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ON SINGULAR INITIAL VALUE PROBLEMS

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Abstract *This paper is devoted to the study of singular initial value problems including Lane Emden equations. The singular initial value problem have been investigated by applying variational iteration (VIM) and homotopy perturbation (HPM) methods coupled with a transformation which is very useful to cope with the singular behavior which occur at* $x=0$. *Several examples are given to confirm the reliability and efficiency of the proposed techniques.*

Keywords Variational iteration method, homotopy perturbation method, singular initial value problems, Lane Emden equation.

1. Introduction

The singular initial value problems [1-10, 21-24, 28-30, 33, 34] are of utmost importance in nonlinear sciences. Several techniques have been employed to tackle such problems, see [1-10, 21-24, 28-30, 33, 34] and the references therein. He [8-17] developed the variational iteration and homotopy perturbation methods which have been applied [3, 5-28, 31, 32] to a wide class of initial and boundary value problems. The basic motivation of this paper is the extension of these very reliable techniques for solving singular initial value problems. The singularity behavior at $x=0$, has been tackled by the introduction of a transformation $u(x)=x\,y(x)$. It is observed that the suggested transformation is very effective and useful. Moreover, in case of variational iteration method (VIM), we convert the second-order initial value problems into a system of two first-order integral equations which in turn makes the identification of Lagrange multiplier uniform and very simple. The homotopy perturbation method (HPM) coupled with the above transformation has been implemented on Lane-Emden equations. Numerical results clearly reveal the complete reliability of the proposed algorithms.

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2. Variational Iteration Method (VIM)

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x) \tag{1}$$

where L is a linear operator, N a nonlinear operator and g(x) is the forcing term. According to the VIM [3, 5-8, 10, 13-18, 22-24, 26-28, 31], we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(s) + N\tilde{u}_n(s) - g(s))ds \tag{2}$$

where λ is a Lagrange multiplier [3, 5-8, 10, 13-18, 22-24, 26-28, 31] , which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta\tilde{u}_n = 0$; (2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [3, 5-8, 10, 13-18, 22-24, 26-28, 31]. For the sake of simplicity and to convey the idea of the technique, we consider the following system of differential equations:

$$x_i^{/}(t) = f_i(t, x_i), \quad i = 1,2,3,....,n, \tag{3}$$

subject to the boundary conditions. $x_i(0) = c_i$, $i = 1,2,3,...,n$. To solve the system by means of the variational iteration method, we rewrite the system (3) in the following form:

$$x_i^{/}(t) = f_i(x_i) + g_i(t), \quad i = 1,2,3,.....,n, \tag{4}$$

subject to the boundary conditions. $x_i(0) = c_i$, $i = 1,2,3,...,n$ and g_i is defined in (1). The correction functional for the nonlinear system (4) can be approximated as

$$
\begin{aligned}
x_1^{(k+1)}(t) &= x_1^{(0)}(t) + \int_0^t \lambda_1 \left(x_1^{/(k)}(T), f_1(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)) - g_1(T) \right) dT, \\
x_2^{(k+1)}(t) &= x_2^{(0)}(t) + \int_0^t \lambda_2 \left(x_2^{/(k)}(T), f_2(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)) - g_2(T) \right) dT, \\
&\vdots \\
x_n^{(k+1)}(t) &= x_n^{(0)}(t) + \int_0^t \lambda_n \left(x_n^{/(k)}(T), f_n(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)) - g_n(T) \right) dT.
\end{aligned}
\tag{5}
$$

where $\lambda_i = \pm 1,\ i = 1,2,3,\ldots,n$ are Lagrange multipliers, $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ denote the restricted variations.

