Neural, Parallel, and Scientific Computations 17 (2009) 351-362

CALCULATING ZEROS OF THE FROBENIUS-EULER POLYNOMIALS

C. S. RYOO[1] AND R. P. AGARWAL[2]

[1]Department of Mathematics, Hannam University, Daejeon 306-791, Korea
[2]Department of Mathematical Sciences, Florida Institute of Technology,
Melbourne, Florida 32901-6975 USA

ABSTRACT. In this paper we observe the behavior of real roots of the Frobenius-Euler polynomials $H_n(u)$ for $u < -1$. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the $H_n(u)$ for $u < -1$. We investigate the zeros of the Frobenius-Euler polynomials $H_n(u)$ for $u < -1$. Finally, we give a table for the solutions of the Frobenius-Euler polynomials $H_n(u)$ for $u < -1$.

Key words: Euler numbers, Euler polynomials, Frobenius-Euler numbers, Frobenius-Euler polynomials

1. INTRODUCTION

In the 21st century, the computing environment would make more and more rapid progress. The importance of numerical simulation and analysis in mathematics is steadily increasing. Several mathematicians have studied the Bernoulli numbers and polynomials, the Euler numbers and polynomials, and the Frobenius-Euler numbers and polynomials (see [1, 2, 3, 4, 5, 6, 8]). These numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics. Using computer, a realistic study for these numbers and polynomials is very interesting. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the Frobenius-Euler polynomials $H_n(u,x)$ for $u < -1$ in complex plane. The outline of this paper is as follows. We introduce the Frobenius-Euler numbers $H_n(u)$ and polynomials $H_n(u,x)$. In Section 2, using a numerical investigation, we observe the beautiful zeros of the Frobenius-Euler polynomials $H_n(u,x)$. Finally, we investigate the structure of the roots of the Frobenius-Euler polynomials $H_n(u,x)$. In Section 3, we shall discuss the more general open problems and observations. First, we introduce the Euler numbers E_n and polynomials $E_n(x)$. The Euler numbers E_n are usually defined by means of the following generating function: The usual Euler

Received July 31, 2009

numbers E_n are defined by the generating function:

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\quad (|t|<2\pi), \tag{1}$$

where the symbol E_k is interpreted to mean that E^k must be replaced by E_k when we expand the one on the left. The Euler polynomials $E_n(x)$ are defined by the generating function:

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{2}$$

Frobenius extended such numbers as E_n to the so-called Frobenius-Euler numbers $H_n(u)$ belonging to an algebraic number u with $|u|>1$. Let u be algebraic number. For $u\in\mathbb{C}$ with $|u|>1$, the Frobenius-Euler numbers $H_n(u)$ belonging to u are defined by the generating function

$$F(u,t)=\frac{1-u}{e^t-u}=\sum_{n=0}^{\infty}H_n(u)\frac{t^n}{n!},(|t|<2\pi), \tag{3}$$

with the usual convention of symbolically replacing H^n by H_n. This relation can be written as

$$H_0(u)=1,\quad (H(u)+1)^k-uH_k(u)=0\quad (k\geq 1).$$

Therefore we have

$$uH_k(u)=\sum_{i=0}^{k}\binom{k}{i}H_i(u),\quad H_k(u)=\frac{1}{u-1}\sum_{i=0}^{k-1}\binom{k}{i}H_i(u),\ \text{for } u\neq 1.$$

We observe that $H_n(-1)=E_n$.

For an algebraic number $u\in\mathbb{C}$ with $|u|>1$, the Frobenius-Euler polynomials $H_n(u,x)$ are defined by

$$F(u,x,t)=\frac{1-u}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}H_n(u,x)\frac{t^n}{n!},\ \text{cf. [1, 3, 6]} \tag{4}$$

usual convention of symbolically replacing H^n by H_n as before. By the above definition, we obtain

$$\begin{aligned}\sum_{l=0}^{\infty}H_l(u,x)\frac{t^l}{l!}&=\frac{1-u}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}H_n(u)\frac{t^n}{n!}\sum_{m=0}^{\infty}x^m\frac{t^m}{m!}\\&=\sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}H_n(u)\frac{t^n}{n!}x^{l-n}\frac{t^{l-n}}{(l-n)!}\right)=\sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n}H_n(u)x^{l-n}\right)\frac{t^l}{l!}.\end{aligned}$$

By using comparing coefficients $\frac{t^l}{l!}$, we have

$$H_n(u,x)=\sum_{k=0}^{n}\binom{n}{k}H_k(u)x^{n-k}.$$

In the special case $x = 0$, we define $H_n(u,0) = H_n(u)$. We also observe that $H_n(-1,x) = E_n(x)$.

