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AN ALMOST SECOND ORDER FEM FOR A WEAKLY COUPLED SYSTEM OF TWO SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS OF REACTION-DIFFUSION TYPE WITH DISCONTINUOUS SOURCE TERM

A. RAMESH BABU AND N. RAMANUJAM

Department of Mathematics, Bharathidasan University
Tiruchirappalli - 620024, Tamilnadu, India

ABSTRACT. In this paper, we consider a Boundary Value Problem(BVP) for a weakly coupled system of two singularly perturbed ordinary differential equations of reaction-diffusion type with discontinuous source term. The solution of this type of problem exhibits boundary and interior layers. A numerical method based on finite element method on Shishkin and Bakhvalov-Shishkin meshes is presented. We derive an error estimate of order $O(N^{-2}\ln^2 N)$ in the maximum norm. Numerical experiments are also presented to support our theoretical results.

Key Words Singularly perturbed problem, Weakly coupled system, Finite element method, Discontinuous source term, Boundary value problem, Layer adapted mesh.

1. INTRODUCTION

Differential equations with a small parameter $(0 < \varepsilon \ll 1)$ multiplying the highest order derivatives, termed as Singularly Perturbed Differential Equations (SPDEs), arise in diverse areas of applied mathematics, including linearized Navier - Stokes equations of high Reynolds number, heat transfer problem with large Peclet number, drift diffusion equations of semiconductor device modelling, chemical reactor theory, etc., In general, this type of equations exhibit boundary and/or interior layers. Standard numerical methods like finite difference and finite element methods on uniform mesh for solving this type of euations fail to produce good approximations to exact solutions. Many authors [1, 2, 3, 6, 14] have developed efficient numerical methods to resolve boundary and interior layers. A good number of articles have been appearing in the past three decades on non-classical methods which cover mostly single second order equation. But, a few authors only have considered system of SPDEs [9, 11, 12, 13].

In this paper, we consider the following system of singularly perturbed second order ordinary differential equations of reaction-diffusion type with discontinuous source

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term:

$$L_1\bar{u} := -\varepsilon u_1''(x) + a_{11}(x)u_1(x) + a_{12}(x)u_2(x) = f_1(x), \quad x \in \Omega^- \cup \Omega^+,$$
$$L_2\bar{u} := -\varepsilon u_2''(x) + a_{21}(x)u_1(x) + a_{22}(x)u_2(x) = f_2(x), \quad x \in \Omega^- \cup \Omega^+, \tag{1.1}$$

$$u_1(0) = 0, \quad u_1(1) = 0, \quad u_2(0) = 0, \quad u_2(1) = 0, \tag{1.2}$$

with conditions on coefficients

$$a_{12}(x) \le 0, \quad a_{21}(x) \le 0, \tag{1.3}$$

$$a_{11}(x) > |a_{12}(x)|, \quad a_{22}(x) > |a_{21}(x)|, \quad \forall x \in \bar{\Omega}, \tag{1.4}$$

and for the matrix $A = [a_{ij}]$

$$\xi A \xi^T \ge \alpha \xi \xi^T \quad \text{for every} \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2. \tag{1.5}$$

Here $\varepsilon > 0$ is a small parameter, $\alpha > 0$, $\Omega = (0,1)$, $\Omega^- = (0,d)$, $\Omega^+ = (d,1)$, $d \in \Omega$, and $u_1, u_2 \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$, $\bar{u} = (u_1, u_2)^T$. Further it is assumed that the source terms f_1, f_2 are sufficiently smooth on $\bar{\Omega} \setminus \{d\}$; both the functions $f_1(x)$ and $f_2(x)$ are assumed to have a single jump discontinuity at the point $d \in \Omega$. That is $f_i(d-) \neq f_i(d+), i = 1, 2$. In general this type of discontinuity gives rise to interior layers in the solution of the problem. Because $f_i, i = 1, 2$ are discontinuous at d the solution $\bar{u}$ of (1.1)–(1.2) does not necessarily have a continuous second derivative at the point d. That is $u_1, u_2 \notin C^2(\Omega)$. But the first derivative of the solution exists and is continuous.

Systems of this kind have applications in electro analytic chemistry when investigating diffusion processes complicated by chemical reactions. The parameters multiplying the highest derivatives characterize the diffusion coefficient of the substances. Other applications include equations of prey-predator population dynamics. As mentioned above, in general, classical numerical methods fail to produce good approximations to singularly perturbed equations. Hence various methods are proposed in the literature in order to obtain numerical solution to singularly perturbed system of second order differential equations subject to Dirichlet type boundary conditions when the source terms are smooth [9]. Motivated by the works of H-G. Roos et al.[8], in the present paper we suggest a numerical method for the above BVP. This method is based on Finite Element Method (FEM) with layer adapted meshes like Shishkin and Bakhvalov-Shishkin meshes. For this method we derive an error estimate of order $O(N^{-2}\ln^2 N)$ for Shishkin mesh, $O(N^{-2})$ for Bakhvalov-Shishkin mesh, in the maximum norm. In order to capture a boundary layer with a numerical method, it is essential that the approximate solutions generated by the numerical method are defined globally at each point of the domain of the exact solution. The numerical solution obtained from a finite element method defined only at the mesh points, is extended it to the whole domain by a simple interpolation process such as piecewise linear interpolation. Because we want our technique to be capable of extension to complex problems in higher dimensions, we only consider the finite element subspaces by piecewise polynomial basis functions.

In this connection we wish to state that the authors from [13] proved almost first order convergence with respect to ε on a Shishkin mesh of the finite difference method with special discretization at the point d. When we compute numerical solutions, it is not desirable to obtain error estimates in L^1 or L^2 norm, as they do not detect the local phenomena such as boundary or interior layer. Therefore the most appropriate norm for the study of singular perturbation problem is the maximum norm [6]. The main significance of this paper is that the error estimate for numerical solution is given in terms of the maximum norm. Now we define the maximum norm of $\bar{u} = (u_1, u_2)$ as

$$\| \bar{u} \|_\infty = \max\{\| u_1 \|_\infty, \| u_2 \|_\infty\}, \quad \| u_1 \|_\infty = \max_{x\in[0,1]} |u_1(x)|,$$
$$\| u_2 \|_\infty = \max_{x\in[0,1]} |u_2(x)|, \| \bar{u} \|_{\infty[x_{i-1},x_i]} = \max\{\| u_1 \|_{\infty[x_{i-1},x_i]}, \| u_2 \|_{\infty[x_{i-1},x_i]}\},$$
$$\| u_1 \|_{\infty[x_{i-1},x_i]} = \max_{x\in[x_{i-1},x_i]} |u_1(x)|, \| u_2 \|_{\infty[x_{i-1},x_i]} = \max_{x\in[x_{i-1},x_i]} |u_2(x)|.$$

Further we define

$$|\bar{u}(x)| = |(u_1(x), u_2(x))| = \max(|u_1(x)|, |u_2(x)|).$$

Remark 1.1. Through out this paper, C denotes generic constant that is independent of the parameter ε and N, the dimension of the discrete problem.

Lemma 1.2. [13] *The solution $\bar{u}$ of the problem* (1.1)–(1.2) *can be decomposed of smooth part $\bar{v}$ and layer part $\bar{w}$ as $\bar{u} = \bar{v} + \bar{w}$, $\bar{v} = (v_1, v_2)$, $\bar{w} = (w_1, w_2)$. Then, for each k, $0 \le k \le 3$, we have*

$$|v_i^{(k)}(x)| \le \begin{cases} C(1 + \varepsilon^{(1-k/2)} e_1(x)), & x \in \Omega^-, \\ C(1 + \varepsilon^{(1-k/2)} e_2(x)), & x \in \Omega^+, i = 1, 2, \end{cases}$$
$$|w_i^{(k)}(x)| \le \begin{cases} C(\varepsilon^{(-k/2)} e_1(x)), & x \in \Omega^-, \\ C(\varepsilon^{(-k/2)} e_2(x)), & x \in \Omega^+, i = 1, 2. \end{cases}$$

where $e_1(x) = e^{-x\sqrt{\frac{\gamma}{\varepsilon}}} + e^{-(d-x)\sqrt{\frac{\gamma}{\varepsilon}}}$, $e_2(x) = e^{-(x-d)\sqrt{\frac{\gamma}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\gamma}{\varepsilon}}}$ *and* $\gamma = \min_{x\in\bar{\Omega}}\{a_{11}(x) + a_{12}(x), a_{21}(x) + a_{22}(x)\}$.

This paper is organized as follows. Section 2 presents a weak formulation of the BVP (1.1)–(1.2) and describes a finite element discretization of the problem. Section 3 presents a role of projection operator on approximation space and error representation. It also includes an analysis of the corresponding scheme on Shishkin and Bakhvalov-Shishkin meshes and an interpolation error in the maximum norm. In

Section 4 we present a detailed error analysis of the projection operator, consistency part and other error terms. The paper concludes with numerical examples.