For $\lambda_i = -1, \quad i = 1,2,3,\ldots,n$ we have the following iterative schemes

$$
\begin{aligned}
x_1^{(k+1)}(t) &= x_1^{(0)}(t) - \int_0^t \left(x_1'^{(k)}(T), f_1(x_1^{(k)}(T), x_2^{(k)}(T), \ldots, x_n^{(k)}(T)) - g_1(T) \right) dT, \\
x_2^{(k+1)}(t) &= x_2^{(0)}(t) - \int_0^t \left(x_2'^{(k)}(T), f_2(x_1^{(k)}(T), x_2^{(k)}(T), \ldots, x_n^{(k)}(T)) - g_2(T) \right) dT, \\
&\vdots \\
x_n^{(k+1)}(t) &= x_n^{(0)}(t) - \int_0^t \left(x_n'^{(k)}(T), f_n(x_1^{(k)}(T), x_2^{(k)}(T), \ldots, x_n^{(k)}(T)) - g_n(T) \right) dT.
\end{aligned}
\tag{6}
$$

If we start with the initial approximations $x_i(0) = c_i, \quad i = 1,2,3,\ldots,n$ then the approximations can be completely determined; finally we approximate the solution $x_i(t) = \underset{k\to\infty}{Lim}\, x_i^{(n)}(t)$ by the nth term $x_i^{(n)}(t)$ for $i = 1,2,3,\ldots,n.$

3. Homotopy Perturbation Method (HPM)

To explain the He's homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0, \tag{7}$$

where L is any integral or differential operator. We define a convex homotopy H (u, p) by

$$H(u,p) = (1-p)F(u) + pL(u), \tag{8}$$

where F (u) is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u,p) = 0, \tag{9}$$

we have

$$H(u,0) = F(u), \qquad H(u,1) = L(u).$$

This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point H (v_0, 0) to a solution function H (*f*, 1). The embedding parameter monotonically increases from zero to unit as the trivial problem F (u) = 0 is continuously deforms the original problem L (u) = 0. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter [8-12, 19-21, 23-25, 32]. The homotopy perturbation method uses the homotopy parameter *p* as an expanding parameter [8-12] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots, \tag{10}$$

if p $\to$ 1, then (10) corresponds to (8) and becomes the approximate solution of the form,

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \tag{11}$$

It is well known that series (11) is convergent for most of the cases and also the rate of convergence is dependent on L (u); see [8-12]. We assume that (11) has a unique solution. The comparisons of like powers of *p* give solutions of various orders.

4. Numerical Applications

In this section, we apply the variational iteration method (VIM) coupled with the transformation for solving singular initial value problems. We also apply homotopy perturbation method (HPM) along with the similar transformation to solve Lane-Emden equations.

Example 4.1 Consider the following linear singular initial value problem

$$y'' + \frac{2}{x} y' + y = 6 + 12x + x^2 + x^3 \tag{12}$$

with initial conditions

$$y(0) = 0, \qquad y'(0) = 0. \tag{13}$$

Using the transformation $u(x) = x\, y(x)$, the singular initial value problem (12, 13) can be converted

to the following non-singular second order initial value problem

$$u''(x) + u(x) = 6x + 12x^2 + x^3 + x^4, \tag{14}$$

with initial conditions

$$u(0) = 0, \; u'(0) = 0. \tag{15}$$

Now, using the transformation $\frac{du}{dx} = z(x)$, the non singular initial value problem (14, 15) can be

rewritten as the following system of differential equations

$$\begin{cases} \frac{du}{dx} = z(x), \\ \frac{dz}{dx} = 6x + 12x^2 + x^3 + x^4 - u, \end{cases}$$

with boundary conditions

$$u(0) = 0, \qquad z(0) = 0.$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1,\ i = 1, 2.$

$$\begin{cases} u^{(k+1)}(x) = \lambda_1 \int_0^x z^{(k)}(t)dt, \\ z^{(k+1)}(x) = 3x^2 + 4x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 - \lambda_2 \int_0^x u^{(k)}(t)dt. \end{cases}$$

