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PARAMETER UNIFORM NUMERICAL SCHEMES FOR SECOND ORDER SINGULARLY PERTURBED DIFFERENTIAL DIFFERENCE EQUATIONS WITH LAYER BEHAVIOR

V. P. RAMESH AND M. K. KADALBAJOO

Department of Mathematics & Statistics, I. I. T. Kanpur, India
vpramesh@gmail.com   kadal@iitk.ac.in

ABSTRACT. A linear second order singularly perturbed differential difference equation without turning point is considered as a model problem. Simple upwind, Midpoint upwind and a Hybrid algorithm are analysed on a Shishkin mesh. Theoretical error bounds are established and proved that hybrid algorithm yields second order approximation. Numerical experiments support these theoretical results.

Keywords. Singularly perturbed convection diffusion problems, Differential difference equation, Finite difference scheme, Hybrid method, Non-uniform mesh.

1. INTRODUCTION

We consider boundary value problems(BVPs) for linear second order singularly perturbed differential difference equations(DDEs) on the domain $\Omega = (0, 1)$

$$-\epsilon^2 u''(x) + a(x)u'(x) + \alpha(x)u(x-\delta) + w(x)u(x) + \beta(x)u(x+\eta) = f(x) \tag{1.1}$$

subject to the interval conditions:

$$u(x) = \phi(x) \quad -\delta(\epsilon) \le x \le 0 \tag{1.2a}$$

$$u(x) = \gamma(x) \quad 1 \le x \le 1 + \eta(\epsilon) \tag{1.2b}$$

where ϵ is a small parameter $0 < \epsilon << 1$, the delay parameter $\delta(\epsilon)(0 < \delta(\epsilon) < 1)$ is of $\circ(\epsilon)$, the advance parameter $\eta(\epsilon)(0 < \eta(\epsilon) < 1)$ is of $\circ(\epsilon)$, and $a(x)$, $\alpha(x)$,$\beta(x)$, w(x), f(x), $\phi(x)$ and $\gamma(x)$ are smooth functions of x and are assumed, for simplicity, to be independent of ϵ and

$$\alpha(x) + \beta(x) + w(x) \ge \theta > 0 \tag{1.3}$$

The solution $u(x)$ is continuous on [0,1], continuously differentiable on (0,1) and also statisfies (1.1) and (1.2).

For the retarded arguments equal to zero (i.e., $\delta = 0 = \eta$), the solution of the above problem exhibits layer behavior of width $O(\epsilon^2 \ln(\epsilon^{-2}))$ on the left or the right

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depending on the value of $a(x)$. The boundary layer occurs at $x = 0$ when $a(x) < 0$ on $0 \leq x \leq 1$ and occurs at $x = 1$ when $a(x) > 0$ on $0 \leq x \leq 1$. If $a(x)$ changes sign in $0 < x < 1$, then the solution exhibit turning point behavior. Such problems are not analyzed in this paper. In this we have considered the model with layer behavior and without any turning points.

DDEs of above type arise naturally in the theoretical analysis of neuronal variability. In the brain, time delays arise because interneural distances and axonal conduction times are finite. Details about various models for determining the responses of a neuron to random synaptic input can be found in [9], [10]. On the theoretical side there have been many advanced model of nerve membrane potential in the presence of random synaptic inputs. Due to the analytic difficulties in solving any realistic model, computer simulation has played an important role as a first step. In 1965, Stein introduced a model for neuron activity which incorporates some of the physiological features of real neurons [3], [4]. The model and its modifications have been used as a basis for many studies devoted to the theoretical description of neuronal activites [12], [16], [15].

The concept of the model is to calculate the expected time to firing of a nerve impulse when there is Poisson excitation. Between two jumps caused by the input process, the membrane potential decays exponentially $(-xu'(x))$ to the resting level with a membrane time. By the term membrane potential, we mean the membrane depolarization from the resting level at the trigger zone. If there are inputs that can be modeled as Weiner process with variance σ and drift parameter μ then the calculation of the expected time u(x), given the initial membrane potential $x \in (x_1, x_2)$ can be formulated as a linear second order differential difference equation

$$-\frac{\sigma^2}{2}u''(x) + (\mu - x)u'(x) + \lambda_I u(x - a_I) + \lambda_E u(x + a_E) - (\lambda_E + \lambda_I)u(x) = 1,$$

where the values at $x = x_1$ and $x = x_2$ corresponds to the inhibitory reversal potential and to the threshhold value of membrane potential for action potential generation, respectively. The reaction terms corresponds to excitatory and inhibitory synaptic inputs which are assumed to be poissonian. This excitatory synaptic input contributes to the membrane potential by a_E with intensity λ_E and similarly the inhibitory synaptic inputs contributes to the membrane potential by a_I with intensity λ_I. a_E and a_I are small quantities and could depend on voltage. A convincing neurophysiological demonstration of Stein's model can be found in the following literature [3], [11], [7].

One of the principle difficulties with the application of this model lies in solving the delay differential equations that form the mathematical expression of the model [16]. Though there have been extensive studies of the properties of the solutions of many kinds of functional equations [1], [2] a little progress has been made on equations of type (1.1) with both forward(advance) and backward(delay) delays. These applications motivates the approximation of DDEs of Stein's model type. Stein [3] approximated the solution of his model using monte carlo techniques. Others who approximated the Stein's model(DDE) are Tuckwell and Richter [11], Tuckwell and Cope [8] and Wilbur and Rinzel [16]. Lange and Miura considered a linear second order differential equation with a small delay as well as advance in the reaction term with layer behavior in [5]. For small delays authors used taylors series for the approximation of the retarded arguments and showed how the delay affects the layer solution. In [17], Kadalbajoo and Sharma discussed standard upwind, fitted operator and fitted mesh for a similar problem.

The intension of this paper is to improve the rate of convergence and the accuracy. The numerical scheme should also be uniformly accurate in ϵ, δ & η and the solution cost should not grow with decreasing ϵ or with increasing δ & η. The standard finite difference scheme and the second order central difference scheme on a uniform mesh are not going to be successful in this case. The central difference scheme would lead to an oscillating (non-physical) solution unless the number of grid points is very high. Fig. 1-3 shows how the computational cost increases for smaller values of ϵ. Hence it may not be practically possible to compute the solution for smaller values of ϵ using central difference scheme on an uniform mesh.