2. ANALYTICAL RESULTS

A standard weak formulation of (1.1)–(1.2) is: Find $\bar{u} = (u_1, u_2) \in (H_0^1(\Omega))^2$ such that

$$B(\bar{u}, \bar{v}) = f^*(\bar{v}), \quad \forall \bar{v} \in (H_0^1(\Omega))^2, \tag{2.1}$$

with

$$B(\bar{u}, \bar{v}) := (B_1(\bar{u}, \bar{v}), B_2(\bar{u}, \bar{v})) \quad \text{and} \quad f^*(\bar{v}) := (f_1^*(\bar{v}), f_2^*(\bar{v})),$$

where

$$B_1(\bar{u}, \bar{v}) := \varepsilon(u_1', v_1') + (a_{11}u_1 + a_{12}u_2, v_1), \tag{2.2}$$

$$B_2(\bar{u}, \bar{v}) := \varepsilon(u_2', v_2') + (a_{21}u_1 + a_{22}u_2, v_2), \tag{2.3}$$

and

$$f_1^*(\bar{v}) = (f_1, v_1),$$
$$f_2^*(\bar{v}) = (f_2, v_2).$$

Here $H_0^1(\Omega)$ denotes the usual Sobolev space and $(\cdot, \cdot)$ is the inner product on $L^2(\Omega)$. Now we define a norm on $(H_0^1(\Omega))^2$ associated with the bilinear form $B(\cdot, \cdot)$, called energy norm as

$$|||\bar{u}||| = [\varepsilon(|u_1|_1^2 + |u_2|_1^2) + \alpha(\|u_1\|_0^2 + \|u_2\|_0^2)]^{1/2},$$

where $\|u\|_0 := (u, u)^{1/2}$ is the standard norm on $L^2(\Omega)$, while $|u|_1 := \|u'\|_0$ is the usual semi-norm on $H_0^1(\Omega)$. We also use the notation $\|\bar{u}\|_0 = (\|u_1\|_0^2 + \|u_2\|_0^2)^{1/2}$ for the norm in $(L^2(\Omega))^2$. B is a bilinear functional defined on $(H_0^1(\Omega))^2$. We now prove that it is coercive with respect to $|||.|||$, that is

$$| \; B(\bar{u}, \bar{u}) \; | \geq \frac{1}{2}|||\bar{u}|||^2,$$

where $|B(\bar{u}, \bar{u})|^2 = B_1(\bar{u}, \bar{u})^2 + B_2(\bar{u}, \bar{u})^2$.

Lemma 2.1. *A bilinear functional B satisfies the coercive property with respect to $|||.|||$.*

Proof. Let $\bar{u} \in (H_0^1(\Omega))^2$. Then

$$\begin{aligned} |B(\bar{u}, \bar{u})| &= \sqrt{B_1(\bar{u}, \bar{u})^2 + B_2(\bar{u}, \bar{u})^2} \\ &\geq \frac{1}{2}[|B_1(\bar{u}, \bar{u})| + |B_2(\bar{u}, \bar{u})|] \\ &= \frac{1}{2}[\varepsilon(u_1', u_1') + \varepsilon(u_2', u_2') + (a_{11}u_1 + a_{12}u_2, u_1) + (a_{21}u_1 + a_{22}u_2, u_2)] \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{2}[\varepsilon(|u_1|_1^2+|u_2|_1^2)+\int_0^1 \alpha(u_1^2+u_2^2)dx] \\ &= \frac{1}{2}[\varepsilon(|u_1|_1^2+|u_2|_1^2)+\alpha(u_1,u_1)+\alpha(u_2,u_2)] \end{aligned}$$

$$|B(\bar{u},\bar{u})| \geq \frac{1}{2}[\varepsilon(|u_1|_1^2+|u_2|_1^2)+\alpha(\|u_1\|_0^2+\|u_2\|_0^2)].$$

Therefore we have

$$|B(\bar{u},\bar{u})| \geq \frac{1}{2}|||\bar{u}|||^2.$$

□

Also we observe that B is continuous in the energy norm, that is, $| B(\bar{u},\bar{v}) | \leq \beta|||\bar{u}|||\cdot|||\bar{v}|||$ for some $\beta > 0$. Further f^* is a bounded linear functional on $(H_0^1(\Omega))^2$. By Lax-Milgram Theorem [14] we conclude that the problem (2.1) has a unique solution.

2.1. **Discretization of Weak Problem.** Let $\bar{\Omega}_\varepsilon^N = \{x_0, x_1, \ldots, x_N\}$ to be the set of mesh points x_i, for some positive integer N. For $i \in \{1,2,\ldots,N\}$ we set $h_i = x_i - x_{i-1}$ to be the local mesh step size, and for $i \in \{1,2,\ldots,N\}$ let $\bar{h}_i = (h_i + h_{i+1})/2$. We use linear finite elements with a lumping process. That is, for discretization of (2.2) and (2.3)

$$B_{1h}(\bar{u},\bar{v}) := \varepsilon(u_1',v_1') + \sum_{i=1}^{N-1}\bar{h}_i a_{11}(x_i)u_{1,i}v_{1,i} + \sum_{i=1}^{N-1}\bar{h}_i a_{12}(x_i)u_{2,i}v_{1,i},$$

$$B_{2h}(\bar{u},\bar{v}) := \varepsilon(u_2',v_2') + \sum_{i=1}^{N-1}\bar{h}_i a_{21}(x_i)u_{1,i}v_{2,i} + \sum_{i=1}^{N-1}\bar{h}_i a_{22}(x_i)u_{2,i}v_{2,i}$$

and $f_k^*(\bar{v})$ is replaced by $\sum_{i=1,i\neq\frac{N}{2}}^{N-1}\bar{h}_i f_{k,i}v_{k,i} + \frac{1}{2}(h_{\frac{N}{2}}f_{k,\frac{N}{2}-1}v_{k,\frac{N}{2}-1} + h_{\frac{N}{2}+1}f_{k,\frac{N}{2}+1}v_{k,\frac{N}{2}+1})$ for $k = 1,2$. and $u_{k,i} = u_k(x_i)$. Then we have

$$\begin{aligned} B_h(\bar{u},\bar{v}) &:= (B_{1h}(\bar{u},\bar{v}), B_{2h}(\bar{u},\bar{v})), \\ \text{and} \qquad f_h^*(\bar{v}) &:= (f_{1h}^*(\bar{v}), f_{2h}^*(\bar{v})). \end{aligned}$$

Now the discrete problem of (2.1) is: Find $\bar{u}_h \in V_h^2$ such that

$$B_h(\bar{u}_h,\bar{v}_h) = f_h^*(\bar{v}_h), \quad \forall \bar{v}_h \in V_h^2, \tag{2.4}$$

where $V_h^2 = V_h \times V_h$, V_h is a finite dimensional subspace of $H_0^1(\Omega)$ and the basis functions of V_h are given by

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h_i}, & x \in [x_{i-1},x_i] \\ \frac{x_{i+1}-x}{h_{i+1}}, & x \in [x_i,x_{i+1}] \\ 0, & x \notin [x_{i-1},x_{i+1}]. \end{cases}$$

Then $\{\bar{\Phi}_i\}_{i=1}^{2N-2}$ where $\bar{\Phi}_i = (\phi_i, 0)$ for $i = 1(1)N-1$ and $\bar{\Phi}_i = (0, \phi_{N-i+1})$ for $i = N(1)2N-2$, is a basis function of V_h^2. Here we define a discrete energy norm on V_h^2 associated with a bilinear form $B_h(\cdot,\cdot)$ as

$$|||\bar{u}_h|||_{V_h} = [\varepsilon(|u_{1h}|_1^2 + |u_{2h}|_1^2) + \alpha(\|u_{1h}\|^2_{0,\Omega_\varepsilon^N} + \|u_{2h}\|^2_{0,\Omega_\varepsilon^N}]^{1/2}$$

where $\|u_h\|^2_{0,\Omega_\varepsilon^N} := \sum_{k=1}^{N-1} \bar{h}_k u_{h,k}^2$ and $\|\bar{u}_h\|_{0,\Omega_\varepsilon^N} := [\|u_{1h}\|^2_{0,\Omega_\varepsilon^N} + \|u_{2h}\|^2_{0,\Omega_\varepsilon^N}]^{1/2}$ is a discrete norm on V_h^2. B_h is a bilinear functional defined on V_h^2 and it is coercive with respect to $|||.|||_{V_h}$. That is, $|B_h(\bar{u}_h, \bar{u}_h)| \geq \varsigma |||\bar{u}_h|||^2_{V_h}$, for some $\varsigma > 0$. We can also prove that it is continuous and f_h^* is bounded linear functional on V_h^2. By Lax-Milgram Theorem, we conclude that the discrete problem (2.4) admits a unique solution [14].

The difference scheme corresponding to the discrete problem (2.4) is

$$\bar{L}^N(\bar{U}_i) := \begin{cases} (L_1^N \bar{U}_i, L_2^N \bar{U}_i) = (\bar{h}_i f_{1,i}, \bar{h}_i f_{2,i}), \quad i = 1(1)N-1, \quad i \neq \frac{N}{2}, \\ (L_1^N \bar{U}_{\frac{N}{2}}, L_2^N \bar{U}_{\frac{N}{2}}) = (\frac{1}{2}[h_{\frac{N}{2}} f_{1,\frac{N}{2}-1} + h_{\frac{N}{2}+1} f_{1,\frac{N}{2}+1}], \\ \qquad\qquad \frac{1}{2}[h_{\frac{N}{2}} f_{2,\frac{N}{2}-1} + h_{\frac{N}{2}+1} f_{2,\frac{N}{2}+1}]), \\ \bar{U}_0 = \bar{U}_N = \bar{0} = (0,0), \end{cases} \tag{2.5}$$

where

$$L_1^N \bar{U}_i = -\varepsilon\left(\frac{U_{1,i+1} - U_{1,i}}{h_{i+1}} - \frac{U_{1,i} - U_{1,i-1}}{h_i}\right) + \bar{h}_i a_{11}(x_i) U_{1,i} + \bar{h}_i a_{12}(x_i) U_{2,i}$$
$$L_2^N \bar{U}_i = -\varepsilon\left(\frac{U_{2,i+1} - U_{2,i}}{h_{i+1}} - \frac{U_{2,i} - U_{2,i-1}}{h_i}\right) + +\bar{h}_i a_{21}(x_i) U_{1,i} + \bar{h}_i a_{22}(x_i) U_{2,i}$$

and $\bar{U}_i = (U_{1,i}, U_{2,i})$, $U_{1,i} = U_1(x_i)$, $f_{1,i} = f_1(x_i)$ and similarly for $U_{2,i}$, $f_{2,i}$, $i = 1(1)N-1$.