Consequently, following approximants are obtained

$$\begin{cases} u^{(0)}(x) = 0, \\ z^{(0)}(x) = 3x^2 + 4x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5, \end{cases}$$

$$\begin{cases} u^{(1)}(x) = x^3 + x^4 + \frac{1}{20}x^5 + \frac{1}{30}x^6, \\ z^{(1)}(x) = 3x^2 + 4x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5, \end{cases}$$

$$\begin{cases} u^{(2)}(x) = x^3 + x^4 + \frac{1}{20}x^5 + \frac{1}{30}x^6, \\ z^{(2)}(x) = 3x^2 + 4x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 - \left(\frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{120}x^6 + \frac{1}{210}x^7\right), \end{cases}$$

$$\begin{cases} u^{(3)}(x) = x^3 + x^4 + \frac{1}{20}x^5 + \frac{1}{30}x^6 - \left(\frac{1}{20}x^5 + \frac{1}{30}x^6 + \frac{1}{840}x^7 + \frac{1}{1680}x^8\right), \\ z^{(3)}(x) = 3x^2 + 4x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 - \left(\frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{120}x^6 + \frac{1}{210}x^7\right), \end{cases}$$

$$\begin{cases} u^{(4)}(x) = x^3 + x^4 + \frac{1}{20}x^5 + \frac{1}{30}x^6 - \left(\frac{1}{20}x^5 + \frac{1}{30}x^6 + \frac{1}{840}x^7 + \frac{1}{1680}x^8\right), \\ z^{(4)}(x) = 3x^2 + 4x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 - \left(\frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{120}x^6 + \frac{1}{210}x^7\right) + \frac{1}{120}x^6 + \frac{1}{210}x^7 \\ \qquad + \frac{1}{6720}x^8 + \frac{1}{13520}x^9, \end{cases}$$

$$\begin{cases} u^{(5)}(x) = x^3 + x^4 + \frac{1}{20}x^5 + \frac{1}{30}x^6 - \left(\frac{1}{20}x^5 + \frac{1}{30}x^6 + \frac{1}{840}x^7 + \frac{1}{1680}x^8\right) + \frac{1}{840}x^7 + \frac{1}{1680}x^8 \\ \qquad + \frac{1}{60840}x^9 + \frac{1}{135200}x^{10}, \\ z^{(5)}(x) = 3x^2 + 4x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 - \left(\frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{120}x^6 + \frac{1}{210}x^7\right) + \frac{1}{120}x^6 + \frac{1}{210}x^7 \\ \qquad + \frac{1}{6720}x^8 + \frac{1}{13520}x^9, \end{cases}$$

$$\begin{cases} u^{(6)}(x) = x^3 + x^4 + \frac{1}{20}x^5 + \frac{1}{30}x^6 - \left(\frac{1}{20}x^5 + \frac{1}{30}x^6 + \frac{1}{840}x^7 + \frac{1}{1680}x^8\right) + \frac{1}{840}x^7 + \frac{1}{1680}x^8 \\ \qquad + \frac{1}{60840}x^9 + \frac{1}{135200}x^{10}, \\ z^{(6)}(x) = 3x^2 + 4x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 - \left(\frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{120}x^6 + \frac{1}{210}x^7\right) + \frac{1}{120}x^6 + \frac{1}{210}x^7 \\ \qquad + \frac{1}{6720}x^8 + \frac{1}{13520}x^9 - \left(\frac{1}{6720}x^8 + \frac{1}{15840}x^9 + \frac{1}{604800}x^{10} + \frac{1}{1487200}x^{11}\right), \end{cases}$$