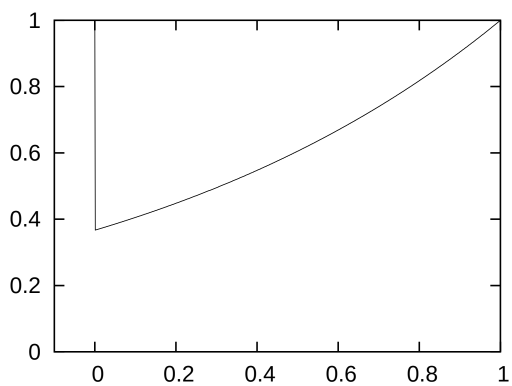

Figure 1: exact solution of Ex: 4.1 for $\epsilon = 0.003$ & $\delta = 1.6 * 10^{-3}$

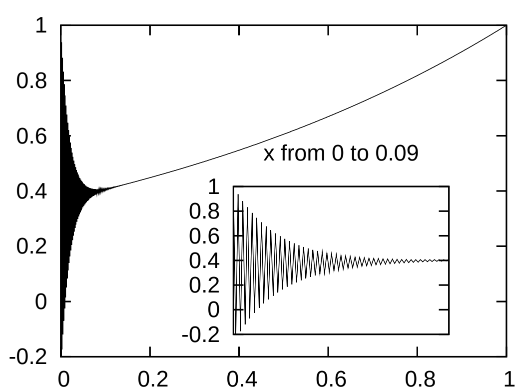

Figure 2: $N = 1024$

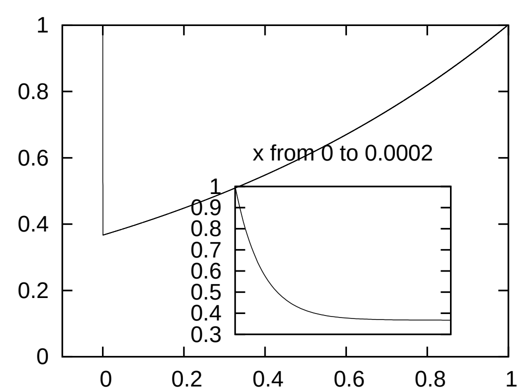

Figure 3: $N = 10, 48, 576$

Numerical solution of Ex: 4.1 using L_{cd} on uniform mesh with $\epsilon = 0.003$ & $\delta = 1.6 * 10^{-3}$

We are going to extend the results of [18] to more complicated DDEs of type (1.1). In this we will consider a midpoint upwind scheme on a priori adaptive Shishkin mesh and proved that it improves the convergence outside the layer region. To improve the convergence through out the domain we consider a hyrbid scheme. The considered methods is shown to have almost second order error estimate in the maximum norm.

The remaining part of the paper is organized as follows. In Section 2, we discuss a priori estimates for the retarded arguments and certain differentiability properties of the solution. In Section 3, we discuss the numerical formulation of the schemes and also about the calculation of parameters for shishkin mesh. In Section 4, we discuss the convergence of the difference scheme. In Section 5, numerical results and discussion are given. Through out this paper, C denotes generic positive constant that is independent of ϵ and in the case of discrete problems it is also independent of N which may take different values. $\|\cdot\|$ denotes the global maximum norm over the appropriate domain as defined above. Ω=(0,1) and $\overline{\Omega}$=(0,1)$\cup\{0,1\}$=[0,1]

2. A PRIORI ESTIMATES

We consider the case when the delays δ, η are $\circ(\epsilon)$(note that the coefficient of the diffusion term is ϵ^2 not ϵ) and use simple Taylor series through second order derivative to approximate the retarded arguments ([7, page 3], [6, page 275])

$$u(x-\delta) \approx u(x) - \delta u'(x) + \frac{\delta^2}{2}u''(x) \tag{2.1}$$

$$u(x+\eta) \approx u(x) + \eta u'(x) + \frac{\eta^2}{2}u''(x) \tag{2.2}$$

Eqs. (1.1) & (1.2) yields the following ODE

$$L_\epsilon u(x) = f(x) \tag{2.3}$$

where

$$L_\epsilon = \{-\epsilon^2 + \frac{\delta^2}{2}\alpha(x) + \frac{\eta^2}{2}\beta(x)\}\frac{d^2}{dx^2} + (a(x) - \delta\alpha(x) + \eta\beta(x))\frac{d}{dx} + (\alpha(x) + \beta(x) + w(x))I$$

subject to the boundary conditions:

$$u(0) = \phi(0) = \phi_0 \tag{2.4a}$$

$$u(1) = \gamma(1) = \gamma_1 \tag{2.4b}$$

Eq (2.3) differs from eq (1.1) by terms of $\bigcirc(\delta^3 u''', \eta^3 u''')$. When the shifts are sufficiently small, then eq (2.3) is a good approximation to eq (1.1). We consider the case when the solution of (2.3), (2.4) exhibits layer behavior on the right side of the interval. It is assumed that

$$a(x) - \delta\alpha(x) + \eta\beta(x) \geq M > 0 \ \forall x \in [0,1] \tag{2.5}$$

where M is a positive constant. The other case, i.e., when $a(x) - \delta\alpha(x) + \eta\beta(x) \leq -M < 0 \; \forall x \in [0,1]$ the layer occurs at the left side can be treated similarly. Here we assume that $\alpha(x) \leq M_1$, $\beta(x) \leq M_2$ and $-\epsilon^2 + \frac{\delta^2}{2}\alpha(x) + \frac{\eta^2}{2}\beta(x) < 0$ and with the condition (2.5) the solution exhibits a layer near $x = 1$.

3. THE DISCRETIZATION

Let $\{x_i\}_{i=0}^N$ be our computational domain and N, the number of grid points in the domain be an even positive integer. For each $i \geq 1$, we define $h_i = x_i - x_{i-1}$ and $x_{i-\frac{1}{2}} = (x_i + x_{i-1})/2$.

Because of a boundary layer of thickness σ located at $x = 1$, we want to use a fine mesh in the subinterval $[\sigma, 1]$ and a coarse mesh outside the layer region $[0, 1-\sigma]$. We set

$$\sigma = \min\{\frac{1}{2}, C(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)\log(N)\}, \tag{3.1}$$

where $C = 1/\theta$. We call σ the transition point from the coarse to the fine mesh. The fine and the coarse meshsizes are defined as $h_1 = (1-\sigma)/(N/2)$ and $h_2 = \sigma/(N/2)$. Since, N being an even positive integer 50% of the grid points will be in the layer region. The above a priori adaptive piecewise uniform mesh is referred to as Shishkin mesh and is given by

$$x_i = \begin{cases} \frac{2(1-\sigma)}{N}i \quad for\ 1 \leq i \leq N/2 \\ 1 - \sigma + \frac{2\sigma}{N}(i - N/2)\ for\ \frac{N}{2} \leq i \leq N \end{cases} = \begin{cases} h_1 i \quad for\ 1 \leq i \leq N/2 \\ \sigma + h_2(i - N/2)\ for\ \frac{N}{2} \leq i \leq N \end{cases}$$

If the transition point σ is chosen independently of N, then one cannot obtain a convergence result that is uniform in ϵ [20]. Similarly, we can generate a Shishkin mesh when the layer is located at $x = 0$.

Let $U = \{U_i\}_{i=0}^N$ be any given function defined on the computational mesh, we shall approximate the first-order and second-order derivatives at the grid point x_i as follows: The forward and backward divided difference operators are

$$D_+U_i = \frac{U_{i+1} - U_i}{h_{i+1}} \quad and \quad D_-U_i = \frac{U_i - U_{i-1}}{h_i}$$

respectively. The central difference operator D_0 is given by

$$D_0U_i = \frac{1}{h_i h_{i+1}(h_i + h_{i+1})}(-h_{i+1}^2 u_{i-1} + (h_{i+1}^2 - hi^2)u_i + h_i^2 u_{i+1})$$

The second order derivative $D_+D_-U_i$ is

$$D_+D_-U_i = \frac{2}{h_i h_{i+1}(h_i + h_{i+1})}(h_{i+1}u_{i-1} - (h_{i+1} + h_i)u_i + h_i u_{i+1})$$

While D_0U_i is second order consistent on any mesh, just as on equidistant meshes, but this is not the case for $D_+D_-U_i$: a term $(h_{i+1} - h_i)u'''(x_i)$, which arises in the consistency error analysis, is only of first-order on arbitary mesh. For more details, the reader can refer to [14].