3. ERROR ANALYSIS - I

Now the given discrete problem is: Find $\bar{u}_h \in V_h^2 \subset (H_0^1(\Omega))^2$ such that

$$B_h(\bar{u}_h, \bar{v}_h) = f_h^*(\bar{v}_h), \quad \forall \bar{v}_h \in V_h^2. \tag{3.1}$$

Since the above discrete problem admits a unique solution and some interpolant $\bar{u}^I \in V_h^2$ of $\bar{u}$ is well defined. We define a biorthogonal basis of V_h^2 with respect to B_h to be the set of functions $\{\bar{\Lambda}^j\}_{j=1}^{2N-2}$ where $\bar{\Lambda}^j = (\lambda_1^j, \lambda_2^j)$ for $j = 1(1)2N-2$, which satisfies the condition

$$B_h(\bar{\Phi}_i, \bar{\Lambda}^j) = (\delta_{ij}, \delta_{ij}) \quad \text{for} \quad i,j = 1(1)2N-2. \tag{3.2}$$

In otherwords

$$\begin{aligned} B_{1h}(\bar{\Phi}_i, \bar{\Lambda}^j) &= \delta_{ij} \quad \text{for} \quad i,j = 1(1)2N-2, \\ B_{2h}(\bar{\Phi}_i, \bar{\Lambda}^j) &= \delta_{ij}, \end{aligned}$$

where δ_{ij} is the Kronecker symbol. Then the components u_{1h} and u_{2h} can be uniquely represented as

$$\begin{aligned} u_{1h} &= \sum_{i=1}^{N-1} B_{1h}((u_{1h}, u_{2h}), (\lambda_1^i, \lambda_2^i))\phi_i \\ \text{and} \quad u_{2h} &= \sum_{i=1}^{N-1} B_{2h}((u_{1h}, u_{2h}), (\lambda_1^{N+i-1}, \lambda_2^{N+i-1}))\phi_i. \end{aligned}$$

Define linear transformations $P_1, P_2 : (H_0^1(\Omega))^2 \longrightarrow V_h$ such that

$$\begin{aligned} P_1\bar{u} &:= \sum_{i=1}^{N-1} B_{1h}((u_1, u_2), (\lambda_1^i, \lambda_2^i))\phi_i \\ \text{and} \quad P_2\bar{u} &:= \sum_{i=1}^{N-1} B_{2h}((u_1, u_2), (\lambda_1^{N+i-1}, \lambda_2^{N+i-1}))\phi_i. \end{aligned}$$

Let $\bar{P} = (P_1, P_2)$ and $\bar{u}_h \in V_h^2$. Then

$$\begin{aligned} \bar{P}\bar{u}_h &= (P_1\bar{u}_h, P_2\bar{u}_h) \\ &= \Big(\sum_{i=1}^{N-1} B_{1h}((u_{1h}, u_{2h}), (\lambda_1^i, \lambda_2^i))\phi_i, \\ &\qquad \sum_{i=1}^{N-1} B_{2h}((u_{1h}, u_{2h}), (\lambda_1^{N+i-1}, \lambda_2^{N+i-1}))\phi_i\Big) \\ &= (u_{1h}, u_{2h}). \end{aligned}$$

$$\text{That is,} \quad \bar{P}\bar{u}_h = \bar{u}_h, \qquad \forall \bar{u}_h \in V_h^2.$$

Hence $\bar{P}$ is a projection operator on V_h^2. Now, the error $\bar{u} - \bar{u}_h$ can be written as,

$$\bar{u} - \bar{u}_h = \bar{u} - \bar{u}^I + \bar{P}(\bar{u}^I - \bar{u}) + \bar{P}\bar{u} - \bar{u}_h. \tag{3.3}$$

We estimate this error in the rest of this section.

3.1. **Shishkin and Bakhvalov-Shishkin Meshes.** For the discretization described above we shall use meshes of the general type introduced in [5], but here adapted for the boundary layers at $x = 0$ and $x = 1$ and the interior layers at $x = d$. Let $N > 8$ be a positive even integer and

$$\sigma_1 = \min\{\frac{d}{4}, \tau\sqrt{\frac{\varepsilon}{\gamma}}\ln N\}, \quad \sigma_2 = \min\{\frac{1-d}{4}, \tau\sqrt{\frac{\varepsilon}{\gamma}}\ln N\}, \quad \tau \geq 2.$$

Our mesh will be equidistant on $\bar{\Omega}_S$, where

$$\Omega_S = (\sigma_1, d - \sigma_1) \cup (d + \sigma_2, 1 - \sigma_2)$$

and graded on $\bar{\Omega}_0$ where

$$\Omega_0 = (0, \sigma_1) \cup (d - \sigma_1, d) \cup (d, d + \sigma_2) \cup (1 - \sigma_2, 1).$$

We choose the transition points to be

$$x_{N/8} = \sigma_1, \quad x_{3N/8} = d - \sigma_1, \quad x_{N/2} = d, \quad x_{5N/8} = d + \sigma_2, \quad x_{7N/8} = 1 - \sigma_2.$$

Because of the specific layers, here we have to use four mesh generating functions φ_1, φ_2, φ_3 and φ_4 which are all continuous and piecewise continuously differentiable, with the following properties: φ_1 and φ_3 are monotonically increasing, φ_2 and φ_4 are monotonically decreasing functions and

$$\begin{aligned} \varphi_1(0) &= 0, & \varphi_1(1/8) &= \ln N, \\ \varphi_2(3/8) &= \ln N, & \varphi_2(1/2) &= 0, \\ \varphi_3(1/2) &= 0, & \varphi_3(5/8) &= \ln N, \\ \varphi_4(7/8) &= \ln N, & \varphi_4(1) &= 0. \end{aligned}$$

The mesh points are

$$x_i = \begin{cases} 2\sqrt{\frac{\varepsilon}{\gamma}}\varphi_1(t_i), & i = 0(1)N/8, \\ \sigma_1 + \frac{4}{N}(d - 2\sigma_1)(i - N/8), & i = N/8 + 1(1)3N/8, \\ d - 2\sqrt{\frac{\varepsilon}{\gamma}}\varphi_2(t_i), & i = 3N/8 + 1(1)N/2, \\ d + 2\sqrt{\frac{\varepsilon}{\gamma}}\varphi_3(t_i), & i = N/2 + 1(1)5N/8, \\ d + \sigma_2 + \frac{4}{N}(1 - d - 2\sigma_2)(i - 5N/8), & i = 5N/8 + 1(1)7N/8, \\ 1 - 2\sqrt{\frac{\varepsilon}{\gamma}}\varphi_4(t_i), & i = 7N/8 + 1(1)N, \end{cases}$$

where $t_i = i/N$. We define new functions ψ_1, ψ_2, ψ_3 and ψ_4 by

$$\varphi_i = -\ln \psi_i, \quad i = 1(1)4.$$

There are several mesh-characterizing functions ψ_i in the literature, but we shall use only those which correspond to Shishkin mesh and Bakhvalov-Shishkin mesh with the following properties

$$\begin{aligned} \max |\psi'| &= C \ln N \quad \text{for Shishkin mesh,} \\ \max |\psi'| &= C \quad \text{for Bakhvalov-Shishkin mesh.} \end{aligned}$$

For Shishkin mesh we take

$$\begin{aligned} \psi_1(t) &= e^{-8t \ln N}, & \psi_2(t) &= e^{-4(1-2t) \ln N}, \\ \psi_3(t) &= e^{-4(2t-1) \ln N}, & \psi_4(t) &= e^{-8(1-t) \ln N}, \end{aligned}$$

whereas for Bakhvalov-Shishkin mesh

$$\begin{aligned} \psi_1(t) &= 1 - 8(1 - N^{-1})t, & \psi_2(t) &= 1 - 4(1 - N^{-1})(1 - 2t), \\ \psi_3(t) &= 1 - 4(1 - N^{-1})(2t - 1), & \psi_4(t) &= 1 - 8(1 - N^{-1})(1 - t). \end{aligned}$$

The set of interior mesh points is denoted by $\Omega_\varepsilon^N = \bar{\Omega}_\varepsilon^N \setminus \{x_0, x_{N/2}, x_N\}$. Also, for the both meshes, on the coarse part Ω_S we have

$$h_i \le CN^{-1}.$$

It is well known that on the layer part [8]

$$h_i \le C\sqrt{\varepsilon}N^{-1}\ln N \quad \text{for Shishkin mesh}$$

and

$$h_i \le C(\sqrt{\varepsilon} + N^{-1}) \quad \text{for Bakhvalov-Shishkin mesh.}$$

In the later analysis, the following estimates of $e_1(x)$ and $e_2(x)$ will be used [8]:

$$e_1(x) \le \begin{cases} C, & x \in \Omega^- \cap \Omega_0 \\ CN^{-\tau}, & x \in \Omega^- \cap \Omega_S. \end{cases} \qquad e_2(x) \le \begin{cases} C, & x \in \Omega^+ \cap \Omega_0 \\ CN^{-\tau}, & x \in \Omega^+ \cap \Omega_S. \end{cases} \tag{3.4}$$

3.2. **Interpolation Error.** To derive an error estimate, we consider the interpolation error in the maximum norm. Let $f \in C^2[x_{i-1}, x_i]$ be arbitrary and f^I a piecewise linear interpolant to f on Ω. Then from the classical theory, we have

$$|(f^I - f)(x)| \le 2\int_{x_{i-1}}^{x_i} |f''(t)|(t - x_{i-1})dt, \quad x \in [x_{i-1}, x_i].$$

Now we compute the interpolation error for $u_i, i = 1, 2$ seperately.