$\vdots$

Cancellation of the noise terms gives the solution

$$u(x) = x^3 + x^4,$$

and consequently

$$y(x) = x^2 + x^3.$$

Example 4.2 Consider the following Lane-Emden type equation

$$y'' + \frac{2}{x}y' + \sin y = 0 \tag{16}$$

with boundary conditions

$$y(0) = 1, \qquad y'(0) = 0 \tag{17}$$

Using the transformation $u(x) = x\,y(x)$, the singular initial value problem (16, 17) can be converted to the following non-singular second order initial value problem

$$u''(x) + x\sin\left(\frac{u(x)}{x}\right) = 0, \tag{18}$$

with initial conditions

$$u(0) = 0, \qquad u'(0) = 1 \tag{19}$$

Now, using the transformation $\frac{du}{dx} = z(x)$, the non singular initial value problem (18, 19) can be converted to the following system of differential equations

$$\begin{cases} \dfrac{du}{dx} = z(x), \\ \dfrac{dz}{dx} = -x\sin\dfrac{1}{x}(u(x)), \end{cases}$$

with

$$u(0) = 0, \qquad z(0) = 1.$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1,\ i = 1, 2.$

$$\begin{cases} u^{(k+1)}(x) = \int_0^x z^{(k)}(x)dx, \\ z^{(k+1)}(x) = 1 - \int_0^x x\sin\dfrac{1}{x}(u^{(k)}(x))dx, \end{cases}$$

Consequently, we obtain the following approximants

$$\begin{cases} u^{(0)}(x) = 0, \\ z^{(0)}(x) = 1, \end{cases}$$

$$\begin{cases} u^{(1)}(x) = x, \\ z^{(1)}(x) = 0, \end{cases}$$

$$\begin{cases} u^{(2)}(x) = x, \\ z^{(2)}(x) = 1 - \dfrac{1}{2!}x^2\sin 1, \end{cases}$$

$$\begin{cases} u^{(3)}(x) = x - \dfrac{1}{3!}x^3 \sin 1, \\ z^{(3)}(x) = 1 - \dfrac{1}{2!}x^2 \sin 1, \end{cases}$$

$$\begin{cases} u^{(4)}(x) = x - \dfrac{1}{3!}x^3(\sin 1), \\ z^{(4)}(x) = 1 - \dfrac{1}{2!}x^2(\sin 1) + \dfrac{1}{22}(\sin 1)(\cos 1)x^4, \end{cases}$$

$$\begin{cases} u^{(5)}(x) = x - \dfrac{1}{3!}x^3(\sin 1) + \dfrac{1}{120}x^5(\sin 1)(\cos 1), \\ z^{(5)}(x) = 1 - \dfrac{1}{2!}x^2(\sin 1) + \dfrac{1}{22}(\sin 1)(\cos 1)x^4, \end{cases}$$

$$\begin{cases} u^{(6)}(x) = x - \dfrac{1}{3!}x^3(\sin 1) + \dfrac{1}{120}x^5(\sin 1)(\cos 1), \\ z^{(6)}(x) = 1 - \dfrac{1}{2!}x^2(\sin 1) + \dfrac{1}{22}(\sin 1)(\cos 1)x^4 + (\sin 1)\left(\dfrac{7}{3024}(\sin 1)^2 - \dfrac{7}{5040}(\cos 1)^2\right)x^6, \end{cases}$$

$$\begin{cases} u^{(7)}(x) = x - \dfrac{1}{3!}x^3(\sin 1) + \dfrac{1}{120}x^5(\sin 1)(\cos 1) + \left(\dfrac{1}{3024}(\sin 1)^2 - \dfrac{1}{5040}(\cos 1)^2\right)x^7(\sin 1), \\ z^{(7)}(x) = 1 - \dfrac{1}{2!}x^2(\sin 1) + \dfrac{1}{22}(\sin 1)(\cos 1)x^4 + (\sin 1)\left(\dfrac{7}{3024}(\sin 1)^2 - \dfrac{7}{5040}(\cos 1)^2\right)x^6 \\ \qquad + \left(\dfrac{113\times 9}{3265920}(\sin 1)^2 + \dfrac{1\times 9}{362880}(\cos 1)^2\right)x^8(\sin 1)(\cos 1), \end{cases}$$