3.1. **Upwind Scheme:** L^h_{up}**.** To approximate the solution of Eq. (2.3) and (2.4) we discretize the equation(2.3) using a standard upwind finite difference method on the above mentioned shishkin's mesh given by

Find $U^N \in \mathbb{R}^{N+1} := \{Y = (y_0, y_1, \ldots, y_N) \in \mathbb{R}^{N+1} : y_0 = \phi_0 \textit{ and } y_N = \gamma_1\}$ such that

(3.2)

$$L^h_{up}U^N_i = \{-\epsilon^2 + \frac{\delta^2}{2}\alpha_i + \frac{\eta^2}{2}\beta_i\}D^+D^-U^N_i + (a_i - \delta\alpha_i + \eta\beta_i)D^-U^N_i + (\alpha_i + \beta_i + w_i)U_i = f_i$$

$\Longrightarrow$

$$U^N_0 = \phi_0$$

$$-E_iU^N_{i-1} + F_iU^N_i - G_iU^N_{i+1} = H_i \quad i = 1, 2, \ldots, N-1$$

$$U^N_N = \gamma_1$$

where

$$\begin{aligned}
E_i &= \frac{2\{\epsilon^2 - \frac{\delta^2}{2}\alpha_i - \frac{\eta^2}{2}\beta_i\}}{h_i(h_i + h_{i+1})} + \frac{(a_i - \delta\alpha_i + \eta\beta_i)}{h_{i+1}} \\
F_i &= \frac{2\{\epsilon^2 - \frac{\delta^2}{2}\alpha_i - \frac{\eta^2}{2}\beta_i\}}{h_ih_{i+1}} + \frac{(a_i - \delta\alpha_i + \eta\beta_i)}{h_{i+1}} + (\alpha_i + \beta_i + w_i) \\
G_i &= \frac{2\{\epsilon^2 - \frac{\delta^2}{2}\alpha_i - \frac{\eta^2}{2}\beta_i\}}{h_{i+1}(h_i + h_{i+1})} \\
H_i &= f_i
\end{aligned}$$

Assume (1.3), (2.5), It is very clear that the upwind operator L^h_{up} satisfies the following conditions

$$E_i > 0, \quad G_i > 0 \quad F_i > E_i + G_i \tag{3.4}$$

It follows from the above inequalities that the matrix system is an M-matrix and has an inverse. From this one can prove the uniform boundedness of the difference equations [22]. It can be proved [17] that the nodal errors e^N_i satisfies

$$\|e^N_i\| = \|u(x_i) - U^N_i\| \le CN^{-1}(\ln\ N)^2$$

where u is the exact solution of the model (2.3),(2.4), U^N is the solution of (3.2) and the constant C is independent of ϵ, δ, η and the mesh. The main disadvantage of this upwind scheme is its low order of accuracy. In order to improve the order of accuracy we discuss a simple midpoint upwind scheme which has proved to be popular in many problems.

3.2. **Midpoint Upwind Scheme: L^h_{mp}.** To keep the proof as simple as possible, we restrict our model (2.3) to

$$(3.5)\quad L_1 u = \left(-\epsilon^2 + \frac{\delta^2}{2}\alpha(x) + \frac{\eta^2}{2}\beta(x)\right)\frac{d^2u}{dx^2} + (a(x) - \delta\alpha(x) + \eta\beta(x))\frac{du}{dx} = f(x)$$

subject to the boundary conditions (2.4). In this we use midpoint upwind method on a shishkin mesh to approximate the solution of Eqs. (3.5) and (2.4) and is given by Find $U^N \in \mathbb{R}^{N+1} := \{Y = (y_0, y_1, \ldots, y_N) \in \mathbb{R}^{N+1} : y_0 = \phi(0) \; and \; y_N = \gamma_1\}$ such that

$$(3.6)\;\; L^h_{mp}U^N_i := \{-\epsilon^2 + \frac{\delta^2}{2}\alpha_{i-\frac{1}{2}} + \frac{\eta^2}{2}\beta_{i-\frac{1}{2}}\}D^+D^-U^N_i + (a - \delta\alpha + \eta\beta)_{i-\frac{1}{2}}D^-U^N_i = f_{i-\frac{1}{2}}$$

and $a_{i-1/2} = (a_i + a_{i-1})/2$ as defined before. In the more general case the reaction term should be replaced by the average, $(\alpha + \beta + w)_{i-\frac{1}{2}}U_{i-\frac{1}{2}}$. See [23], [24] for details of the scheme.

Lemma 3.1. *The system $L^h_{mp}U^N_i = f_{i-1/2},\ 1 \le i \le N-1$, with given U^N_0 and U^N_N has a solution. If $L^h_{mp}U^N_i \le L^h_{mp}B_i,\ 1 \le\ i\ \le\ N-1$, and if $U^N_0 \le B_0$, $U^N_N \le B_N$, then $U^N_i \le B_i$, $1 \le\ i\ \le N-1$.*

Proof. The equations $L^h_{mp}U_i = f_{i-1/2},\ 1 \le i \le N-1$ may be considered as a system of $N-1$ linear equations with $U_i, 1 \le i \le N-1$ where for $i = 1$ and $i = N-1$, the terms involving U_0 and U_N are moved to the right hand side. It is an easy computation to verify that the matrix is diagonally dominant and has non positive off diagonal entries. Hence, the matrix is an irreducible M matrix, and so has a positive inverse. Hence, the solution U_i, $1 \le i \le N-1$ exists. The rest of the proof is very simple. For more details the reader can refer [19]. We say $\{B_i\}$ is a barrier function for $\{U^N_i\}$. □

Lemma 3.2. *Let $z_i = 1 + x_i$, for $0 \le i \le N$. Then there exists a positive constant C such that $L^h_{mp}z_i \ge C$ for $1 \le i \le N-1$.*

Proof. This proof is a easy computation, it enables us to give a bound. □

If $s(x)$ is a smooth function. We now consider the truncation error of the operator L^h_{mp} applied to s at $x_{i-\frac{1}{2}}$ define as $\tau := L^h_{mp}(s_i) - (L_1 s)(x_{i-1/2})$, where $s_i = s(x_i)$.

Lemma 3.3. *There exists a constant $C > 0$ such that*

$$|\tau| \le C(\epsilon^2 - M_1\frac{\delta^2}{2} - M_2\frac{\eta^2}{2})\int_{x_{i-1}}^{x_{i+1}} |s^{(3)}(t)|dt + Ch_i\int_{x_{i-1}}^{x_i} |s^{(3)}(t)|dt.$$

Proof. By repeated use of the fundamental theorem of calculus, or by Peano's theorem one can obtain the proof as in [19]. □

Lemma 3.4. *For $i = 0, 1, \dots, N$, we set*

$$R_i = \prod_{j=1}^{i} \left(1 + \frac{Mh_j}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right)$$

Let C be any positive contant independent of ϵ, δ, η and of the mesh. Then for $i = 1, 2, \dots, N-1$, we have

$$L^h_{mp}R_i \geq \frac{C}{max\{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2, h_i\}}R_i$$

Proof. It is easy to verify that

$$\frac{R_i - R_{i-1}}{h_i} = \frac{M}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}R_{i-1},$$