Lemma 3.1. *If $\sqrt{\varepsilon} \le CN^{-1}$ and for the Shishkin mesh, we have*

$$|u_i(x) - u_i^I(x)| \le \begin{cases} CN^{-2}\ln^2 N, & x \in \Omega_0 \\ CN^{-2}, & x \in \Omega_S \end{cases}$$

and for the Bakhavalov-Shishkin mesh it holds

$$|u_i(x) - u_i^I(x)| \le CN^{-2}, \quad x \in \Omega^- \cup \Omega^+, \quad i = 1, 2.$$

Proof. We now give a proof for the case $i = 1$ for the Shishkin mesh. To prove the estimate we use the decomposition of solution as smooth and layer components and triangle inequality

$$|(u_1 - u_1^I)(x)| \le |(v_1 - v_1^I)(x)| + |(w_1 - w_1^I)(x)|. \tag{3.5}$$

On Shishkin mesh, let $x \in [x_{i-1}, x_i] \subset \Omega^- \cap \Omega_S$. Then by using (3.4) we have to compute the first term of (3.5)

$$\begin{aligned} |(v_1 - v_1^I)(x)| &\le 2\int_{x_{i-1}}^{x_i} |v_1''(t)|(t - x_{i-1})dt \\ &\le 2C\int_{x_{i-1}}^{x_i} (t - x_{i-1})dt + 2C\int_{x_{i-1}}^{x_i} \mid e_1(t) \mid (t - x_{i-1})dt \\ &\le C\frac{h_i^2}{2} + CN^{-\tau}\frac{h_i^2}{2} \end{aligned}$$

$$|(v_1 - v_1^I)(x)| \le CN^{-2}.$$

Again the second term of (3.5) will be

$$\begin{aligned} |(w_1 - w_1^I)(x)| &\le 2\|w_1(x)\|_{L_\infty[x_{i-1},x_i]} \\ &\le CN^{-\tau} \\ |(w_1 - w_1^I)(x)| &\le CN^{-2}. \end{aligned}$$

Now let $x \in [x_{i-1}, x_i] \subset \Omega^- \cap \Omega_0$ we have

$$\begin{aligned} |(v_1 - v_1^I)(x)| &\le 2\int_{x_{i-1}}^{x_i} |v_1''(t)|(t - x_{i-1})dt \\ &\le 2C\int_{x_{i-1}}^{x_i} (t - x_{i-1})dt + 2C\int_{x_{i-1}}^{x_i} \mid e_1(t) \mid (t - x_{i-1})dt \\ &\le C\frac{h_i^2}{2} \\ |(v_1 - v_1^I)(x)| &\le CN^{-2}, \end{aligned}$$

and also the layer component will be

$$\begin{aligned} |(w_1 - w_1^I)(x)| &\le 2\int_{x_{i-1}}^{x_i} |w_1''(t)|(t - x_{i-1})dt \\ &\le 2C\varepsilon^{-1}\int_{x_{i-1}}^{x_i} \mid e_1(t) \mid (t - x_{i-1})dt \\ &\le C\varepsilon^{-1}\frac{h_i^2}{2} \\ &\le C\varepsilon^{-1}(\sqrt{\varepsilon}N^{-1}\ln N)^2 \\ |(w_1 - w_1^I)(x)| &\le CN^{-2}\ln^2 N. \end{aligned}$$

Similarly we can obtain a similar estimate for $x \in \Omega^+$.

To prove estimates on Bakhvalov-Shishkin mesh, we follow the above procedure. If $x \in [x_{i-1}, x_i] \subset \Omega_S$ then $h_i \le CN^{-1}$ and if $x \in [x_{i-1}, x_i] \subset \Omega_0$ then $h_i \le C(\sqrt{\varepsilon} + N^{-1})$. Using the fact that $\max|\psi'| = C$ and $\sqrt{\varepsilon} \le CN^{-1}$ we can arrive the required result. □

Then the interpolation error of $\bar{u}$ in maximum norm is

$$\|\bar{u} - \bar{u}^I\|_\infty \le \begin{cases} CN^{-2}\ln^2 N, & \text{for Shishkin mesh,} \\ CN^{-2}, & \text{for Bakhalov-Shishkin mesh.} \end{cases} \tag{3.6}$$

4. **ERROR ANALYSIS - II**

Let $x_k \in \bar{\Omega}_\varepsilon^N$ be a mesh point. From equation (3.3), the second term at the points of the mesh is

$$\bar{P}(\bar{u}^I - \bar{u})(x_k) = (P_1(\bar{u}^I - \bar{u})(x_k), P_2(\bar{u}^I - \bar{u})(x_k)).$$

Each of the components of the above will be estimated seperately. We have

$$\begin{aligned}P_1(\bar{u}^I-\bar{u})(x_k) &= B_{1h}(((u_1^I-u_1),(u_2^I-u_2)),(\lambda_1^k,\lambda_2^k))\\ &= \varepsilon((u_1^I-u_1)',(\lambda_1^k)')+\sum_{i=1}^{N-1}\bar{h}_i a_{11}(x_i)(u_1^I-u_1)(x_i)\lambda_{1,i}^k\\ &\quad+\sum_{i=1}^{N-1}\bar{h}_i a_{12}(x_i)(u_2^I-u_2)(x_i)\lambda_{1,i}^k.\end{aligned}$$

Using integration by parts and the fact that $(\lambda_1^k)''=0,\quad u_1^I(x_i)=u_1(x_i),\quad u_2^I(x_i)=u_2(x_i)$ for $i=1(1)N-1$, we have $P_1(\bar{u}^I-\bar{u})(x_k)=0$. Similarly $P_2(\bar{u}^I-\bar{u})(x_k)=0$. Therefore, we have

$$\bar{P}(\bar{u}^I-\bar{u})(x_k)=\bar{0}. \tag{4.1}$$

The remaining part of this section is devoted for the estimation of the third term of the error representation (3.3). For this representation, we first need the following L^1- estimates of discrete Green's functions λ_1^j and λ_2^j.

Lemma 4.1. *On an arbitrary mesh, the discrete Green's functions $(\lambda_1^j,\lambda_2^j)$ for B_h satisfy $\|\lambda_1^j\|_{L^1(\Omega)}\le C$ and $\|\lambda_2^j\|_{L^1(\Omega)}\le C$, where $\|\lambda_1^j\|_{L^1(\Omega)}=\sum_{i=1}^N\int_{x_{i-1}}^{x_i}|\lambda_1^j|dx$ and $\|\lambda_2^j\|_{L^1(\Omega)}=\sum_{i=1}^N\int_{x_{i-1}}^{x_i}|\lambda_2^j|dx$.*

Proof. Following the procedure adopted in [8, 10], we can prove this theorem. □

Let $\bar{K}=(K_1,K_2)=\bar{P}\bar{u}-\bar{u}_h=((P_1\bar{u}-u_{1h}),(P_2\bar{u}-u_{2h}))$. That is, $K_1=P_1\bar{u}-u_{1h}$ and $K_2=P_2\bar{u}-u_{2h}$. Now,

$$\begin{aligned}K_1 &= P_1\bar{u}-u_{1h}\\ &= \sum_{i=1}^{N-1}B_{1h}((u_1,u_2),(\lambda_1^i,\lambda_2^i))\phi_i-\sum_{i=1}^{N-1}B_{1h}((u_{1h},u_{2h}),(\lambda_1^i,\lambda_2^i))\phi_i\\ &= \sum_{i=1}^{N-1}B_{1h}((u_1,u_2),(\lambda_1^i,\lambda_2^i))\phi_i+\sum_{i=1}^{N-1}f_1^*((\lambda_1^i,\lambda_2^i))\phi_i-\sum_{i=1}^{N-1}B_1((u_1,u_2),(\lambda_1^i,\lambda_2^i))\phi_i\\ &\quad-\sum_{i=1}^{N-1}B_{1h}((u_{1h},u_{2h}),(\lambda_1^i,\lambda_2^i))\phi_i\\ &= \sum_{i=1}^{N-1}(B_{1h}-B_1)((u_1,u_2),(\lambda_1^i,\lambda_2^i))-\sum_{i=1}^{N-1}f_{1h}^*((\lambda_1^i,\lambda_2^i))\phi_i+\sum_{i=1}^{N-1}f_1^*((\lambda_1^i,\lambda_2^i))\phi_i.\\ K_1 &= \sum_{i=1}^{N-1}(B_{1h}-B_1)((u_1,u_2),(\lambda_1^i,\lambda_2^i)\phi_i+\sum_{i=1}^{N-1}(f_1^*-f_{1h}^*)((\lambda_1^i,\lambda_2^i))\phi_i,\end{aligned}$$

where $u_1(x_i)=u_{1,i}$, $u_2(x_i)=u_{2,i}$ and $a_{12}(x_i)=a_{12,i}$. Then we have

$$\begin{aligned}K_1(x_k)&=(B_{1h}-B_1)((u_1,u_2),(\lambda_1^k,\lambda_2^k))+(f_1^*-f_{1h}^*)((\lambda_1^k,\lambda_2^k))\\ &=B_{1h}((u_1,u_2),(\lambda_1^k,\lambda_2^k))\end{aligned}$$

