \quad \|\partial_{x_j x_j} v(t,x)\|_\infty \le \widetilde{C}$$

where $j = 1,\dots,n$.

Then

$$\|u_h^{(r,m)} - v_h^{(r,m)}\|_\infty \le \widetilde{\eta}_h^{(r)}, \tag{3.11}$$

where

$$\widetilde{\eta}_h^{(r)} = \alpha_0(h)(1+h_0L)^r + \bar{\gamma}(h)\frac{(1+h_0L)^r - 1}{L}$$

and

$$\bar{\gamma}(h) = L\gamma(h) + \widetilde{\gamma}(h), \quad \widetilde{\gamma}(h) = \frac{1}{2}\widetilde{C}h_0 + L\gamma(h) + \widetilde{C}(2 + \|h'\|)\widetilde{M}.$$

Proof. From the assumptions of Lemma we conclude that the operators δ_0, δ satisfy the following conditions

$$\|\delta_0 v^{(r,m)} - \partial_t v^{(r,m)}\|_\infty \le \frac{1}{2}\widetilde{C}h_0,$$

$$\|\delta_j v^{(r,m)} - \partial_{x_j} v^{(r,m)}\|_\infty \le \frac{1}{2}\widetilde{C}\|h'\|, \quad j = 1,\dots,n.$$

It follows from above estimates and from Assumption $H[T_h]$ that

$$\Gamma_{h.i}^{(r,m)} = \delta_0 v_i^{(r,m)} - \partial_t v_i^{(r,m)}$$
$$+ f_i(t^{(r)}, x^{(m)}, v^{(r,m)}, v_{(t^{(r)},x^{(m)})}, \partial_x v_i^{(r,m)})$$
$$- f_i(t^{(r)}, x^{(m)}, v^{(r,m)}, T_h v_{[r,m]}, s_i \delta v_i^{(r,m)} + (1-s_i)\delta v_i^{(r+1,m)}).$$

Then

$$\|\Gamma_h^{(r,m)}\|_\infty \le \widetilde{\gamma}(h).$$

The function $\widetilde{\eta}_h$ is a solution of the problem

$$\eta^{(r+1)} = \eta^{(r)}(1+h_0L) + h_0[L\gamma(h) + \widetilde{\gamma}(h)], \quad 1 \le i \le k.$$

Then from Theorem 3.1 we get the assertion (3.11). □

4 NUMERICAL EXAMPLES

Example 4.1 For $n = 2$, $k = 1$ we put

$$E = \{(t, x, y) : t \in [0, a], \quad x \in [-1, 1], \quad y \in [-1, 1]\}. \tag{4.1}$$

Consider the differential integral equation

$$\partial_t z(t, x, y) = \partial_x z(t, x, y) - \partial_y z(t, x, y) \tag{4.2}$$
$$- \sin \Big[\partial_x z(t, x) - \partial_y z(t, x, y) + t^2 \int_x^1 z(t, \tau, y) d\tau$$
$$+ t^2 \int_{-1}^{y} z(t, x, \tau) d\tau - t e^{t(1-y)} - t e^{t(x+1)}\Big] - (2t - x + y) e^{t(x-y)}.$$

with the initial boundary condition

$$z(0, x, y) = 1, \quad x \in [-1, 1], \quad y \in [-1, 1], \tag{4.3}$$
$$z(t, 1, y) = e^{t(1-y)}, \quad t \in [0, a], \quad y \in [-1, 1],$$
$$z(t, x, -1) = e^{t(x+1)}, \quad t \in [0, a], \quad x \in [-1, 1].$$

The exact solution of this problem is known. It is $v(t, x) = e^{t(x-y)}$. Put $h = (h_0, h_1, h_2)$ stand for the steps of the mesh on E.

Difference method for the problem (4.2), (4.3) has the form

$$z^{(r+1,m_1,m_2)} = z^{(r,m_1,m_2)} + h_0 \Big[s\delta_1 z^{(r,m_1,m_2)} + (1-s)\delta_1 z^{(r+1,m_1,m_2)} \tag{4.4}$$
$$- s\delta_2 z^{(r,m_1,m_2)} - (1-s)\delta_2 z^{(r+1,m_1,m_2)} - \sin \Big(s\delta_1 z^{(r,m_1,m_2)} + (1-s)\delta_1 z^{(r+1,m_1,m_2)}$$
$$- s\delta_2 z^{(r,m_1,m_2)} - (1-s)\delta_2 z^{(r+1,m_1,m_2)} + (t^{(r)})^2 \int_{x^{(m_1)}}^1 z(t^{(r)}, \tau, y^{(m_2)}) d\tau$$
$$+ (t^{(r)})^2 \int_{-1}^{y^{(m_2)}} z(t^{(r)}, x^{(m_1)}, \tau) d\tau - t^{(r)} \exp(t^{(r)}(1 - y^{(m_2)})) - t^{(r)} \exp(t^{(r)}(x^{(m_1)} + 1)) \Big)$$
$$- (2t^{(r)} - x^{(m_1)} + y^{(m_2)}) z^{(r,m_1,m_2)} \Big].$$

We put $s = s_1 = s_2$. Let us denote by $z_h : E_h \to R$ the solution of the explicit difference problem corresponding to (4.2), (4.3) which we get from (4.4) with $s = 1$. By $u_h : E_h \to R$ we denote the solution of the implicit difference problem received form (4.4) with $s = 0$. If we put in (4.4) $s = 0.5$ we get the implicit difference method and by its solution we denote $\tilde{u}_h : E_h \to R$.