Since

$$\begin{aligned}\sum_{l=0}^{\infty} H_l(u,x+y)\frac{t^l}{l!} &= \frac{1-u}{e^t-u}e^{(x+y)t} = \sum_{n=0}^{\infty} H_n(u,x)\frac{t^n}{n!}\sum_{m=0}^{\infty} y^m\frac{t^m}{m!} \\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} H_n(u,x)\frac{t^n}{n!}y^{l-n}\frac{t^{l-n}}{(l-n)!}\right) \\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n}H_n(u,x)y^{l-n}\right)\frac{t^l}{l!},\end{aligned}$$

we have the following theorem.

Theorem 1. The Frobenius-Euler polynomials $H_n(u,x)$ satisfies the following relation:

$$H_l(u,x+y) = \sum_{n=0}^{l}\binom{l}{n}H_n(u,x)y^{l-n}.$$

Because

$$\frac{\partial}{\partial x}F(u,x,t) = tF(u,x,t) = \sum_{n=0}^{\infty}\frac{d}{dx}H_n(u,x)\frac{t^n}{n!},$$

it follows the important relation

$$\frac{d}{dx}H_n(u,x) = nH_{n-1}(u,x).$$

We have the integral formula as follows:

$$\int_a^b H_{n-1}(u,x)dx = \frac{1}{n}(H_n(u,b) - E_n(u,a)).$$

By using computer, the Frobenius-Euler polynomials $H_n(u,x)$ can be determined explicitly. A few of them are

$$H_0(u,x) = 1, \quad H_1(u,x) = \frac{1-x+ux}{-1+u},$$

$$H_2(u,x) = \frac{1+u-2x+2ux+x^2-2ux^2+u^2x^2}{(-1+u)^2},$$

$$H_3(u,x) = \frac{1+4u+u^2-3x+3u^2x+3x^2-6ux^2+3u^2x^2-x^3+3ux^3-3u^2x^3+u^3x^3}{(-1+u)^3},$$

$\ldots$

For example, setting $u = -2$, we get

$H_0(2, x) = 1, \quad H_1(2, x) = 1/3(-1 + 3x),$

$H_2(2, x) = 1/9(-1 - 6x + 9x^2),$

$H_3(2, x) = 1/9(1 - 3x - 9x^2 + 9x^3),$

$H_4(2, x) = 1/27(5 + 12x - 18x^2 - 36x^3 + 27x^4),$

$H_5(2, x) = 1/81(-7 + 75x + 90x^2 - 90x^3 - 135x^4 + 81x^5),$

$H_6(2, x) = 1/81(-49 - 42x + 225x^2 + 180x^3 - 135x^4 - 162x^5 + 81x^6),$

$H_7(2, x) = 1/243(-53 - 1029x - 441x^2 + 1575x^3 + 945x^4 - 567x^5 - 567x^6 + 243x^7).$

2. ZEROS OF THE FROBENIUS-EULER POLYNOMIALS $H_n(u, x)$

In this section, we display the shapes of the Frobenius-Euler polynomials $H_n(u, x)$ and we investigate the zeros of the Frobenius-Euler polynomials $H_n(u, x)$ for $u < -1$. For $n = 1, \ldots, 10$, we can draw a plot of the Frobenius-Euler polynomials $H_n(u, x)$, respectively. This shows the ten plots combined into one. We display the shape of $H_n(-2, x)$, $-3 \le x \le 3$ (Figure 1). We investigate the beautiful zeros of the $H_n(u, x)$