$$
\begin{aligned}
&- B_1((u_1,u_2),(\lambda_1^k,\lambda_2^k)) + f_1^*((\lambda_1^k,\lambda_2^k)) - f_{1h}^*((\lambda_1^k,\lambda_2^k)) \\
&= \varepsilon(u_1',(\lambda_1^k)') + \sum_{i=1}^{N-1}\bar{h}_i a_{11,i}u_{1,i}\lambda_{1,i}^k + \sum_{i=1}^{N-1}\bar{h}_i a_{12,i}u_{2,i}\lambda_{1,i}^k - \varepsilon(u_1',(\lambda^k)') \\
&\quad - \int_0^1 a_{11}(x)u_1(x)\lambda^k(x)dx - \int_0^1 a_{12}(x)u_2(x)\lambda^k(x)dx \\
&\quad + \int_0^1 f_1(x)\lambda_1^k(x)dx - \sum_{i=1}^{N-1}\bar{h}_i f_{1,i}\lambda_{1,i}^k
\end{aligned}
$$

$$
\begin{aligned}
K_1(x_k) &= (\sum_{i=1}^{N-1}\bar{h}_i a_{11,i}u_{1,i}\lambda_{1,i}^k - \int_0^1 a_{11}(x)u_1(x)\lambda^k(x)dx) + (\sum_{i=1}^{N-1}\bar{h}_i a_{12,i}u_{2,i}\lambda_{1,i}^k \\
&\quad - \int_0^1 a_{12}(x)u_2(x)\lambda_1^k(x)dx) + (\int_0^1 f_1(x)\lambda_1^k(x)dx - \sum_{i=1}^{N-1}\bar{h}_i f_{1,i}\lambda_{1,i}^k).
\end{aligned}
$$

Similarly we get

$$
\begin{aligned}
K_2(x_k) &= (\sum_{i=1}^{N-1}\bar{h}_i a_{21,i}u_{1,i}\lambda_{2,i}^{N+k-1} - \int_0^1 a_{21}(x)u_1(x)\lambda_2^{N+k-1}(x)dx) + (\sum_{i=1}^{N-1}\bar{h}_i a_{22,i}u_{2,i}\lambda_{2,i}^{N+k-1} \\
&\quad - \int_0^1 a_{22}(x)u_2(x)\lambda_2^{N+k-1}(x)dx) + (\int_0^1 f_2(x)\lambda_2^{N+k-1}(x)dx - \sum_{i=1}^{N-1}\bar{h}_i f_{2,i}\lambda_{2,i}^{N+k-1}).
\end{aligned}
$$

Now we define

$$
\begin{aligned}
K_1^*(x_k) &= \sum_{i=1}^{N-1}\bar{h}_i a_{11,i}u_{1,i}\lambda_{1,i}^k - \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(a_{11}(x)u_1(x))^I\lambda_1^k(x)dx + \sum_{i=1}^{N-1}\bar{h}_i a_{12,i}u_{2,i}\lambda_{1,i}^k \\
&\quad - \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(a_{12}(x)u_2(x))^I\lambda_1^k(x)dx + \sum_{i=1}^{N}\int_{x_{i=1}}^{x_i} f_1^I(x)\lambda_1^k(x)dx - \sum_{i=1}^{N-1}\bar{h}_i f_{1,i}\lambda_{1,i}^k.
\end{aligned}
$$

Then we can write $K_1(x_k)$ as

$$
\begin{aligned}
K_1(x_k) &= K_1^*(x_k) + \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}((a_{11}(x)u_1(x))^I - (a_{11}(x)u_1(x)))\lambda_1^k(x)dx \\
&\quad + \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}((a_{12}(x)u_2(x))^I - (a_{12}(x)u_2(x)))\lambda_1^k(x)dx \\
&\quad - \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}((f_1(x))^I - f_1(x))\lambda_1^k(x)dx.
\end{aligned} \tag{4.2}
$$

The later sums of $K_1(x_k)$ can be bounded by

$$
\begin{aligned}
& \mid \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}((a_{11}(x)u_1(x))^I - (a_{11}(x)u_1(x)))\lambda_1^k(x)dx \mid \\
\le\ & C(\|u_1 - u_1^I\|_\infty\|a_{11}\|_\infty + \|a_{11}^I - a_{11}\|_\infty\|u_1\|_\infty)\|\lambda_1^k\|_{L^1(\Omega)} \\
\le\ & C(\|u_1 - u_1^I\|_\infty + N^{-2}\|u_1\|_\infty)\|\lambda_1^k\|_{L^1(\Omega)}
\end{aligned}
$$

$$\begin{aligned}&\leq C(\|u_1 - u_1^I\|_\infty + N^{-2})\\ &\leq CN^{-2}\max|\psi'|^2,\end{aligned}$$

$$|\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}((a_{11}(x)u_1(x))^I - (a_{11}(x)u_1(x)))\lambda_1^k(x)dx| \leq CN^{-2}\max|\psi'|^2, \tag{4.3}$$

$$|\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}((a_{12}(x)u_2(x))^I - (a_{12}(x)u_2(x)))\lambda_1^k(x)dx| \leq CN^{-2}\max|\psi'|^2, \tag{4.4}$$

and

$$|\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(f_1 - f_1^I)\lambda_1^k(x)dx\,| \quad \leq CN^{-2}\|\lambda_1^k\|_{L^1(\Omega)} \quad \leq CN^{-2}. \tag{4.5}$$

If we define $K_2^*(x_k)$ similar to $K_1^*(x_k)$, then we can write

$$\begin{aligned}K_2(x_k) = K_2^*(x_k) &+ \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}((a_{21}(x)u_1(x))^I - (a_{21}(x)u_1(x)))\lambda_2^{N+k-1}(x)dx\\ &+\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}((a_{22}(x)u_2(x))^I - (a_{22}(x)u_2(x)))\lambda_2^{N+k-1}(x)dx\\ &-\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}((f_2(x))^I - f_2(x))\lambda_2^{N+k-1}(x)dx.\end{aligned} \tag{4.6}$$

We can also estimate the later sums of $K_2(x_k)$ as done for $K_1(x_k)$. In the pointwise errors, $K_1(x_k)$ and $K_2(x_k)$ it remains only to estimate the expressions $K_1^*(x_k)$ and $K_2^*(x_k)$. First we write $K_1^*(x_k)$ and $K_2^*(x_k)$ in the form

$$K_1^*(x_k) = \langle (a_{11}u_1)^I, \lambda_1^k\rangle_h + \langle (a_{12}u_2)^I, \lambda_1^k\rangle_h - \langle f_1^I, \lambda_1^k\rangle_h, \tag{4.7}$$

$$K_2^*(x_k) = \langle (a_{21}u_1)^I, \lambda_2^{N+k-1}\rangle_h + \langle (a_{22}u_2)^I, \lambda_2^{N+k-1}\rangle_h - \langle f_2^I, \lambda_2^{N+k-1}\rangle_h, \tag{4.8}$$

where

$$\langle g, \omega^k\rangle_h = \sum_{i=1}^{N-1}\bar{h}_i g(x_i)\omega_i^k - \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} g(x)\omega^k(x)dx,$$

for a piecewise linear function g, not neccessarily continuous. For integrals in the previous formula, we use Simpson's rule

$$\langle g, \omega^k\rangle_h = \frac{1}{6}\sum_{i=1}^{N-1}(h_i(g_i^- - g_{i-1}^+) - h_{i+1}(g_{i+1}^- - g_i^+))\omega_i^k. \tag{4.9}$$

In order to estimate $K_1^*(x_k)$ and $K_2^*(x_k)$, we start with the decomposition of the solution $\bar{u}$. Hence we seperately analyze smooth part $\bar{v}$ and the layer part $\bar{w}$. Now the equation (4.7) can be rewritten as

$$K_1^*(x_k) = \langle (\varepsilon v_1'')^I, \lambda_1^k\rangle_h + \langle (a_{11}w_1 + a_{12}w_2)^I, \lambda_1^k\rangle_h. \tag{4.10}$$

The first term in the above expression containing the regular component v_1 can be easily estimated. In fact,

$$| \langle (\varepsilon v_1'')^I, \lambda_1^k \rangle_h | \leq \frac{\varepsilon}{6} \sum_{i=1}^{N-1} (h_i(v''^{-}_{1,i} - -v''^{+}_{1,i-1}) - h_{i+1}(v''^{-}_{1,i+1} - v''^{+}_{1,i}))\lambda_{1,i}^k.$$

Then we have

$$\begin{aligned} | \langle (\varepsilon v_1'')^I, \lambda_1^k \rangle_h | &\leq \frac{\varepsilon}{6} \sum_{i=1}^{N-1} (h_i^2 \|v_1'''\|_{L^\infty[x_{i-1},x_i]} + h_{i+1}^2 \|v_1'''\|_{L^\infty[x_i,x_{i+1}]} \lambda_{1,i}^k \\ &\leq \frac{\varepsilon}{3}(CN^{-1}) \|v_1'''\|_{L^\infty(\Omega)} \sum_{i=1}^{N-1} (\frac{h_i + h_{i+1}}{2}) \lambda_{1,i}^k \\ &\leq C\sqrt{\varepsilon} N^{-1} \sum_{i=1}^{N-1} \bar{h}_i \lambda_{1,i}^k \\ &\leq C\sqrt{\varepsilon} N^{-1} \|\lambda_1^k\|_{L^1(\Omega)}, \end{aligned}$$

by using (1.2), $h_i \leq CN^{-1}, i = 1(1)N-1$ and $\|\lambda_1^k\|_{L^1(\Omega)} \leq C$. Finally, we get

$$| \langle (\varepsilon v_1'')^I, \lambda_1^k \rangle_h | \leq C\sqrt{\varepsilon} N^{-1}. \tag{4.11}$$

Let us denote the coefficient in $< (a_{11}w_1 + a_{12}w_2)^I, \lambda_1^k >_h$ corresponding to $\lambda_{1,i}^k$ by m_i. Depending on the values of index i, we consider different cases. In general, $g_i^\pm$ denotes right-limit and left-limit of a function g at a mesh point x_i.