Write

$$\eta_h^{(r)} = \frac{1}{(2N+1)^2} \sum_{m_1=-N}^{N} \sum_{m_2=-N}^{N} |z_h^{(r,m_1,m_2)} - v^{(r,m_1,m_2)}|, \tag{4.5}$$

$$\bar{\eta}_h^{(r)} = \frac{1}{(2N+1)^2} \sum_{m_1=-N}^{N} \sum_{m_2=-N}^{N} |u_h^{(r,m_1,m_2)} - v^{(r,m_1,m_2)}|, \tag{4.6}$$

$$\tilde{\eta}_h^{(r)} = \frac{1}{(2N+1)^2} \sum_{m_1=-N}^{N} \sum_{m_2=-N}^{N} |\tilde{u}_h^{(r,m_1,m_2)} - v^{(r,m_1,m_2)}|, \tag{4.7}$$

The numbers $\eta_h^{(r)}$, $\bar{\eta}_h^{(r)}$, $\tilde{\eta}_h^{(r)}$ are the arithmetical mean of the errors with fixed $t^{(r)}$. The values of the functions $\eta_h^{(r)}$, $\bar{\eta}_h^{(r)}$, $\tilde{\eta}_h^{(r)}$ are listed in the tables. We write "x" for $\eta_h > 100$.

Table of errors for $h = (0.01, 0.01, 0.01)$

	η_h	$\tilde{\eta}_h$
$t = 0.5$	0.001554	0.001664
$t = 1.0$	0.002772	0.003799
$t = 1.5$	0.004161	0.009324
$t = 2.0$	x	0.011336
$t = 2.5$	x	0.189344

Table of errors for $h = (0.05, 0.005, 0.005)$

	$\bar{\eta}_h$
$t = 0.1$	0.001631
$t = 0.2$	0.003236
$t = 0.3$	0.004798
$t = 0.4$	0.006284
$t = 0.5$	0.006986

Our experiments have the following property. The explicit method for steps $h = (0.01, 0.01, 0.01)$ which are not satisfy the condition (CFL)

$$1 - h_0 \sum_{j=1}^{2} \frac{1}{h_j} s |\partial_{q_j} f(P)| \geq 0, \tag{4.8}$$

with parameter $s = 1$, is not stable. The difference method with $s = 0.5$ for the same steps gives better results. For the steps $h = (0.05, 0.005, 0.005)$ which are not satisfy the condition (4.8) with $s = 0.5$, the difference method is not stable. The implicit difference method, which we received with $s = 0$, is stable aside from steps of the mesh.

Example 4.2 For $n = 2$, $k = 1$ we put

$$E = \{(t, x, y) : t \in [0, a], \quad x \in [-1, 1], \quad y \in [-1, 1]\}. \tag{4.9}$$

Consider the differential integral equation

$$\partial_t z(t, x, y) = \partial_x z(t, x, y) + \cos\Big[\partial_x z(t, x, y) - tyz(t, x, y)\Big] \tag{4.10}$$

$$-\partial_y z(t,x,y) - \sin\left[\partial_y z(t,x,y) - txz(t,x,z)\right] + z(t, \frac{x-y}{2}, \frac{x+y}{2})$$
$$+(xy - ty + tx)z(t,x,y) - \exp\,(\frac{1}{4}t(x^2 - y^2)) - 1.$$

with the initial boundary condition

$$z(0,x,y) = 1, \quad x \in [-1,1], \quad y \in [-1,1], \tag{4.11}$$
$$z(t,1,y) = e^{ty}, \quad t \in [0,a], \quad y \in [-1,1],$$
$$z(t,x,-1) = e^{-tx}, \quad t \in [0,a], \quad x \in [-1,1].$$

The exact solution of this problem is known. It is $v(t,x) = e^{txy}$. Put $h = (h_0, h_1, h_2)$ stand for the steps of the mesh on E.