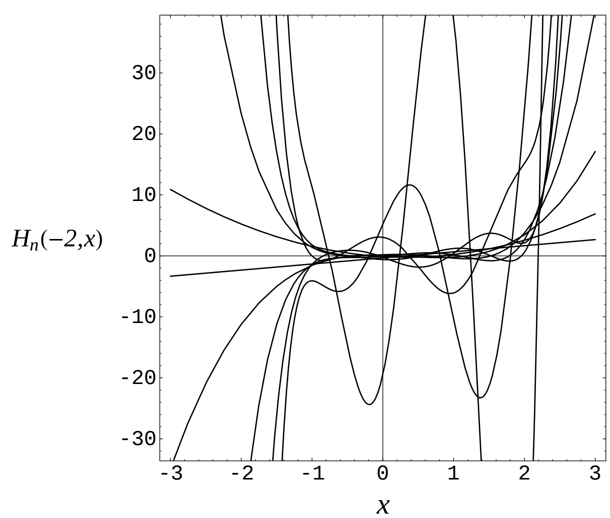

FIGURE 1. Curve of $H_n(-2, x)$

by using a computer. We plot the zeros of the Frobenius-Euler polynomials $H_n(-2, x)$ for $n = 10, 15, 25, 30$ and $x \in \mathbb{C}$ (Figure 2).

Our numerical results for approximate solutions of real zeros of $H_n(u, x)$ are displayed (Tables 1, 2).

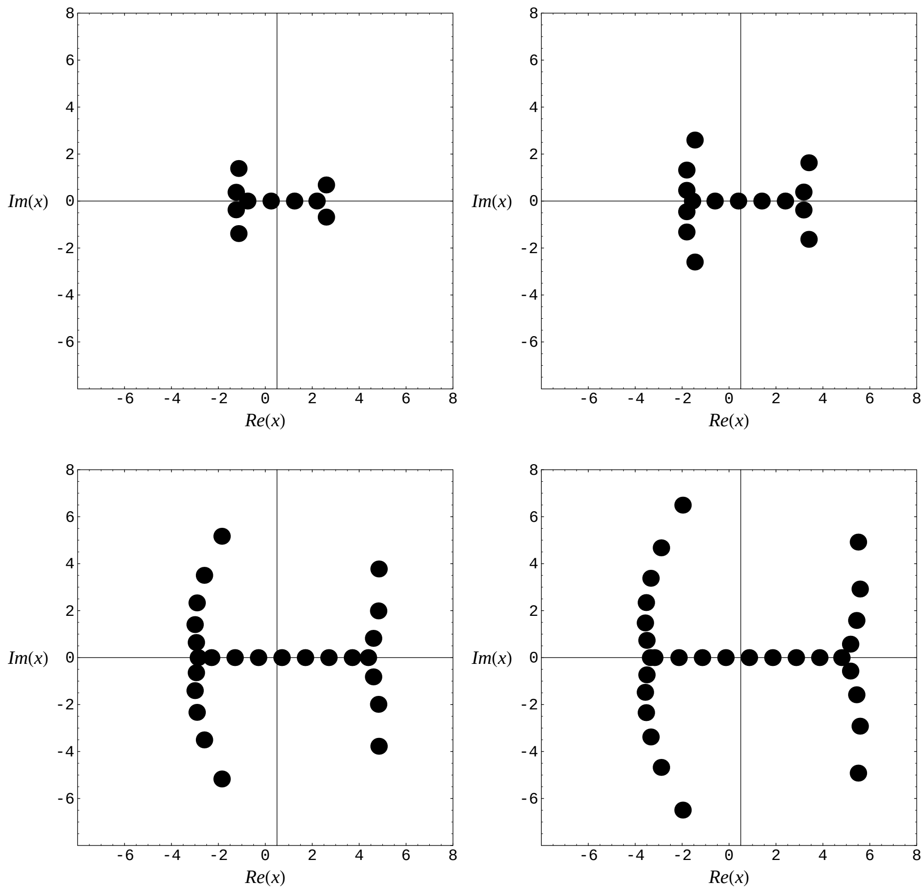

FIGURE 2. Zeros of $H_n(-2, x)$ for $n = 10, 15, 25, 30$

Table 1. Numbers of real and complex zeros of $H_n(u, x)$

degree n	$u = -1$		$u = -2$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	4	0	2	2
5	5	0	3	2
6	2	4	4	2
7	3	4	3	4
8	4	4	4	4
9	5	4	3	6
10	6	4	4	6
11	3	8	5	6