Now,

$$\begin{aligned} L^h_{mp}R_i &= \frac{2(-\epsilon^2 + \frac{\delta^2}{2}\alpha_{i-\frac{1}{2}} + \frac{\eta^2}{2}\beta_{i-\frac{1}{2}})M(R_i - R_{i-1})}{(h_i + h_{i+1})\left(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2\right)} + \frac{(a_{i-\frac{1}{2}} - \delta\alpha_{i-\frac{1}{2}} + \eta\beta_{i-\frac{1}{2}})M}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}R_{i-1} \\ &= \frac{MR_i\left(a_{i-\frac{1}{2}} - \delta\alpha_{i-\frac{1}{2}} + \eta\beta_{i-\frac{1}{2}} - \frac{2Mh_i(\epsilon^2 - \frac{\delta^2}{2}\alpha_{i-\frac{1}{2}} - \frac{\eta^2}{2}\beta_{i-\frac{1}{2}})}{(h_i + h_{i+1})(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\delta^2}{2}M_2)}\right)}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2 + Mh_i} \end{aligned}$$

from which the result follows. □

Lemma 3.5.

$$e^{-\beta(1-x_i)/(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)} \leq \prod_{j=i+1}^{N}\left(1 + \frac{Mh_j}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right)^{-1}$$

for each i.

Proof.

$$e^{-Mh_j/(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)} = \left(e^{Mh_j/(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)}\right)^{-1} \leq \left(1 + \frac{Mh_j}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right)^{-1} \forall j$$

The above inequality is true for each j. Now we multiply these inequalities for $j = i+1, \dots, N$, we get

$$e^{-\beta(1-x_i)/(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)} \leq \prod_{j=i+1}^{N}\left(1 + \frac{Mh_j}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right)^{-1}$$

hence the result. □

Now to prove the parameter uniform convergence of the scheme we need more information about the exact solution of (2.3), (2.4), which is obtained by decomposing the solution u into smooth and singular components as

$$u = v + w$$

where the smooth component v can be written as $v(x) = v_0(x) + \epsilon v_1(x) + \epsilon^2 v_2(x)$, where $v_0(x)$ for $x \in \Omega$ is the solution of the reduced problem of (2.3), (2.4)

$$(3.7)\quad (a(x) - \delta\alpha(x) + \eta\beta(x))v_0'(x) + (\alpha(x) + \beta(x) + w(x))v_0(x) = f(x),\ v_0(1) = u(1)$$

and $v_1(x)$ for $x \in \Omega$ satisfies the boundary value problem

$$(3.8)\quad (a(x) - \delta\alpha(x) + \eta\beta(x))v_1'(x) + (\alpha(x) + \beta(x) + w(x))v_1(x) = -v_0''(x),\ v_1(1) = 0$$

and $v_2(x)$ satisfies the boundary value problem

$$L_1 v_2(x) = -v_1''(x),\ x \in\ \Omega,\ v_2(0) = 0,\ v_2(1) = 0 \tag{3.9}$$

Thus the smooth component $v(x)$ is the solution of

$$L_1 v(x) = f(x),\ x \in\ \Omega,\ v(0) = v_0(0) + \epsilon v_1(0), v(1) = u(1) \tag{3.10}$$

The singular component w is the solution of the homogenous problem

$$L_1 w(x) = 0, \quad x \in \Omega, \quad |w(0)| \le Ce^{-M/(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)}, \quad |w(1)| \le C$$

Theorem 3.6. *Let $u = v + w\ \forall x \in [0,1]$ be the decomposition of the solution of our model* (2.3), (2.4). *For sufficiently small ϵ and for any finite q and $0 < x < 1$ we have the following bounds*

$$\|v^{(k)}\| \le C \quad for\ 0 \le k \le q \tag{3.11a}$$

$$\|w^{(k)}\| \le C(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)^{-k} e^{-M(1-x)/(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)} \tag{3.11b}$$

Proof. The proof of inequality (3.11a) trivially follows from equation(3.7). See [21] for a proof. □

Lemma 3.7. *There exists a constant C such that*

$$\prod_{j=i+1}^{N}\left(1 + \frac{Mh_j}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right)^{-1} \le CN^{-4(1-i/N)}\ for\ N/2 \le i < N$$

Proof. Suppose $N/2 \le i < N$. By [19]

$$\begin{aligned}\prod_{j=i+1}^{N}\left(1 + \frac{Mh_j}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right)^{-1} &\le e^{-\beta(1-x_i)/(Mh_2 + \epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)} \\ &= e^{-4(N-i)N^{-1}\ln N/(1+4N^{-1}\ln N)} \\ &= N^{-4(N-i)N^{-1}/(1+4N^{-1}\ln N)} \\ &= N^{-4(1-i/N)}N^{16(i-1/N)N^{-1}\ln N/(1+4N^{-1}\ln N)}\end{aligned}$$

It is easy to verify that $N^{16(i-1/N)N^{-1}\ln N/(1+4N^{-1}\ln N)}$ is bounded for any $N \ge 2$ from which the result follows. □

Now we will derive the error estimates by estimating the errors of smooth part v and the layer part w separately.

Theorem 3.8. *We have the following error estimate for the midpoint upwind scheme* (3.6)*:*

$$\|u - U^N\| \leq \begin{cases} CN^{-1}(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2 + N^{-1}) & for\ 0 \leq i < 3N/4, \\ CN^{-1}(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2 + N^{-4(1-i/N)}) \ln N & for\ 3N/4 \leq i \leq N, \end{cases}$$

Proof. As in Theorem (3.6) we split the computed solution $\{U_i^N\}_{i=0}^N$ as $U_i^N = V_i^N + W_i^N$ for $i = 0, 1, \ldots, N$ where

$$L_{mp}^h V_i^N = f_{i-1/2}, \quad V_0^N = v(0),\ V_N^N = v(1),$$

and

$$L_{mp}^h W_i^N = 0, \quad W_0^N = w(0),\ W_N^N = w(1).$$

Hence the nodal errors can be estimated separately in V^N and W^N.

Let us compute the nodal error for the smooth part $\{V_i^N\}$. With lemmas 3.3 and theorem 3.6 we get

(3.12)
$$|L_{mp}^h(v_i - V_i^N)| = |L_{mp}^h(V_i^N) - (L_1 v)(x_{i-1/2})| \leq C(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2 + h_i)(h_i + h_{i+1})$$

Now let $B_i := CN^{-1}(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2 + N^{-1})(1 - x_i)$ be the barrier function and with lemmas 3.1 and 3.2 we get

$$|v_i - V_i| \leq CN^{-1}(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2 + N^{-1}) \tag{3.13}$$

Now lets analyze the error of the singular component. We know that

$$\begin{aligned} |W_0^N| = |w(0)| \leq Ce^{\overline{\frac{-M}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}}} &= C\prod_{j=1}^{N} e^{-Mh_j/(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)} \\ &\leq C\prod_{j=1}^{N}\left(1 + \frac{Mh_j}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right)^{-1} \end{aligned}$$

and $|W_N^N| = w(1) \leq C$ and $L_{mp}^h W_i^N = 0$ for $i = 1, 2, \ldots, N-1$. Now lets use lemma 3.4 and define

$$B_i := C\left[\prod_{j=1}^{N}\left(1 + \frac{Mh_j}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right)^{-1}\right] S_i$$

a barrier function for $\{W_i^N\}$. By the discrete maximum principle we get

$$|W_i^N| \leq B_i = C\prod_{j=1}^{N}\left(1 + \frac{Mh_j}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right)^{-1} \quad \forall i$$

Therefore by lemma 3.5 and theorem 3.6 we get

$$|w_i - W_i^N| \leq C \prod_{j=i+1}^{N} \left(1 + \frac{Mh_j}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right)^{-1} \quad \forall i \tag{3.14}$$

and on the coarser part

$$|w_i - W_i^N| \leq CN^{-2} \quad for\ 0 \ < \ i \ \leq N/2 \tag{3.15}$$

and on the finer part we compute the error as we did for the smooth component, i.e by using the consistency and barrier function, but we restrict our domain to $[1-\sigma, 1]$ rather than $[0, 1]$.