$$\begin{cases} u^{(8)}(x) = x - \dfrac{1}{3!}x^3(\sin 1) + \dfrac{1}{120}x^5(\sin 1)(\cos 1) + \left(\dfrac{1}{3024}(\sin 1)^2 - \dfrac{1}{5040}(\cos 1)^2\right)x^7(\sin 1) \\ \qquad + \left(\dfrac{113}{3265920}(\sin 1)^2 + \dfrac{1}{362880}(\cos 1)^2 x^8(\sin 1)(\cos 1)\right), \\ z^{(8)}(x) = 1 - \dfrac{1}{2!}x^2(\sin 1) + \dfrac{1}{22}(\sin 1)(\cos 1)x^4 + (\sin 1)\left(\dfrac{7}{3024}(\sin 1)^2 - \dfrac{7}{5040}(\cos 1)^2\right)x^6 \\ \qquad + \left(\dfrac{113\times 9}{3265920}(\sin 1)^2 + \dfrac{1\times 9}{362880}(\cos 1)^2\right)x^8(\sin 1)(\cos 1) \\ \qquad + \left(\dfrac{1781\times 11}{898128000}(\sin 1)^2(\cos 1)^2 - \dfrac{11}{39916800}(\cos 1)^4 - \dfrac{19\times 11}{23950080}(\sin 1)^4\right)x^{10}(\sin 1), \end{cases}$$

$\vdots$

The series solution is given by

$$u(x) = x - \frac{1}{3!}(\sin 1)\,x^3 + \frac{1}{5!}(\cos 1)(\sin 1)x^5 + (\sin 1)\left(\frac{1}{3024}(\sin 1)^2 - \frac{1}{5040}(\cos 1)^2\right)x^7$$
$$+(\sin 1)(\cos 1)\left(-\frac{113}{3265920}(\sin 1)^2 + \frac{1}{362880}(\cos 1)^2\right)x^9$$
$$+(\sin 1)\left(\frac{1781}{89812800}(\sin 1)^2(\cos 1)^2 - \frac{1}{39916800}(\cos 1)^4 - \frac{19}{23950080}(\sin 1)^4\right)x^{11} + \cdots,$$

and, consequently,

$$y(x) = 1 - \frac{1}{3!}(\sin 1)\,x^2 + \frac{1}{5!}(\cos 1)(\sin 1)x^4 + (\sin 1)\left(\frac{1}{3024}(\sin 1)^2 - \frac{1}{5040}(\cos 1)^2\right)x^6$$
$$+(\sin 1)(\cos 1)\left(-\frac{113}{3265920}(\sin 1)^2 + \frac{1}{362880}(\cos 1)^2\right)x^8$$
$$+(\sin 1)\left(\frac{1781}{89812800}(\sin 1)^2(\cos 1)^2 - \frac{1}{39916800}(\cos 1)^4 - \frac{19}{23950080}(\sin 1)^4\right)x^{10} + \cdots.$$

Example 4.3 Consider the following second order singular initial value problem

$$y'' + \frac{2}{x}y' + \sinh y = 0 \qquad (20)$$

with initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \qquad (21)$$

Using the transformation $u(x) = x\,y(x)$, the singular initial value problem (20, 21) can be converted to the following non-singular second order initial value problem

$$u''(x) = x \sinh\,(\frac{1}{x}u\,(x)) = 0, \qquad (22)$$

with boundary conditions

$$u(0) = 0, \qquad u'(0) = 1. \qquad (23)$$

Now, using the transformation $\frac{du}{dx} = z(x)$, the non-singular initial value problem (22, 23) can be converted to the following system of differential equations

$$\begin{cases} \dfrac{du}{dx} = z(x), \\ \dfrac{dz}{dx} = -x \sinh \dfrac{1}{x}(u(x)), \end{cases}$$

with $\quad u(0) = 0, \qquad z(0) = 1.$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1,\ i = 1, 2.$

$$\begin{cases} u^{(k+1)}(x) = \int_0^x z^{(k)}(x)dx, \\ z^{(k+1)}(x) = 1 + \int_0^x -x\sinh\frac{1}{x}(u^{(k)}(x))dx, \end{cases}$$