We plot the zeros of the Frobenius-Euler polynomials $H_n(u, x)$ for $n = 30, u = -5, -10, -20, -30$ and $x \in \mathbb{C}$ (Figure 3). We also plot the zeros of the Frobenius-

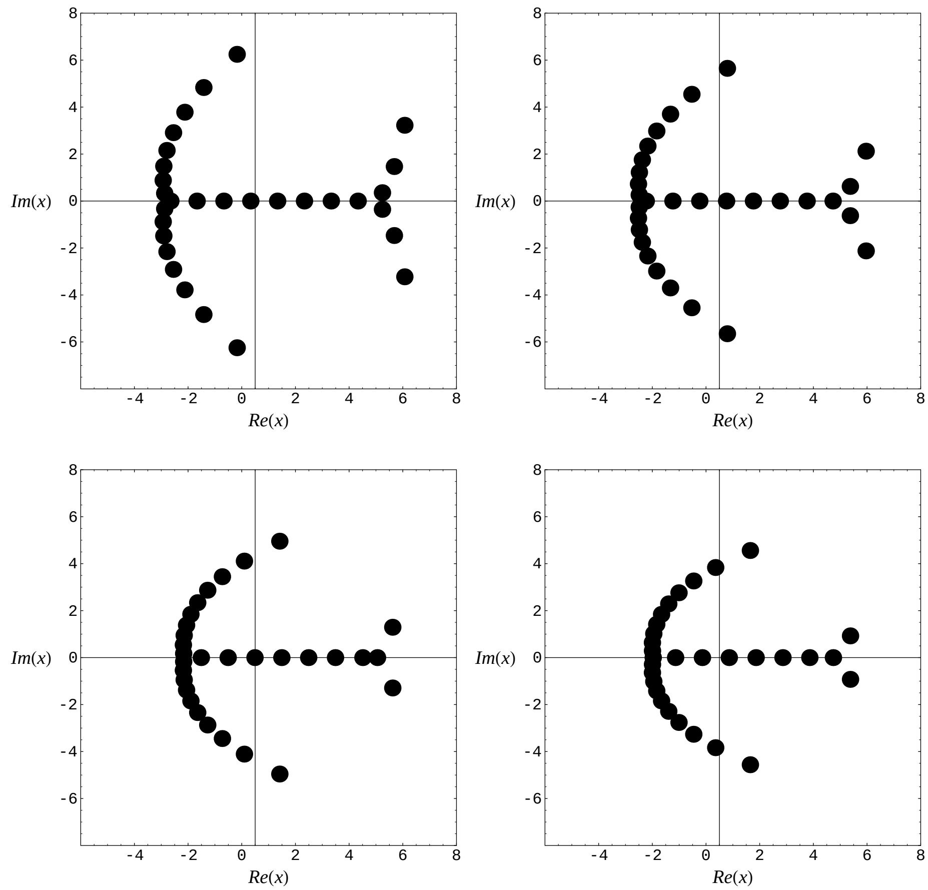

FIGURE 3. Zeros of $H_{30}(u,x)$ for $u=-5,-10,-20,-30$

Euler polynomials $H_n(u,x)$ for $n=25, u=-1.5,-1.2,-1.01,-1.001$ and $x\in\mathbb{C}$ (Figure 4).