Now taking $i = N/2$ in (3.15) we get

$$|w_{N/2} - W_{N/2}^N| \leq CN^{-2} \quad and \quad |w_N - W_N^N| = 0.$$

It is easy to verify that lemma 3.3 for $N/2 < i < N$ can be modified to (using lemma 3.3 [19])

$$|\tau_2| \leq C \int_{x_{i-1}}^{x_{i+1}} \left[(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)|s^{(3)}(t)| + |s^{(2)}(t)|\right] dt. \tag{3.16}$$

$$\begin{aligned}
|L_{mp}^h(w_i - W_i^N)| &\leq C(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)^{-2} \int_{x_{i-1}}^{x_{i+1}} e^{-\beta(1-x)/(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)} dt \\
&= C(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)^{-1} \left[e^{\frac{-M(1-x_{i+1})}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}} - e^{\frac{-M(1-x_{i-1})}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}} \right] \\
&= C(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)^{-1} e^{\frac{-M(1-x_i)}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}} \sinh\left(\frac{Mh_2}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right) \\
&\leq \left(C(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)^{-1} N^{-1} \ln N\right) e^{-\beta(1-x_i)/(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)} \\
&\leq C(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)^{-1} N^{-1} \ln N \prod_{j=i+1}^{N} \left(1 + \frac{Mh_2}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right)^{-1}
\end{aligned}$$

By Lemma 3.5 and since $\sinh x \leq Cx$ for $x \in [0, 1]$.

Now let the barrier function be

$$B_i := C\left\{N^{-2} + (N^{-1}\ln N)\left[\prod_{j=1}^{N}\left(1 + \frac{Mh_2}{\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2}\right)^{-1}\right] R_i\right\}$$

for $i = N/2, \ldots, N$ and C is chosen sufficiently large. Now in this case we have already replaced the interval $[0, 1]$ by $[1 - \sigma, 1]$ and its easy to verify the discrete maximum principle. So by using this principle, with the above barrier function, we get

$$|w_i - W_i^N| \leq B_i$$

Now by lemma 3.7 for $i = N/2, \ldots, N$ we get

$$|w_i - W_i^N| \leq C \max\left\{N^{-2}, N^{-5+4i/N \ln N}\right\} \tag{3.17}$$

From (3.13), (3.15) and (3.17) gives the required result. □

Thus for small ϵ, that for $\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2 \leq CN^{-1}$ the above midpoint scheme gives second order convergence outside the layer region, while on the layer region the order of convergence is at most first order. Hence this L^h_{mp} approximates the solution better outside the layer region than the simple upwind L^h_{up}.

3.3. **Central difference: L^h_{cd}.** In this we use a second order central difference approximation to the convection term of Eq. (2.3) and is given by

Find $U^N \in \mathbb{R}^{N+1} := \{Y = (y_0, y_1, \ldots, y_N) \in \mathbb{R}^{N+1} : y_0 = \phi(0) \ and \ y_N = \gamma_1\}$ such that

(3.18)

$$L^h_{up}U^N_i = \{\epsilon^2 + \frac{\delta^2}{2}\alpha_i + \frac{\eta^2}{2}\beta_i\}D^+D^-U^N_i + (a_i - \delta\alpha_i + \eta\beta_i)D^0U^N_i + (\alpha_i + \beta_i + w_i)U_i = f_i$$

L^h_{cd} is not always stable when applied throught out the domain since the central difference approximation to the convection term would lead to non-physical oscillation unless the mesh size (h) is very very small. The behavior of L^h_{cd} scheme on uniform mesh is shown in Figure 2 & 3. But, one of the very important factors affecting the accuracy of the discrete solution is the choice of discretization schemes for the first derivative convection term. In our hyrbid scheme we use central difference in the boundary layer region (fine mesh region) and midpoint upwind schemes outside the boundary layer region (coarse mesh region).

3.4. **Hyrbid scheme L^h_{hy}.** In this subsection we define difference schemes which are hybrid of mid-point upwind and central difference schemes. Lets define a hybrid scheme L^h_{hy} as,

Find $U^N \in \mathbb{R}^{N+1} := \{Y = (y_0, y_1, \ldots, y_N) \in \mathbb{R}^{N+1} : y_0 = \phi(0) \ and \ y_N = \gamma_1\}$ such that

$$L^h_{hy}U^N_i := \begin{cases} L^h_{mp} \ for \ 1 \leq i \leq \frac{N}{2} \\ L^h_{cd} \ for \ \frac{N}{2} < i \leq N-1 \end{cases} \tag{3.19}$$

Here we use midpoint upwind approximating the grid points outside the layer region and central difference scheme inside the layer region. Since we are applying central difference scheme on the fine part of the shishkin mesh, in order to attain the stability we need the following mild condition

Theorem 3.9. *Assume that*

$$\frac{N}{\ln N} \geq 2\frac{\max(a(x) - \delta\alpha(x) + \eta\beta(x))}{\min(a(x) - \delta\alpha(x) + \eta\beta(x))} \tag{3.20}$$

Then the hybrid scheme (3.19) *satisfies the error estimate*

$$|u(x_i) - U_i^N| \leq \begin{cases} CN^{-1}((\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2) + N^{-1}) & 0 \leq i \leq N/2 \\ CN^{-2}\ln^2 N \quad N/2 < i \leq N. \end{cases} \tag{3.21}$$

Proof:- The argument is similar to the previous theorem. The condition (3.20) guarantees that the matrix of the discrete problem is an M-matrix. Proof follows the same way as we did for the midpoint upwind scheme except when estimating $|w_i - W_i^N|$ the error of the singular component on the fine part instead of (3.16) we use the following one [19].

$$|\tau_3| \leq Ch \int_{x_{i-1}}^{x_{i+1}} \left[(\epsilon^2 - \frac{\delta^2}{2}M_1 - \frac{\eta^2}{2}M_2)|s^{(4)}(t)| + |s^{(3)}(t)| \right] dt \quad for\ N/2 \ < \ i \ < \ N$$

which is true for the central scheme and is the key to get better accuracy compared to other schemes.