Case 1: When $\frac{N}{8} + 1 \leq i \leq \frac{3N}{8} - 1$ or $\frac{5N}{8} + 1 \leq i \leq \frac{7N}{8} - 1$. That is, $[x_{i-1}, x_{i+1}] \subset \Omega_S$. The coefficient m_i can be estimated by

$$\begin{aligned} |m_i| &= | h_i(a_{11,i}^- w_{1,i}^- - a_{11,i-1}^+ w_{1,i-1}^+) - h_{i+1}(a_{11,i+1}^- w_{1,i+1}^- - a_{11,i}^+ w_{1,i}^+) | \\ &\quad + | h_i(a_{12,i}^- w_{2,i}^- - a_{12,i-1}^+ w_{2,i-1}^+) - h_{i+1}(a_{12,i+1}^- w_{2,i+1}^- - a_{12,i}^+ w_{2,i}^+) | \\ &\leq C\bar{h}_i[\|w_1\|_{L_\infty[x_{i-1},x_{i+1}]} + \|w_2\|_{L_\infty[x_{i-1},x_{i+1}]}] \\ &\leq C\bar{h}_i[\max_{x\in\Omega_S} |e_1(x)| + \max_{x\in\Omega_S} |e_2(x)|], \quad \text{from (1.2) and (3.4).} \end{aligned}$$

$$| m_i | \leq C\bar{h}_i N^{-\tau}. \tag{4.12}$$

Case 2: When $1 \leq i \leq \frac{N}{8} - 1$ or $\frac{3N}{8} + 1 \leq i \leq \frac{N}{2} - 1$ or $\frac{N}{2} + 1 \leq i \leq \frac{5N}{8} - 1$ or $\frac{3N}{8} + 1 \leq i \leq N - 1$. That is, the subinterval $[x_{i-1}, x_{i+1}] \subset \Omega_0$. The layer part will be calculated by estimating m_i. We have

$$\begin{aligned} m_i &= h_i(a_{11,i}^- w_{1,i}^- - a_{11,i-1}^+ w_{1,i-1}^+) - h_{i+1}(a_{11,i+1}^- w_{1,i+1}^- - a_{11,i}^+ w_{1,i}^+) \\ &\quad + h_i(a_{12,i}^- w_{2,i}^- - a_{12,i-1}^+ w_{2,i-1}^+) - h_{i+1}(a_{12,i+1}^- w_{2,i+1}^- - a_{12,i}^+ w_{2,i}^+) \\ &= h_i(-a_{11,i+1}w_{1,i+1} + 2a_{11,i}w_{1,i} - a_{11,i-1}w_{1,i-1}) \\ &\quad + (h_i - h_{i+1})(a_{11,i+1}w_{1,i+1} - a_{11,i}w_{1,i}) \\ &\quad + h_i(-a_{12,i+1}w_{2,i+1} + 2a_{12,i}w_{2,i} - a_{12,i-1}w_{2,i-1}) \end{aligned}$$

$$
\begin{aligned}
&+ (h_i - h_{i+1})(a_{12,i+1}w_{2,i+1} - a_{12,i}w_{2,i})\\
&= a_{11,i}(h_i(-w_{1,i+1} + 2w_{11,i} - w_{1,i-1}) + (h_i - h_{i+1})(w_{1,i+1} - w_{1,i}))\\
&+ h_i(a_{11,i} - a_{11,i-1})\\
&(w_{1,i-1} - w_{1,i}) + h_{i+1}(a_{11,i+1} - a_{11,i})(w_{1,i} - w_{1,i+1})\\
&+ w_{1,i}(-h_{i+1}a_{11,i+1} + (h_i + h_{i+1})\\
&a_{11,i} - h_i a_{11,i-1})a_{12,i}(h_i(-w_{2,i+1} + 2w_{21,i} - w_{2,i-1}) + (h_i - h_{i+1})(w_{2,i+1} - w_{2,i}))\\
&+ h_i(a_{12,i} - a_{12,i-1})(w_{2,i-1} - w_{2,i}) + h_{i+1}(a_{12,i+1} - a_{12,i})(w_{2,i} - w_{2,i+1})\\
&+ w_{2,i}(-h_{i+1}a_{12,i+1} + (h_i + h_{i+1})a_{12,i} - h_i a_{12,i-1}).
\end{aligned}
$$

Using the Taylor's expansion for each of the terms in the previous expression yields

$$
\begin{aligned}
h_i a_{11,i}(-w_{1,i+1} + 2w_{1,i} - w_{1,i-1}) &= h_i(h_i - h_{i+1})a_{11,i}w'_{1,i}\\
&\quad - \frac{h_i^3}{2}a_{11,i}w''_1(\theta_i) - \frac{h_i h_{i+1}^2}{2}a_{11,i}w''_1(\theta_{i+1}),\\
h_i a_{12,i}(-w_{2,i+1} + 2w_{2,i} - w_{2,i-1}) &= h_i(h_i - h_{i+1})a_{12,i}w'_{2,i}\\
&\quad - \frac{h_i^3}{2}a_{12,i}w''_2(\theta_i) - \frac{h_i h_{i+1}^2}{2}a_{12,i}w''_2(\theta_{i+1}),\\
(h_i - h_{i+1})a_{11,i}(w_{1,i+1} - w_{1,i}) &= h_{i+1}(h_i - h_{i+1})a_{11,i}w_1^{'}(\xi_{i+1}),\\
(h_i - h_{i+1})a_{12,i}(w_{2,i+1} - w_{2,i}) &= h_{i+1}(h_i - h_{i+1})a_{12,i}w_2^{'}(\xi_{i+1}),\\
h_i(a_{11,i} - a_{11,i-1})(w_{1,i-1} - w_{1,i}) &= -h_i^3 a_{11}^{'}(\rho_i)w_1^{'}(\xi_i),\\
h_i(a_{12,i} - a_{12,i-1})(w_{2,i-1} - w_{2,i}) &= -h_i^3 a_{12}^{'}(\rho_i)w_2^{'}(\xi_i),\\
h_{i+1}(a_{11,i+1} - a_{11,i})(w_{1,i} - w_{1,i+1}) &= -h_{i+1}^3 a_{11}^{'}(\rho_{i+1})w'_1(\xi_{i+1}),\\
h_{i+1}(a_{12,i+1} - a_{12,i})(w_{2,i} - w_{2,i+1}) &= (h_i^2 - h_{i+1}^2)a_{12,i}^{'}w_{2,i}\\
&\quad - \frac{1}{2}(h_i^3 a''_{12}(\eta_k) + h_{i+1}^3 a''_{12}(\eta_{i+1}))w_{2,i},
\end{aligned}
$$

where $\theta_i, \xi_i, \rho_i, \eta_i \in [x_{i-1}, x_i]$.

To derive an estimate for $|m_i|$, we need the following lemma.

Lemma 4.2. *For the points $x_{i-1}, x_i, x_{i+1} \in \Omega_0$, $x_i \neq d = x_{\frac{N}{2}}$ of the mesh with $\tau \geq 2$ the following holds*

$$
\begin{aligned}
|\ (h_i - h_{i+1})(w_{1,i+1} - w_{1,i})\ | &\leq Ch_{i+1}N^{-2},\\
|\ (h_i - h_{i+1})(w_{2,i+1} - w_{2,i})\ | &\leq Ch_{i+1}N^{-2},\\
|\ (h_i - h_{i+1})w_{1,i}^{'}\ | &\leq CN^{-2}\\
\textit{and} \quad |\ (h_i - h_{i+1})w_{2,i}^{'}\ | &\leq CN^{-2}.
\end{aligned}
$$

Proof. Let $x_{i-1},\, x_i,\, x_{i+1} \in \bar{\Omega}_0 \cap \Omega^-$ and $x_i \neq d = x_{\frac{N}{2}}$

$$| \ h_i - h_{i+1} \ |= 2\sqrt{\frac{\varepsilon}{\gamma}} N^{-1} \ | \ \phi^{'}(\rho_i) - \phi^{'}(\rho_{i+1}) \ |$$

for $\rho_i,\, \rho_{i+1} \in (t_{i-1}, t_{i+1})$. Also $| \ w_{1,i+1} - w_{1,i} \ |= h_{i+1} \ | \ w^{'}_{1,i}(\alpha_{i+1}) \ |, \quad \alpha_{i+1} \in (x_i, x_{i+1})$

$$\begin{aligned} | \ (h_i - h_{i+1})(w_{1,i+1} - w_{1,i}) \ | &\leq C\sqrt{\varepsilon} h_{i+1} N^{-2} \ | \ \phi_1''(\psi_i) \ || \ w_1^{'}(\alpha_{i+1}) \ | \\ &\leq C\sqrt{\varepsilon} h_{i+1} N^{-2} (\frac{\psi^{'}(x_i)}{\psi_1(x_i)})^2 e_1(\alpha_{i+1}) \varepsilon^{\frac{-1}{2}} \\ &\leq C h_{i+1} N^{-2} (\frac{\max \psi^{'}}{\psi_1(x_i)})^2 e_1(\alpha_{i+1}) \\ &\leq C h_{i+1} N^{-2} (\psi_1(t_{i+1}))^{-2} e_1(\alpha_{i+1}). \end{aligned}$$