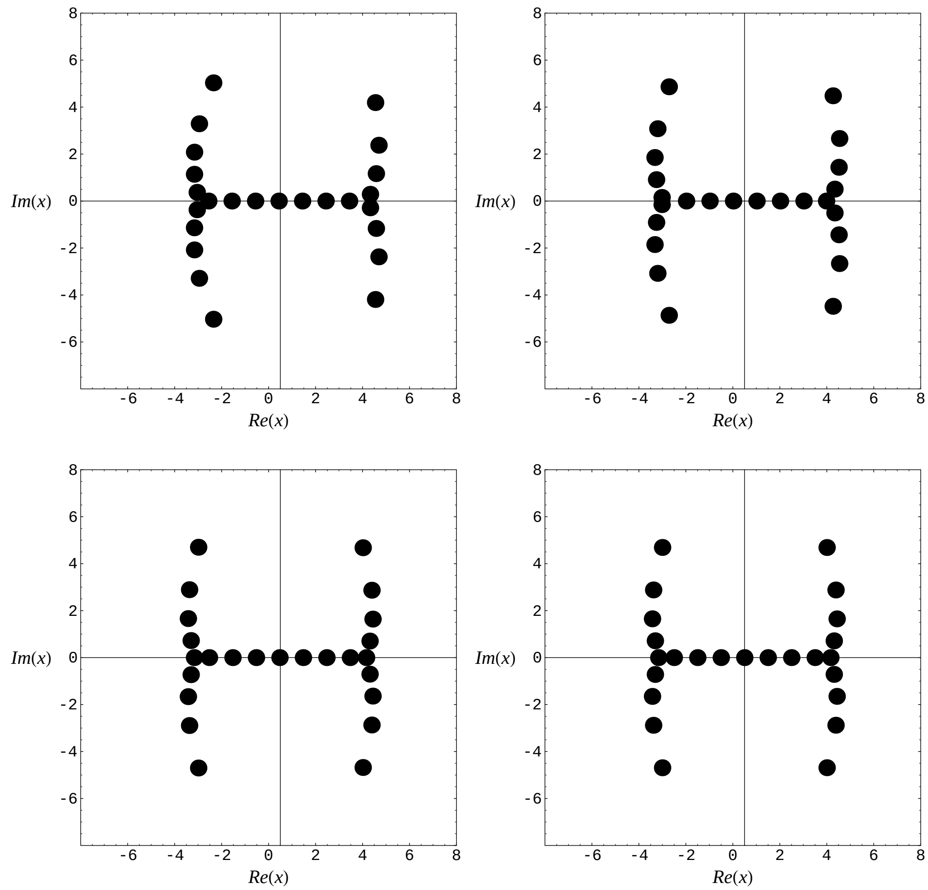

FIGURE 4. Zeros of $H_{25}(u, x)$ for $u = -1.5, -1.2, -1.01, -1.001$

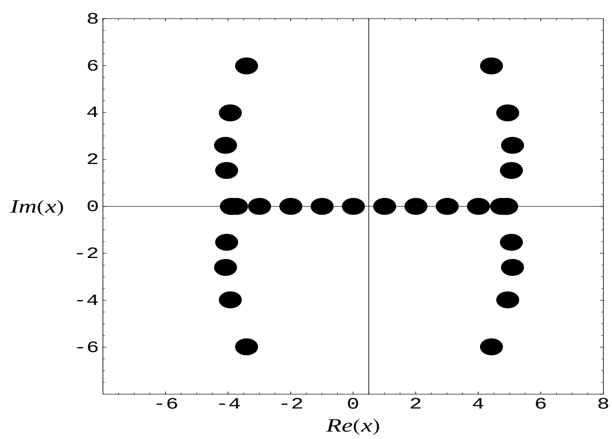

FIGURE 5. Zeros of $H_{25}(-1, x) = E_n(x)$

Stacks of zeros of $H_n(-2, x)$ for $1 \leq n \leq 30$ from a 3-D structure are presented (Figure 6). We observe a remarkably regular structure of the complex roots of the Frobenius-Euler polynomials $H_n(u, x)$. We hope to verify a remarkably regular structure of the complex roots of the Frobenius-Euler polynomials $H_n(u, x)$ (Table 1).