3.5. **Maximum pointwise error($E^N_{\epsilon,\delta}$), ϵ-uniform error(E^N), Rate of convergence ($p^N_{\epsilon,\delta}$), ϵ-uniform rate of convergence(p^N).** We use the following definitions to compute the maximum pointwise error and the rate of convergence [25]. The maximum pointwise error $E^N_{\epsilon,\delta}$ for a fixed ϵ,δ and N the number of grid points is calculated by comparing the exact solution u(x) with the corresponding values of the approximations U^N generated by using the methods discussed in the pervious subsections of this section and is given by

$$E^N_{\epsilon,\delta} = \max_{x_i \in \overline{\Omega}} |u(x_i) - U_i^N|$$

If the exact solution is not known, then the maximum pointwise error is estimated using the double mesh principle [26] defined by

$$E^N_{\epsilon,\delta} = \max_{1 \leq i \leq N-1} |U_i^N - U_{2i}^{2N}|.$$

where U_{2i}^{2N} is the solution obtained on a mesh containing the same N number of shishkin mesh used to compute $\{U_i^N\}$ and N more mesh points are added by selecting the mid points of all $\{x_i\}'s$, i.e., $x_{i+1/2} = (x_i + x_{i+1})/2$ for $i = 0, 1, 2, \ldots, N-1$.

For any value of N, the ϵ-uniform errors are calculated using

$$E^N = \max_{\epsilon,\delta}\{E^N_{\epsilon,\delta}\} \tag{3.22}$$

With the calculated Errors, the rate of convergence of the scheme is calculated using a standard technique given by

$$p^N_{\epsilon,\delta} = \log_2 \left(\frac{E^N_{\epsilon,\delta}}{E^{2N}_{\epsilon,\delta}} \right) = \frac{\log\left(E^N_{\epsilon,\delta} - E^{2N}_{\epsilon,\delta}\right)}{\log 2}$$

The ϵ-uniform rate of convergence is calculated using

$$p^N = \log_2\left(\frac{E^N}{E^{2N}}\right) = \frac{\log\left(E^N - E^{2N}\right)}{\log 2}$$

4. NUMERICAL RESULTS

In this section we consider two examples to show the efficiency of the schemes on a priori adaptive shiskin mesh. First example is a constant coefficient problem with only delay and second one is a variable coeficient problem with both delay and advance.

Example 4.1.

$$-\epsilon^2 u''(x) - u'(x) - 2u(x-\delta) + 3u(x) = 0,$$

under the interval and boundary conditions

$$u(x) = 1 \text{ for } -\delta \le x \le 0 \quad \text{and } u(1) = 1$$

It has a layer at $x = 0$. The Exact solution is known.

Example 4.2.

$$-\epsilon^2 u''(x) + (1 + e^{x^2})u'(x) + (1 + x)e^x u(x-\delta) - u(x) + e^{-x}u(x+\eta) = 100x(1-x),$$

under the interval and boundary conditions

$$u(x) = 1 \text{ for } -\delta \le x \le 0 \quad \text{and } u(1) = -1 \text{ for } 1 \le x \le 1+\eta$$

It has a layer at $x = 1$. The exact solution is not known. We use double mesh principle discussed in section 3.5.

The computed solutions U^N of example 4.1 for different values of delay parameter are shown in figure 1 and shows the effect of delay parameters in our test problem. Tables 1, 2 & 3 shows the computed maximum pointwise error ($E^N_{\epsilon,\delta}$), $\epsilon-$uniform errror (E^N_δ) and $\epsilon-$uniform rate of convergence(p^N_δ) of example 4.1 computed using standard upwind (L^h_{up}), midpoint upwind (L^h_{mp}) and our hybrid method (L^h_{hy}) respectively, defending the advantages of L^h_{hy} over the other upwind methods. From these tables we observe that the maximum pointwise error $E^N_{\epsilon,\delta}$ decreases as N increases for each value of ϵ. We see that the maximum pointwise error stabilize as $\epsilon \to 0$ for each N. Table 4 shows the $\epsilon-$uniform errror (E^N_δ), parameter uniform errors (E^N) and the parameter uniform rate of convergence (p^N) of example 4.1 using L^h_{hy}. Table 5 gives comparison of different schemes proving the advantage of our hybrid scheme in terms of accuracy and the rate of convergence. Similarly for our second test problem (ex: 4.2) the errors $E^N_{\epsilon,\delta}$ are shown in Tables 6, 7 & 8 proving the advantage of our hybrid scheme over other methods. To show the efficiency of the scheme with smaller grid points, we have computed the errors of ex: 4.2 with very high number of grid points.

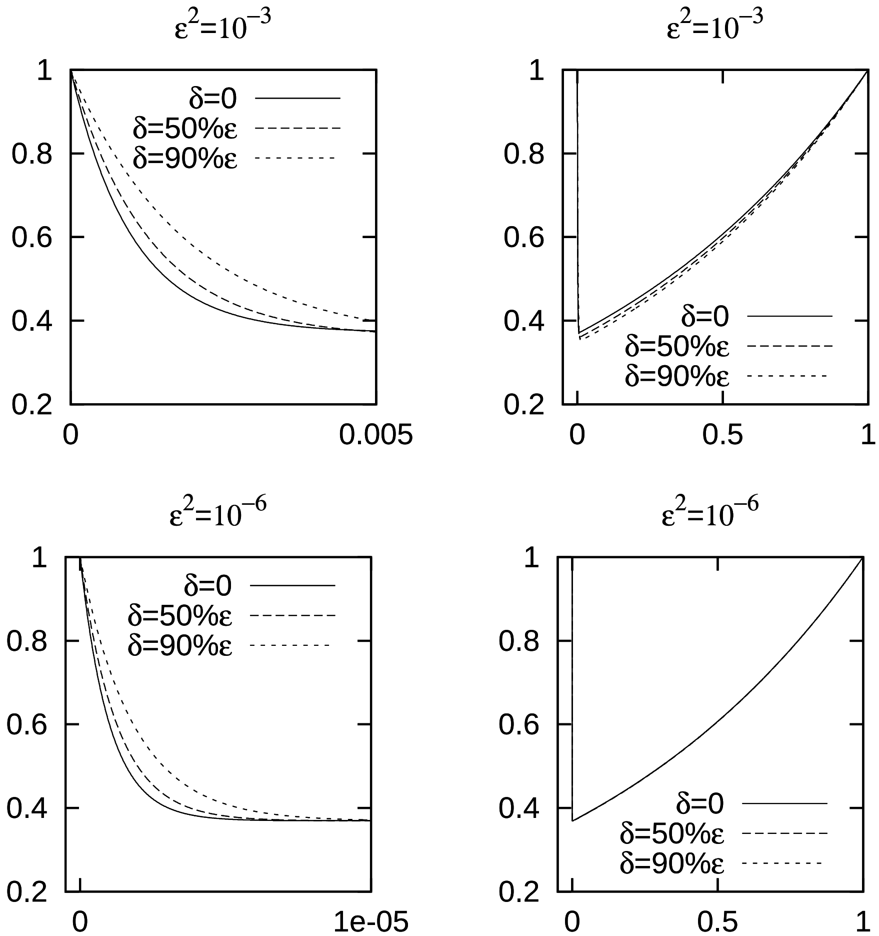

FIGURE 1. Solution of example 4.1 using L_{hy}^h for different ϵ

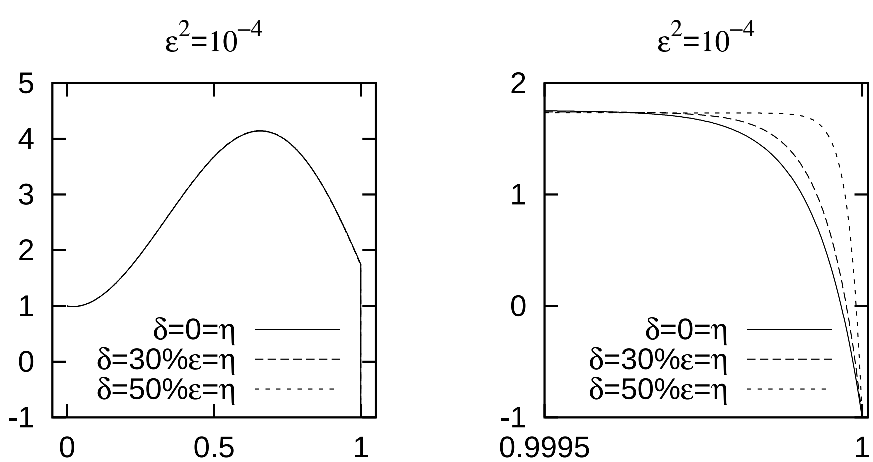

FIGURE 2. Solution of example 4.2 using L_{hy}^h with ($\epsilon^2 = 10^{-4}$)