Using the fact that $\max | \ \psi_1^{'} \ | = C$ and $e_1(\alpha_{i+1}) \leq \psi_1(t_i)^2 + N^{-\tau}$, we have

$$\begin{aligned} | \ (h_i - h_{i+1})(w_{1,i+1} - w_{1,i}) \ | &\leq C h_{i+1} N^{-2} (\psi_1(t_i)^2 + N^{-\tau})(\psi_1(t_{i+1}))^{-2} \\ | \ (h_i - h_{i+1})(w_{1,i+1} - w_{1,i}) \ | &\leq C h_{i+1} N^{-2}, \end{aligned}$$

since $\tau \geq 2$. When $[x_{i-1}, x_{i+1}] \subset [d, d+\sigma_2]$ and $[x_{i-1}, x_{i+1}] \subset [1-\sigma_2, 1]$, the above estimate is also true for these intervals. From the previous analysis, we get

$$\begin{aligned} h_i a_{11,i}(-w_{1,i+1} + 2w_{1,i} - w_{1,i-1}) &\leq C h_i N^{-2} + C h_i N^{-2} \max | \ \psi_1^{'} \ |, \\ h_i a_{12,i}(-w_{2,i+1} + 2w_{2,i} - w_{2,i-1}) &\leq C h_i N^{-2} + C h_i N^{-2} \max | \ \psi_1^{'} \ |, \end{aligned}$$

and

$$\begin{aligned} (h_i - h_{i+1}) a_{11,i}(w_{1,i+1} - w_{1,i}) &\leq C h_{i+1} N^{-2}, \\ (h_i - h_{i+1}) a_{12,i}(w_{2,i+1} - w_{2,i}) &\leq C h_{i+1} N^{-2}. \end{aligned}$$

□

Applying the above Lemma 4.2 to each of the terms in m_i of Case 2, we have

$$|m_i| \leq C\bar{h}_i N^{-2} \max |\psi'|^2. \tag{4.13}$$

Now it remains to prove the estimates at the transition points.

Case 3: When $x_i, i \in \{\frac{N}{8}, \frac{3N}{8}, \frac{5N}{8}, \frac{7N}{8}\}$ and $i \neq \frac{N}{2}$. At these points $w_{1,i}, w_{1,i\pm1}$ and $w_{2,i}, w_{2,i\pm1}$ are bounded by $CN^{-\tau}$. Then, using the expression for $|m_i|$ given in Case 2,

$$| \ m_i \ |\leq C\bar{h}_i N^{-\tau}. \tag{4.14}$$

Case 4: When $i = \frac{N}{2}$. That is, $x_i = d$

$$\begin{aligned} m_i &= h_i(a^-_{11,i} w^-_{1,i} - a^+_{11,i-1} w^+_{1,i-1}) - h_{i+1}(a^-_{11,i+1} w^-_{1,i+1} - a^+_{11,i} w^+_{1,i}) \\ &\quad + h_i(a^-_{12,i} w^-_{2,i} - a^+_{12,i-1} w^+_{2,i-1}) - h_{i+1}(a^-_{12,i+1} w^-_{2,i+1} - a^+_{12,i} w^+_{2,i}) \\ &= h_i(-a_{11,i+1} w_{1,i+1} + a^+_{11,i} w^+_{1,i} + a^-_{11,i} w^-_{1,i} - a_{11,i-1} w_{1,i-1}) \end{aligned}$$

$$h_i(-a_{12,i+1}w_{2,i+1} + a^+_{12,i}w^+_{2,i} + a^-_{12,i}w^-_{2,i} - a_{12,i-1}w_{2,i-1})$$

$$\begin{aligned}|m_i| &\le h_i|(a^+_{11,i} - a_{11,i+1})w^+_{1,i} + (a^-_{11,i} - a_{11,i-1})w^-_{1,i}| + h_i|a_{11+1,i}(w^+_{1,i} - w_{1,i+1})\\ &\quad + a_{11,i-1}(w^-_{1,i} - w_{1,i-1})| + h_i|(a^+_{12,i} - a_{12,i+1})w^+_{2,i} + (a^-_{12,i} - a_{12,i-1})w^-_{2,i}|\\ &\quad + h_i|a_{12+1,i}(w^+_{2,i} - w_{2,i+1}) + a_{12,i-1}(w^-_{2,i} - w_{2,i-1})|\\ &\le Ch_ih_{i+1}|w^+_{1,i}| + Ch_i^2|w^-_{1,i}| + Ch_i(h_i(a_{11,i-1} - a^-_{11,i})\bar{w}'_{1,i} - \frac{1}{2}h^2_{i+1}a^-_{11,i}\bar{w}''_1(\vartheta_i)\\ &\quad + \frac{1}{2}h_i^2a_{11,i-1}\bar{w}''_1(\vartheta_i) + R_1) + Ch_ih_{i+1}|w^+_{2,i}| + Ch_i^2|w^-_{2,i}| + Ch_i(h_i(a_{12,i-1} - a^-_{12,i})\bar{w}'_{2,i}\\ &\quad - \frac{1}{2}h^2_{i+1}a^-_{12,i}\bar{w}''_2(\vartheta_i) + \frac{1}{2}h_i^2a_{12,i-1}\bar{w}''_2(\vartheta_i) + R_2), \quad \vartheta_i \in [x_{i-1}, x_i].\end{aligned}$$

We use the asymptotic expansion of the layer components $w_1 = \bar{w}_1 + R_1$ and $w_2 = \bar{w}_2 + R_2$, that can be derived using the technique from [4]. It can be concluded that the leading part $\bar{w}'_1$ of w'_1 and $\bar{w}'_2$ of w'_2 are continuous at $x = d$, enabling us to use Taylor's expansions for estimating $w^+_{1,i} - w_{1,i+1}, w^-_{1,i} - w_{1,i-1}$ and $w^+_{2,i} - w_{2,i+1}, w^-_{2,i} - w_{2,i-1}$. Since R_1, R_2 contain lower order terms, we have

$$| m_i | \le C\bar{h}_i\sqrt{\varepsilon}N^{-1} + C\bar{h}_i\sqrt{\varepsilon}N^{-2}\max|\psi'|^2 + C\bar{h}_iN^{-2}\max|\psi'|^2, \tag{4.15}$$

and we use the estimate of $\max|\psi'|$ in the above result to obtain

$$| m_i | \le \begin{cases} C\bar{h}_i(\sqrt{\varepsilon} + N^{-1})N^{-1}\ln^2 N, & \text{for Shishkin mesh,}\\ C\bar{h}_i(\sqrt{\varepsilon} + N^{-1})N^{-1}, & \text{for Bakhalov-Shishkin mesh.}\end{cases}$$

Collecting estimates (4.12)–(4.15) from the previously analyzed cases and using $\sqrt{\varepsilon} \le CN^{-1}$, we have

$$\begin{aligned}|< (a_{11}w_1 + a_{12}w_2)^I, \lambda_1^k >_h| &\le \frac{1}{6}\sum_{i=1}^{N-1}|m_i|\lambda^k_{1,i}\\ &\le C(N^{-\tau} + N^{-2}\max|\psi'|)\sum_{i=1}^{N-1}\bar{h}_i\lambda^k_{1,i}\\ &\le CN^{-2}\max|\psi'|\|\lambda_1^k\|_{L^1(\Omega)}\\ &\le CN^{-2}\max|\psi'|,\end{aligned}$$

since $\tau \ge 2$ and $\|\lambda_1^k\|_{L^1(\Omega)} \le C$.

From (4.10), (4.11) and the above estimate, we have

$$K_1^*(x_k) \le C\sqrt{\varepsilon}N^{-1} + CN^{-2}\max|\psi'|. \tag{4.16}$$

A similar estimate is also hold for $K_2^*(x_k)$, from (4.8). Therefore from equations (4.2)–(4.6), (4.16) and $\max|\psi'| = C\ln N$ (in case of Shishkin mesh), for $p = 1, 2$ we have

$$K_p(x_k) \le C\sqrt{\varepsilon}N^{-1} + CN^{-2}\ln^2 N.$$

Since $|\bar{K}(x_i)| = \max(|K_1(x_i)|, |K_2(x_i)|)$, we have

$$|\bar{P}\bar{u}(x_i) - \bar{u}_h(x_i)| = |\bar{K}(x_i)| \leq C\sqrt{\varepsilon}N^{-1} + CN^{-2}\ln^2 N.$$

Therefore we conclude

Lemma 4.3. *Let $\bar{u}$ and $\bar{u}_h$ be solution of the BVP* (1.1)–(1.2) *and* (2.5) *respectively. Then for Shishkin mesh, the pointwise maximum norm of the error satisfies*

$$|\bar{u}(x_i) - \bar{u}_h(x_i)| = |\bar{P}\bar{u}(x_i) - \bar{u}_h(x_i)| \leq C\sqrt{\varepsilon}N^{-1} + CN^{-2}\ln^2 N, \quad x_i \in \Omega_\varepsilon^N. \quad \square$$

Now, since V_h uses linear Lagrange elements, we can easily derive a bound for the error $u_j - u_{jh}$ on each element $[x_{i-1}, x_i], i = 1(1)N, j = 1, 2$. For arbitrary $i \in \{1, 2, ..., N\}$ and $x \in [x_{i-1}, x_i]$, the triangle inequality implies

$$|u_j(x) - u_{jh}(x)| \leq |u_j(x) - u_j^I(x)| + |u_j^I(x) - u_{jh}(x)|, j = 1, 2.$$