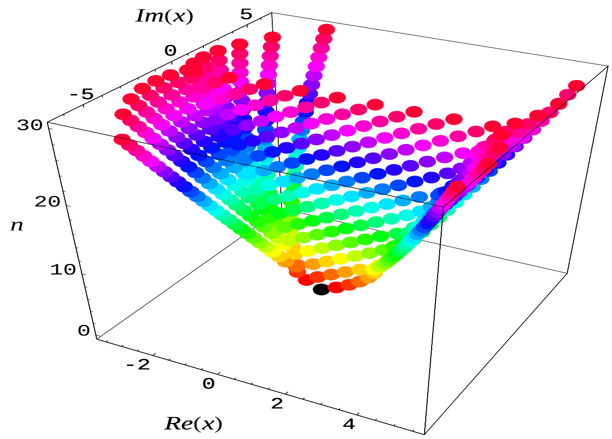

FIGURE 6. Stacks of zeros of $H_n(-2, x), 1 \leq n \leq 30$

This numerical investigation is especially exciting because we can obtain an interesting phenomenon of scattering of the zeros of $H_n(u, x)$. These results are used not only in pure mathematics and applied mathematics, but also used in mathematical physics and other areas. Next, we calculated an approximate solution satisfying $H_n(u, x), x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $H_n(-2, x) = 0, x \in \mathbb{R}$

degree n	x
1	0.33333
2	-0.13807, 0.8047
3	-0.42060, 0.22004, 1.2006
4	0.6547, 1.5273
5	0.08542, 1.0854, 1.7866
6	-0.4719, 0.5160, 1.528, 1.958
7	-0.8424, -0.05293, 0.9471
⋮	⋮

Figures 7 and 8 show the distribution of real zeros of $H_n(u, x)$ for $1 \leq n \leq 30$.

Figure 9 presents the distribution of real zeros of $H_n(-1, x) = E_n(x)$ for $1 \leq n \leq 30$.

3. DIRECTIONS FOR FURTHER RESEARCH

Finally, we shall consider the more general problems. In [5], we observed the behavior of complex roots of the Euler polynomials $E_n(x)$, using numerical investigation. Prove that $E_n(x), x \in \mathbb{C}$, has $Re(x) = 1/2$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions. The obvious

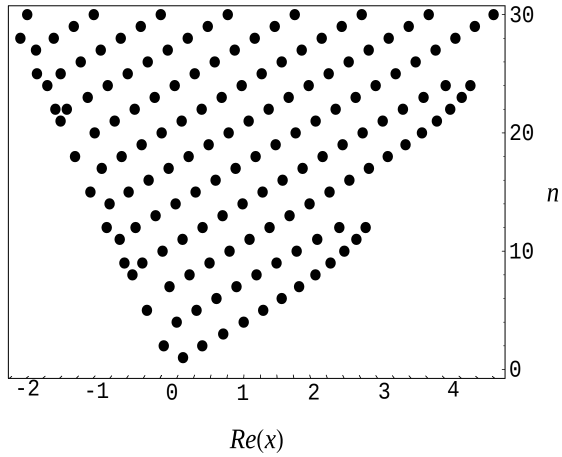

FIGURE 7. Plot of real zeros of $H_n(-10, x), 1 \leq n \leq 30$

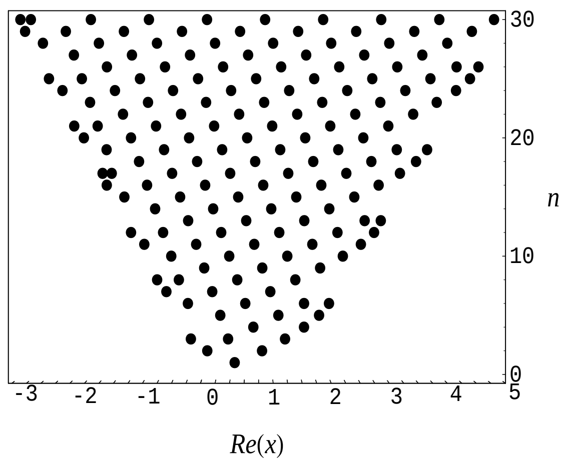

FIGURE 8. Plot of real zeros of $H_n(-2, x), 1 \leq n \leq 30$

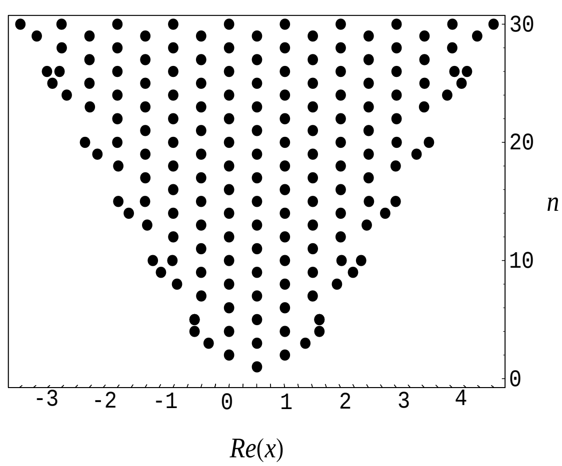

FIGURE 9. Plot of real zeros of $H_n(-1, x) = E_n(x), 1 \leq n \leq 30$

corollary is that the zeros of $E_n(x)$ will also inherit these symmetries.