Difference method for the problem (4.2), (4.3) has the form

$$z^{(r+1,m_1,m_2)} = z^{(r,m_1,m_2)} + h_0\Bigg[s\delta_1 z^{(r,m_1,m_2)} + (1-s)\delta_1 z^{(r+1,m_1,m_2)} \tag{4.12}$$
$$+\cos\left(s\delta_1 z^{(r,m_1,m_2)} + (1-s)\delta_1 z^{(r+1,m_1,m_2)} - t^{(r)}y^{(m_1)}z^{(r,m_1,m_2)}\right)$$
$$-s\delta_2 z^{(r,m_1,m_2)} - (1-s)\delta_2 z^{(r+1,m_1,m_2)}$$
$$-\sin\left(s\delta_2 z^{(r,m_1,m_2)} + (1-s)\delta_2 z^{(r+1,m_1,m_2)} - t^{(r)}x^{(m_2)}z^{(r,m_1,m_2)}\right)$$
$$+z(t^{(r)}, 0.5(x^{(m_1)} - y^{(m_2)}), 0.5(x^{(m_1)} + y^{(m_2)})) + (x^{(m_1)}y^{(m_2)}t^{(r)}y^{(m_2)} + t^{(r)}x^{(m_1)})z^{(r,m_1,m_2)}$$
$$-\exp\left(\frac{1}{4}t^{(r)}((x^{(m_2)})^2 - (y^{(m_2)})^2)\right) - 1\Bigg]$$

We put $s = s_1 = s_2$. Let us denote by $z_h : E_h \to R$ the solution of the explicit difference problem corresponding to (4.10), (4.11) which we get from (4.12) with $s = 1$. By $u_h : E_h \to R$ we denote the solution of the implicit difference problem received form (4.12) with $s = 0$. If we put in (4.12) $s = 0.5$ we get the implicit difference method and by its solution we denote $\tilde{u}_h : E_h \to R$.

The numbers $\eta_h^{(r)}$, $\bar{\eta}_h^{(r)}$, $\tilde{\eta}_h^{(r)}$, given by (4.5)-(4.7) respectively, are the arithmetical mean of the errors with fixed $t^{(r)}$. The values of the functions $\eta_h^{(r)}$, $\bar{\eta}_h^{(r)}$, $\tilde{\eta}_h^{(r)}$ are listed in the tables. We write "x" for $\eta_h > 100$.

Table of errors for $h = (0.01, 0.01, 0.01)$

	η_h	$\tilde{\eta}_h$
$t = 0.1$	0.032645	0.000487
$t = 0.2$	x	0.000890
$t = 0.3$	x	0.001229
$t = 0.4$	x	0.001525
$t = 0.5$	x	0.001769

Table of errors for $h = (0.05, 0.005, 0.005)$

	$\bar{\eta}_h$
$t = 0.1$	0.005775
$t = 0.2$	0.010111
$t = 0.3$	0.013670
$t = 0.4$	0.016837
$t = 0.5$	0.018405

Conclusion from above experiment is that implicit difference method received from (4.12) with $s = 0$, is stable for any choose of steps $h = (h_{0,}, h_1, h_2)$. For stability of the explicit difference method and implicit difference method received from (4.12) with $s = 0.5$, we need satisfying the condition (CFL) given by (4.8).

References

[1] A. Baranowska, Numerical methods for nonlinear first order partial differential equations with deviated variables. *Numer. Methods Partial Differential Equations*, v. 22, no. 3, pp. 708–727, 2006.

[2] A. Baranowska, Z. Kamont, Numerical approximation of first order partial differential equations with deviated variables. *Comment. Math. Prace. Mat.*, v. 43, no. 1, pp. 1–31, 2003.

[3] P. Brandi, Z. Kamont, A. Salvadori, Difference methods for nonlinear first order partial differential equations with mixed initial boundary conditions. *Math. Balkanica*, v. 10, no. 2–3, pp. 249–269, 1996.

[4] Z. Kamont, Finite difference approximations for first order partial differential functional equations. *Ukr. Math. J.*, v. 46, pp. 985–996, 1994.

[5] Z. Kamont, *H*yperbolic Functional Differential Inequalities and Applications. Dordrecht: Kluver Acad. Publ, 1999.

[6] Z. Kamont, K. Prządka, Difference methods for first order partial differential functional equations with initial - boundary condtions. *J. Vycisl. Mat. i Math. Fis.*, v. 31, pp. 1476–1488, 1991.

[7] A. Kępczyńska, Implicit difference functional inequalities and applications. *Math. Balk.*, v. 20, no 2, pp. 167–184, 2006.

[8] M. Malec, Sur une famille bi-paramétrique de schémas des différences finies pour l'équation parabolique sans dérivées mixtes. *Ann. Polon. Math.*, v. 31, pp. 47–54, 1975.

[9] M. Malec, Sur une famille bi-paramétrique des schémas des différences pour les systémes paraboliques. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.*, v. 23, no. 8, pp. 871–875, 1975.

[10] M. Malec, Sur une famille biparamétrique de schémas des différences finies pour un systéme d'équations paraboliques aux dérivées mixtes et avec des conditions aux limites du type de Neumann. *Ann. Polon. Math.*, v. 32, pp. 33–42, 1976.