Consequently, we obtain the following approximants

$$\begin{cases} u^{(0)}(x) = 0, \\ z^{(0)}(x) = 1, \end{cases}$$

$$\begin{cases} u^{(1)}(x) = x, \\ z^{(1)}(x) = 1, \end{cases}$$

$$\begin{cases} u^{(2)}(x) = x, \\ z^{(2)}(x) = 1 - \frac{e^2-1}{12e}x^3, \end{cases}$$

$$\begin{cases} u^{(3)}(x) = x - \frac{e^2-1}{12e}x^3, \\ z^{(3)}(x) = 1 - \frac{e^2-1}{12e}x^3 - \frac{e^4-1}{96e^2}x^5, \end{cases}$$

$$\begin{cases} u^{(4)}(x) = x - \frac{e^2-1}{12e}x^3 + \frac{e^4-1}{480e^2}x^5, \\ z^{(4)}(x) = 1 - \frac{e^2-1}{12e}x^3 - \frac{e^4-1}{96e^2}x^5 - 7\left(\frac{2e^6+3e^2-3e^4-2}{30240e^3}\right)x^6, \end{cases}$$

$$\begin{cases} u^{(5)}(x) = x - \frac{e^2-1}{12e}x^3 + \frac{e^4-1}{480e^2}x^5 - \frac{2e^6+3e^2-3e^4-2}{30240e^3}x^7, \\ z^{(5)}(x) = 1 - \frac{e^2-1}{12e}x^3 - \frac{e^4-1}{96e^2}x^5 - 7\left(\frac{2e^6+3e^2-3e^4-2}{30240e^3}\right)x^6 + 9\left(\frac{61e^8-104e^6+104e^2-61}{26127360e^4}\right)x^8, \end{cases}$$

$\vdots$

The series solution is given by

$$u(x) = x - \frac{e^2+1}{12e}x^3 + \frac{e^4-1}{480e^2}x^5 - \frac{2e^6+3e^2-3e^4-2}{30240e^3}x^7 + \frac{61e^8-104e^6+104e^2-61}{26127360e^4}x^9 + \cdots,$$

and consequently,

$$y(x) = 1 - \frac{e^2+1}{12e}x^2 + \frac{e^4-1}{480e^2}x^4 - \frac{2e^6+3e^2-3e^4-2}{30240e^3}x^6 + \frac{61e^8-104e^6+104e^2-61}{26127360e^4}x^8 + \cdots.$$

Example 4.4 Consider the following nonlinear singular initial value problem

$$y''(x) + \frac{2}{x}y'(x) - 6y(x) = 4y(x)\ln y(x) \tag{24}$$

with initial conditions

$$y(0) = 1, \qquad y'(0) = 0 \tag{25}$$

Using the transformation $u(x) = x\,y(x)$, the singular initial value problem (24, 25) can be converted to

the following non-singular second order initial value problem

$$u'' - 6u(x) = 4x\left(\frac{u(x)}{x}\right)\ln\left(\frac{u(x)}{x}\right), \tag{26}$$

with initial conditions

$$u(0) = 0, \qquad u'(0) = 1. \tag{27}$$

Now, using the transformation $\frac{du}{dx} = z(x)$, the non singular initial value problem (26, 27) can be

converted to the following system of differential equations

$$\begin{cases} \dfrac{du}{dx} = z(x), \\ \dfrac{dz}{dx} = 6u(x) + 4u(x)\ln\dfrac{1}{x}u(x). \end{cases}$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1,\ i = 1, 2.$

$$\begin{cases} u^{(k+1)}(x) = \displaystyle\int_0^x z^{(k)}(t)dt, \\ z^{(k+1)}(x) = 1 + \displaystyle\int_0^x \left(6u^{(k)}(t) + 4u^{(k)}(t)\ln\frac{1}{t}u^{(k)}(t)\right)dt. \end{cases}$$