TABLE 1. $E^N_{\epsilon,\delta=0.5*\epsilon}$ for Ex:4.1 using the method L^h_{up} with N/2 nodes for each subinterval

ϵ^2	N 16	32	64	128	256	512	1024
10^{-1}	3.24E-02	1.85E-02	1.05E-02	5.32E-03	2.68E-03	1.35E-03	6.75E-04
10^{-2}	7.57E-02	4.24E-02	2.23E-02	1.15E-02	6.02E-03	3.19E-03	1.71E-03
10^{-3}	8.10E-02	4.78E-02	2.65E-02	1.42E-02	7.47E-03	3.88E-03	2.00E-03
10^{-4}	8.03E-02	4.75E-02	2.65E-02	1.45E-02	7.80E-03	4.17E-03	2.21E-03
10^{-5}	7.99E-02	4.71E-02	2.64E-02	1.44E-02	7.79E-03	4.19E-03	2.24E-03
10^{-6}	7.97E-02	4.70E-02	2.63E-02	1.44E-02	7.78E-03	4.18E-03	2.24E-03
10^{-7}	7.97E-02	4.70E-02	2.63E-02	1.44E-02	7.77E-03	4.18E-03	2.24E-03
:	:	:	:	:	:	:	:
10^{-10}	7.97E-02	4.70E-02	2.63E-02	1.44E-02	7.77E-03	4.18E-03	2.24E-03
E^N_δ	8.10E-02	4.78E-02	2.65E-02	1.45E-02	7.80E-03	4.19E-03	2.24E-03
p^N_δ	0.7609	0.8510	0.8699	0.8945	0.8965	0.9035	0.9077

TABLE 2. $E^N_{\epsilon,\delta=0.5*\epsilon}$ for Ex:4.1 using the method L^h_{mp} with N/2 nodes for each subinterval

ϵ^2	N 16	32	64	128	256	512	1024
10^{-1}	2.93E-02	1.79E-02	1.05E-02	5.36E-03	2.71E-03	1.36E-03	6.83E-04
10^{-2}	6.37E-02	3.55E-02	1.88E-02	9.81E-03	5.20E-03	2.80E-03	1.52E-03
10^{-3}	6.90E-02	4.05E-02	2.25E-02	1.22E-02	6.47E-03	3.39E-03	1.76E-03
10^{-4}	6.83E-02	4.03E-02	2.26E-02	1.25E-02	6.79E-03	3.67E-03	1.96E-03
10^{-5}	6.79E-02	4.00E-02	2.25E-02	1.24E-02	6.79E-03	3.69E-03	1.99E-03
10^{-6}	6.77E-02	3.99E-02	2.24E-02	1.24E-02	6.77E-03	3.68E-03	1.99E-03
10^{-7}	6.76E-02	3.99E-02	2.24E-02	1.24E-02	6.77E-03	3.68E-03	1.99E-03
:	:	:	:	:	:	:	:
10^{-10}	6.76E-02	3.99E-02	2.24E-02	1.24E-02	6.77E-03	3.68E-03	1.99E-03
E^N_δ	6.90E-02	4.05E-02	2.26E-02	1.25E-02	6.79E-03	3.69E-03	1.99E-03
p^N_δ	0.7687	0.8416	0.8544	0.8804	0.8798	0.8909	0.8952

TABLE 3. $E^N_{\epsilon,\delta=0.5*\epsilon}$ for Ex:4.1 using the method L^h_{hy} with N/2 nodes for each subinterval

ϵ^2	N 16	32	64	128	256	512	1024
10^{-1}	1.82E-03	9.34E-04	4.76E-04	2.40E-04	1.21E-04	6.04E-05	3.02E-05
10^{-2}	5.76E-03	2.68E-03	1.06E-03	3.82E-04	1.31E-04	4.38E-05	1.45E-05
10^{-3}	6.91E-03	3.02E-03	1.14E-03	4.01E-04	1.34E-04	4.34E-05	1.37E-05
10^{-4}	7.44E-03	3.19E-03	1.19E-03	4.15E-04	1.38E-04	4.40E-05	1.37E-05
10^{-5}	7.61E-03	3.25E-03	1.21E-03	4.20E-04	1.39E-04	4.45E-05	1.38E-05
10^{-6}	7.67E-03	3.27E-03	1.21E-03	4.22E-04	1.40E-04	4.47E-05	1.39E-05
10^{-7}	7.70E-03	3.27E-03	1.22E-03	4.23E-04	1.40E-04	4.47E-05	1.39E-05
:	:	:	:	:	:	:	:
10^{-10}	7.70E-03	3.27E-03	1.22E-03	4.23E-04	1.40E-04	4.47E-05	1.39E-05
E^N_δ	7.70E-03	3.27E-03	1.22E-03	4.23E-04	1.40E-04	4.48E-05	1.39E-05
p^N_δ	1.2356	1.4224	1.5282	1.5952	1.6439	1.6884	1.7436

TABLE 4. E^N_δ for $10^{-1} \leq \epsilon^2 \leq 10^{-10}$ and $\delta = \tau * \epsilon$ of Ex:4.1 using the method L^h_{hy}

τ	N 256	512	1,024	2,048	4,096	8,192	16,384
0.0	1.40E-04	4.47E-05	1.39E-05	4.20E-06	1.24E-06	3.51E-07	8.90E-08
0.2	1.40E-04	4.47E-05	1.39E-05	4.21E-06	1.25E-06	3.63E-07	1.00E-07
0.4	1.40E-04	4.47E-05	1.39E-05	4.21E-06	1.25E-06	3.65E-07	1.02E-07
0.5	1.40E-04	4.48E-05	1.39E-05	4.15E-06	1.19E-06	3.04E-07	1.11E-07
0.6	1.40E-04	4.47E-05	1.39E-05	4.19E-06	1.23E-06	3.38E-07	7.62E-08
0.8	1.40E-04	4.47E-05	1.39E-05	4.19E-06	1.23E-06	3.44E-07	8.21E-08
1.0	1.40E-04	4.47E-05	1.39E-05	4.20E-06	1.24E-06	3.55E-07	9.26E-08
E^N	1.40E-04	4.47E-05	1.39E-05	4.21E-06	1.25E-06	3.65E-07	1.02E-07
p^N	1.6471	1.6852	1.7232	1.7519	1.7760	1.8393	–

TABLE 5. Comparison of $E^N_{\epsilon,\delta}$ & $p^N_{\epsilon,\delta}$ using L_{up},L_{mp},L_{hy} for Ex: 4.1 on Shishkin Mesh