The difference between the piecewise linear function u_j^I and u_{jh} at the point x is estimated by

$$\begin{aligned} |u_j^I(x) - u_{jh}(x)| &= |u_j(x_{i-1})\phi_{i-1}(x) + u_j(x_i)\phi_i(x) - u_{jh}(x_{i-1})\phi_{i-1}(x) - u_{jh}(x_i)\phi_i(x)| \\ &\leq |u_j(x_{i-1}) - u_{jh}(x_{i-1})|\phi_{i-1}(x) + |u_j(x_i) - u_{jh}(x_i)|\phi_i(x) \\ &\leq C\sqrt{\varepsilon}N^{-1} + CN^{-2}\ln^2 N, \quad \text{by Lemma 4.3} \quad \text{for} \quad i = 2(1)N-1, \end{aligned}$$

where ϕ_i are functions defined in Section 2.1. The same bound holds for $i = 1$ and $i = N$. Therefore for each interval $[x_{i-1}, x_i]$ we finally obtain the error estimate

$$|u_j(x) - u_{jh}(x)| \leq \|\bar{u} - \bar{u}^I\|_{\infty[x_{i-1},x_i]} + C\sqrt{\varepsilon}N^{-1} + CN^{-2}\ln^2 N, j = 1, 2 \tag{4.17}$$

where $\|\bar{u} - \bar{u}^I\|_{\infty[x_{i-1},x_i]} = \max(\|u_1 - u_1^I\|_{\infty[x_{i-1},x_i]}, \|u_2 - u_2^I\|_{\infty[x_{i-1},x_i]})$.

5. **ERROR ESTIMATE**

The following theorem gives us the result on the maximum norm of the error $\bar{u} - \bar{u}_h$ not just on the mesh points, but on the whole domain $[0, 1]$.

Theorem 5.1. *Let $\bar{u}$ and $\bar{u}_h$ be solution of BVP* (1.1)–(1.2) *and* (2.5) *respectively, $\sqrt{\varepsilon} \leq CN^{-1}$ and $\tau \geq 2$. Then we have*

$$\|\bar{u} - \bar{u}_h\|_\infty \leq \begin{cases} CN^{-2}\ln^2 N, & \textit{for Shishkin mesh,} \\ CN^{-2}, & \textit{for Bakhalov-Shishkin mesh.} \end{cases}$$

Proof. For Shishkin mesh, the theorem follows from the inequality (4.17) and the results on interpolation error (3.6). Similarly we can prove for Bakhalov-Shishkin mesh. $\square$

Remark 5.2. All the results in this paper also hold in case when the functions f_1 and f_2 have more than one point of discontinuity.

6. NUMERICAL EXPERIMENTS

In this section we experimentally verify our theoretical results proved in the previous section.

Example 6.1. Consider the BVP

$$\begin{aligned} -\varepsilon u_1''(x) + 2u_1(x) - u_2(x) &= f_1(x), \quad x \in \Omega^- \cup \Omega^+, \\ -\varepsilon u_2''(x) - u_1(x) + 2u_2(x) &= f_2(x), \quad x \in \Omega^- \cup \Omega^+, \end{aligned} \tag{6.1}$$

$$u_1(0) = 0, \quad u_1(1) = 0, \quad u_2(0) = 0, \quad u_2(1) = 0, \tag{6.2}$$

where

$$f_1(x) = \begin{cases} 1, & 0 \le x \le 0.5, \\ 0.8, & 0.5 \le x \le 1 \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 2, & 0 \le x \le 0.5, \\ 1.8, & 0.5 \le x \le 1. \end{cases}$$

Example 6.2. Consider the BVP

$$\begin{aligned} -\varepsilon u_1''(x) + 2(x+1)^2 u_1(x) - (1+x^3)u_2(x) &= f_1(x), \quad x \in \Omega^- \cup \Omega^+, \\ -\varepsilon u_2''(x) - 2cos(\pi x/4)u_1(x) + 2.2e^{1-x}u_2(x) &= f_2(x), \quad x \in \Omega^- \cup \Omega^+, \end{aligned} \tag{6.3}$$

$$u_1(0) = 0, \quad u_1(1) = 0, \quad u_2(0) = 0, \quad u_2(1) = 0, \tag{6.4}$$

where

$$f_1(x) = \begin{cases} 2e^x, & 0 \le x \le 0.5, \\ 1, & 0.5 \le x \le 1 \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 10x+1, & 0 \le x \le 0.5, \\ 2, & 0.5 \le x \le 1. \end{cases}$$

For our tests, we take $\varepsilon = 2^{-18}$, which is sufficiently small to bring out the singularly perturbed nature of the problem. We measure the accuracy in maximum norm and the rates of convergence r^N are computed using the following formula:

$$r^N = log_2(\frac{E^N}{E^{2N}}),$$

where

$$E^N = \| \bar{u}_h - \bar{u}_{2h}^I \|_\infty,$$

and $\bar{u}_h^I$ denotes the piecewise linear interpolant of $\bar{u}_h$. In Table 1, we present values of E^N, r^N for the solution of the BVPs (6.1)–(6.2) and (6.3)–(6.4) for Shishkin and B-Shishkin meshes respectively. The Figures 1 and 2 depict the numerical solution of the BVP (6.1)–(6.2) for Shishkin mesh with $N = 512$. We compare the values of E^N, r^N for the solution of the same BVP (6.1)–(6.2) for Shishkin mesh using the standard upwind scheme adopted [13]. From these tables, it is obvious that the method presented in this paper performs well. The numerical results are clear illustrations of the convergence estimates derived in the present paper for both the type of meshes.

TABLE 1. Values of E^N and r^N for the solution of the BVPs (6.1)-(6.2) and (6.3)-(6.4) respectively.

N	Shishkin mesh		B-Shishkin mesh		Shishkin mesh		B-Shishkin mesh	
	E^N	r^N	E^N	r^N	E^N	r^N	E^N	r^N
32	7.5324e-03	0.0542	6.8353e-02	2.4161	4.5727e-03	0.0788	1.0557e-01	2.3448
64	7.2545e-03	0.4497	1.2807e-02	2.1880	4.3295e-03	0.1207	2.0782e-02	2.1671
128	5.3117e-03	0.9314	2.8106e-03	2.0923	3.9818e-03	0.5378	4.6273e-03	2.0866
256	2.7851e-03	1.2331	6.5912e-04	2.0460	2.7426e-03	1.1084	1.0894e-03	2.0447
512	1.1848e-03	1.2217	1.5961e-04	2.0230	1.2723e-03	1.1524	2.6405e-04	2.0226
1024	5.0802e-04	-	3.9272e-05	-	5.7236e-04	-	6.4986e-05	-

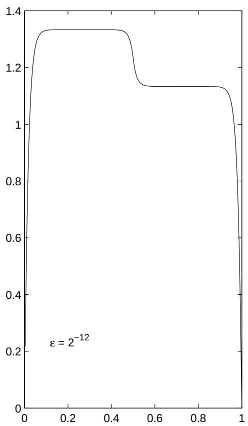

FIGURE 1. Graphs of the first component u_1 of the Example 6.1

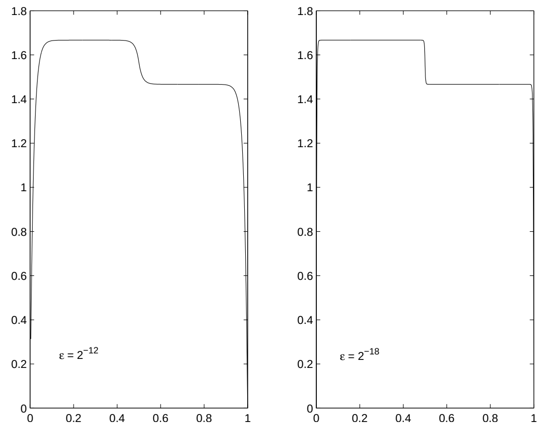

FIGURE 2. Graphs of the second component u_2 of the Example 6.1

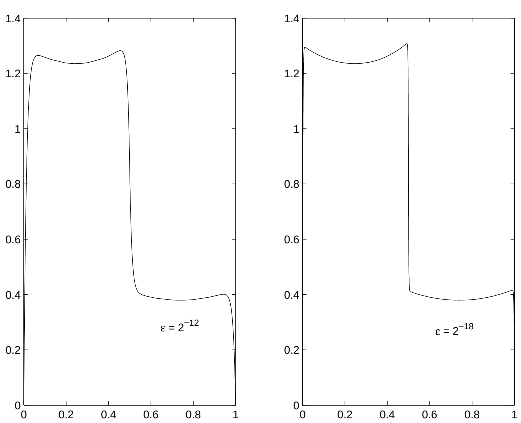

FIGURE 3. Graphs of the first component u_1 of the Example 6.2

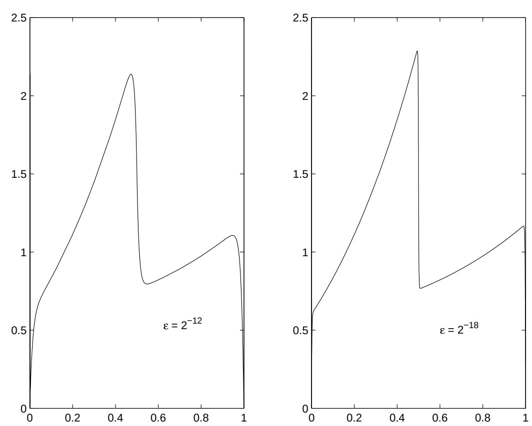

FIGURE 4. Graphs of the first component u_2 of the Example 6.2

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