Proceeding as above, the series solution is given as

$$u(x) = x + x^3 + \frac{1}{2!}x^5 + \frac{1}{3!}x^7 + \frac{1}{4!}x^9 + \frac{1}{5!}x^{11} + \cdots,$$

and consequently**,**

$$y(x) = 1+x^2+\frac{1}{2!}x^4+\frac{1}{3!}x^6+\frac{1}{4!}x^8 +\frac{1}{5!}x^{10}+\cdots,$$

The solution in a closed form is given by

$$y(x) = e^{x^2}.$$

Example 4.5 Consider the white-dwarf equation

$$y''+\frac{2}{x}y'+(y^2-C)^{3/2}=0,$$

with boundary conditions

$$y(0)=1, \qquad y'(0)=0.$$

Using the transformation, $u(x)=x\,y(x),$ the above problem can be re-formulated as

$$u''(x)+x\left(\frac{u^2}{x^2}-C\right)^{3/2}=0.$$

Applying the convex homotopy

$$u_0+pu_1+p^2u_2+\cdots=1-p\int_0^x\int_0^x\left(x\left(\frac{1}{x^2}\left(u_0+pu_1+p^2u_2+\cdots\right)^2-C\right)^{3/2}\right)dx\,dx.$$

Comparing the coefficients of like powers of p

$$p^{(0)}: u_0(x)=x,$$

$$p^{(1)}: u_1(x)=x-\frac{1}{6}(C-1)^{3/2}x^3,$$

$$p^{(2)}: u_2(x)=x-\frac{1}{6}(C-1)^{3/2}x^3+\frac{1}{40}(C-1)^2x^4,$$

$$p^{(3)}: u_3(x)=x-\frac{1}{6}(C-1)^{3/2}x^3+\frac{1}{40}(C-1)^2x^4-\frac{1}{7!}(5(C-1)+14)(C-1)^{5/2}x^7,$$

$$p^{(4)}: u_4(x)=x-\frac{1}{6}(C-1)^{3/2}x^3+\frac{1}{40}(C-1)^2x^4-\frac{1}{7!}(5(C-1)+14)(C-1)^{5/2}x^7 +\frac{1}{3\cdot 9!}(339(C-1)+280)(C-1)^3x^9,$$

$\vdots$

The series solution is given as

$$u(x)=x-\frac{1}{6}(C-1)^{3/2}x^3+\frac{1}{40}(C-1)^2x^4-\frac{1}{7!}(5(C-1)+14)(C-1)^{5/2}x^7+\frac{1}{3\cdot 9!}(339(C-1)+280)(C-1)^3x^9+\cdots,$$

and using the inverse transformation, the required solution is given by

$$y(x)=1-\frac{1}{6}(C-1)^{3/2}x^2+\frac{1}{40}(C-1)^2x^4-\frac{1}{7!}(5(C-1)+14)(C-1)^{5/2}x^6+\frac{1}{3\cdot 9!}(339(C-1)+280)(C-1)^3x^8$$
$$+\frac{1}{5\cdot 11!}(1425(C-1)^2+11436(C-1)+4256)(C-1)^{7/2}x^{10}+\cdots.$$

Example 4.6 Consider the following generalized Lane-Emden equation

$$y''+\frac{n}{x}y'+y^m=0, \qquad n\ge 0,$$

with initial conditions

$$y(0)=1, \qquad y'(0)=0.$$

Using the transformation, $u(x)=x\,y(x)$, the above problem can be re-formulated as

$$u''(x)+\left(\frac{n-2}{1+x}\right)u'(x)+\left(\frac{u}{x}\right)^m=0.$$

Applying the convex homotopy

$$u_0+pu_1+p^2u_2+\cdots=1-p\int_0^x\int_0^x\left(\left(\frac{n-2}{1+x}\right)(u_0'+pu_2''+\cdots)+\left(\frac{1}{x}(u_0+pu_1+p^2u_2+\cdots)\right)^m\right)dx\,dx.$$