N	$\epsilon^2 = 0.0001$ & $\delta = 0.005$					
	Upwind		Midpoint		Hybrid	
	$[0, \sigma)$	$[\sigma, 1]$	$[0, \sigma)$	$[\sigma, 1]$	$[0, \sigma)$	$[\sigma, 1]$
16	8.032E-02	2.191E-02	6.758E-02	4.745E-04	7.437E-03	4.824E-04
	0.759	0.945	0.635	1.954	1.223	1.977
32	4.747E-02	1.138E-02	4.353E-02	1.225E-04	3.187E-03	1.225E-04
	0.838	0.967	0.664	1.976	1.421	1.956
64	2.655E-02	5.820E-03	2.748E-02	3.115E-05	1.190E-03	3.115E-05
	0.875	0.968	0.728	1.965	1.521	1.965
128	1.448E-02	2.975E-03	1.659E-02	7.977E-06	4.146E-04	7.977E-06
	0.892	0.955	0.775	1.935	1.590	1.934
256	7.803E-03	1.535E-03	9.698E-03	2.087E-06	1.377E-04	2.087E-06
	0.904	0.924	0.809	1.878	1.645	1.878
512	4.171E-03	8.091E-04	5.534E-03	5.678E-07	4.404E-05	5.678E-07
	0.916	0.874	0.836	1.783	1.684	1.783
1,024	2.210E-03	4.414E-04	3.100E-03	1.650E-07	1.371E-05	1.650E-07
	0.930	0.812	0.855	1.644	1.714	1.644
2,048	1.160E-03	2.515E-04	1.714E-03	5.278E-08	4.180E-06	5.278E-08
	0.943	0.792	0.871	1.478	1.737	1.477
4,096	6.032E-04	1.453E-04	9.373E-04	1.896E-08	1.254E-06	1.896E-08
	0.953	0.925	0.882	1.314	1.757	1.314
8,192	3.115E-04	7.651E-05	5.085E-04	7.624E-09	3.710E-07	7.624E-09
	0.954	1.115	0.892	1.188	1.771	1.188
16,384	1.608E-04	3.533E-05	2.740E-04	3.347E-09	1.087E-07	3.347E-09
	0.946	1.196	0.900	1.105	1.783	1.103
32,768	8.345E-05	1.542E-05	1.469E-04	1.556E-09	3.158E-08	1.558E-09
	0.938	1.175	0.907	1.039	1.713	1.026
65,536	4.356E-05	6.831E-06	7.835E-05	7.572E-10	9.634E-09	7.704E-10
	0.933	1.120	0.912	0.994	1.6861	1.019
1,31,072	2.282E-05	3.144E-06	4.163E-05	3.802E-10	2.594E-09	3.802E-10
	–	–	–	–	–	–

TABLE 6. $E^N_{\epsilon,\delta=0.5*\epsilon,\eta=0.5*\epsilon}$ for Ex:4.2 using L^h_{up} with N/2 nodes for each subinterval

	N					
	16^2	32^2	64^2	128^2	256^2	512^2
ϵ^2	256	1,024	4,096	16,384	65,536	2,62,144
10^{-1}	2.642E-02	7.405E-03	1.865E-04	4.672E-04	1.169E-04	2.922E-05
10^{-2}	6.995E-02	2.210E-02	6.566E-03	1.888E-03	5.324E-04	1.478E-04
10^{-3}	8.398E-02	2.692E-02	8.056E-03	2.326E-03	6.574E-04	1.832E-04
10^{-4}	8.732E-02	2.814E-02	8.440E-03	2.440E-03	6.902E-04	1.924E-04
10^{-5}	8.824E-02	2.849E-02	8.552E-03	2.473E-03	6.998E-04	1.951E-04
10^{-6}	8.851E-02	2.860E-02	8.586E-03	2.483E-03	7.027E-04	1.957E-04
10^{-7}	8.860E-02	2.863E-02	8.597E-03	2.486E-03	7.036E-04	1.957E-04
10^{-8}	8.863E-02	2.865E-02	8.600E-03	2.487E-03	7.039E-04	1.965E-04
10^{-9}	8.863E-02	2.865E-02	8.601E-03	2.488E-03	7.040E-04	1.963E-04
10^{-10}	8.864E-02	2.865E-02	8.602E-03	2.488E-03	7.041E-04	1.963E-04

TABLE 7. $E^N_{\epsilon,\delta=0.5*\epsilon,\eta=0.5*\epsilon}$ for Ex:4.2 using L^h_{mp} with N/2 nodes for each subinterval

	N					
ϵ^2	256	1,024	4,096	16,384	65,536	2,62,144
10^{-1}	1.367E-02	3.873E-03	9.802E-04	2.458E-04	6.150E-05	1.538E-05
10^{-2}	3.112E-02	1.032E-02	3.156E-03	9.263E-04	2.652E-04	7.448E-05
10^{-3}	3.767E-02	1.259E-02	3.862E-03	1.134E-03	3.247E-04	9.139E-05
10^{-4}	3.979E-02	1.332E-02	4.089E-03	1.201E-03	3.440E-04	9.680E-05
10^{-5}	4.045E-02	1.355E-02	4.162E-03	1.223E-03	3.501E-04	9.846E-05
10^{-6}	4.066E-02	1.362E-02	4.185E-03	1.229E-03	3.520E-04	9.889E-05
10^{-7}	4.072E-02	1.364E-02	4.192E-03	1.232E-03	3.527E-04	9.909E-05
10^{-8}	4.074E-02	1.365E-02	4.194E-03	1.232E-03	3.528E-04	9.932E-05
10^{-9}	4.075E-02	1.365E-02	4.195E-03	1.232E-03	3.529E-04	9.931E-05
10^{-10}	4.075E-02	1.366E-02	4.195E-03	1.233E-03	3.529E-04	9.931E-05

TABLE 8. $E^{N}_{\epsilon,\delta=0.5*\epsilon,\eta=0.5*\epsilon}$ for Ex:4.2 using L^{h}_{hy} with N/2 nodes for each subinterval

	N					
ϵ^2	256	1,024	4,096	16,384	65,536	2,62,144
10^{-1}	1.389E-03	4.279E-04	1.068E-04	2.669E-05	6.671E-06	1.670E-06
10^{-2}	3.273E-03	2.200E-03	7.663E-04	2.094E-04	5.327E-05	1.327E-05
10^{-3}	1.765E-03	2.840E-04	2.361E-04	1.343E-04	4.432E-05	1.198E-05
10^{-4}	1.901E-03	1.882E-04	2.236E-05	1.909E-05	1.976E-05	9.121E-06
10^{-5}	1.959E-03	1.925E-04	1.744E-05	1.722E-06	1.147E-06	2.114E-07
10^{-6}	1.977E-03	1.943E-04	1.755E-05	1.498E-06	1.322E-07	1.899E-08
10^{-7}	1.983E-03	1.949E-04	1.761E-05	1.501E-06	1.286E-07	1.449E-08
10^{-8}	1.985E-03	1.951E-04	1.762E-05	1.503E-06	1.412E-07	1.210E-08
10^{-9}	1.985E-03	1.952E-04	1.763E-05	1.505E-06	1.157E-07	1.203E-08
10^{-10}	1.986E-03	1.952E-04	1.763E-05	1.503E-06	1.174E-07	1.195E-08

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