Comparing the coefficients of like powers of p

$$p^{(0)}:u_0(x)=x,$$

$$p^{(1)}:u_1(x)=x-\frac{1}{2(n+1)}x^3,$$

$$p^{(2)}:u_2(x)=x-\frac{1}{2(n+1)}x^3\,\frac{m}{8(n+1)(n+3)}x^5,$$

$$p^{(3)}:u_3(x)=x-\frac{1}{2(n+1)}x^3\,\frac{m}{8(n+1)(n+3)}x^5+\frac{(2n+4)m^2-(n+3)m}{48(n+1)^2(n+3)(n+5)}x^7,$$

$$p^{(4)}:u_4(x)=x-\frac{1}{2(n+1)}x^3\,\frac{m}{8(n+1)(n+3)}x^5+\frac{(2n+4)m^2-(n+3)m}{48(n+1)^2(n+3)(n+5)}x^7$$
$$+\frac{(6n^2+32n+34)m^2-(97n^2+46n+630m)^2+(2n^2+16n+30)}{384(n+1)^3(n+3)(n+5)(n+7)}x^8,$$

⋮.

The series solution is given by

$$u(x) = x - \frac{1}{2(n+1)}x^3 \frac{m}{8(n+1)(n+3)}x^5 + \frac{(2n+4)m^2 - (n+3)m}{48(n+1)^2(n+3)(n+5)}x^7$$
$$+\frac{(6n^2+32n+34)m^2 - (97n^2+46n+630m)^2 + (2n^2+16n+30)}{384(n+1)^3(n+3)(n+5)(n+7)}x^8 + \cdots,$$

and the inverse transformation will yield

$$y(x) = 1 - \frac{1}{2(n+1)}x^2 + \frac{m}{8(n+1)(n+3)}x^4 + \frac{(2n+4)m^2 - (n+3)m}{48(n+1)^2(n+3)(n+5)}x^6$$
$$+\frac{(6n^2+32n+34)m^2 - (97n^2+46n+630m)^2 + (2n^2+16n+30)}{384(n+1)^3(n+3)(n+5)(n+7)}x^8 + \cdots.$$

For a fixed $n=0$ and $m=0,1,2,$ we obtain the following solutions respectively:

$$\begin{cases} y(x) = 1 - \frac{1}{2}x^2, \\ y(x) = \cos x, \\ y(x) = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{72}x^6 + \frac{1}{504}x^8 + \cdots, \end{cases}$$

where $x=0$ is just an ordinary point. Similarly, for fixed $n=0.5$ and $m=0,1,2,$ we get the following solutions respectively:

$$\begin{cases} y(x) = 1 - \frac{1}{3}x^2, \\ y(x) = 1 - \frac{1}{3}x^2 + \frac{1}{42}x^4 - \frac{1}{1386}x^6 + \frac{1}{83160}x^8 + \cdots, \\ y(x) = 1 - \frac{1}{3}x^2 + \frac{1}{21}x^4 - \frac{13}{2079}x^6 + \frac{23}{31185}x^8 + \cdots. \end{cases}$$

Finally for $n=1$ and $m=0,1,2,$ we obtain the following solutions respectively:

$$\begin{cases} y(x) = 1 - \frac{1}{4}x^2, \\ y(x) = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \frac{1}{147456}x^8 + \cdots, \\ y(x) = 1 - \frac{1}{4}x^2 + \frac{1}{32}x^4 - \frac{1}{288}x^6 + \frac{13}{36864}x^8 + \cdots. \end{cases}$$

5. Conclusion In this paper, we applied variational iteration and homotopy perturbation methods coupled with a very useful transformation for solving second-order singular initial value problems and Lane-Emden equations. The proposed algorithms proved to be very effective and convenient. Numerical results show the complete reliability of the suggested iterative schemes.

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