$$\text{If } E_n(x_0) = 0, \text{ then } E_n(1 - x_0) = 0 = E_n(x_0^*) = E_n(1 - x_0^*) \tag{5}$$

$*$ denotes complex conjugation (see [5], Figures 5, 9). Prove that $E_n(x) = 0$ has n distinct solutions. If $E_{2n+1}(x)$ has $Re(x) = 1/2$ and $Im(x) = 0$ reflection symmetries, and $2n + 1$ non-degenerate zeros, then $2n$ of the distinct zeros will satisfy (5). If the remaining one zero is to satisfy (5) too, it must reflect into itself, and therefore it must lie at $1/2$ (see Figure 9), the center of the structure of the zeros, ie.,

$$E_n(1/2) = 0 \quad \forall \text{ odd } n.$$

Prove that $H_n(u, x) = 0$ has n distinct solutions, ie., all the zeros are non-degenerate. Find the numbers of complex zeros $C_{H_n(u,x)}$ of $H_n(u, x), Im(x) \neq 0$. Since n is the degree of the polynomial $H_n(u, x)$, the number of real zeros $R_{H_n(u,x)}$ lying on the real plane $Im(x) = 0$ is then $R_{H_n(u,x)} = n - C_{H_n(u,x)}$, where $C_{H_n(u,x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{H_n(u,x)}$ and $C_{H_n(u,x)}$. Find the equation of envelope curves bounding the real zeros lying on the plane. We prove that $H_n(u, x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions. If $H_n(u, x) = 0$, then $H_n(u, x^*) = 0$, where $*$ denotes complex conjugate (see Figures 2, 3, 4). Observe that the structure of the zeros of the Euler polynomials $E_n(x)$ resembles the structure of the zeros of the Frobenius-Euler polynomials $H_n(u, x)$ as $u \to -1$ (see Figures 4, 5, 7, 8, 9). In order to study the Frobenius-Euler polynomials $H_n(u, x)$, we must understand the structure of the Frobenius-Euler polynomials $H_n(u, x)$. Therefore, using computer, a realistic study for the Frobenius-Euler polynomials $H_n(u, x)$ is very interesting. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the Frobenius-Euler polynomials $H_n(u, x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [4, 5, 6, 7].

REFERENCES

1. Kim, T.(2008). *Euler numbers and polynomials associated with zeta function, Abstract and Applied Analysis*, Art. ID 581582.
2. Kim, T., Jang, L. C., Pak, H. K. (2001). *A note on q-Euler and Genocchi numbers* , *Proc. Japan Acad.*, v.77 A, pp. 139–141.
3. Liu, G.(2006). *Congruences for higher-order Euler numbers*, *Proc. Japan Acad.*, v. 82 A, pp. 30–33.
4. Ryoo, C.S. (2008). *A numerical computation on the structure of the roots of q-extension of Genocchi polynomials*, *Applied Mathematics Letters*, v.21, pp. 348–354.
5. Ryoo, C.S., Y. S. Yoo (2009). *A note on Euler numbers and polynomials*, *Journal of Concrete and Applicable Mathematics*, v.7(4), pp. 341–348.

6. Ryoo, C.S. (2009). *Calculating zeros of the q-Euler polynomials* , *Proceeding of the Jangjeon Mathematical Society*, v.12, pp. 253–259.
7. Ryoo, C.S., Kim, T., Agarwal, R.P. (2006). *A numerical investigation of the roots of q-polynomials*, *Inter. J. Comput. Math.*, v.83, pp. 223–234.
8. Tsumura, H. (1987). *On a p-adic interpolation of generalized Euler Numbers and its applications*, *Tokyo J. Math.*, v.10, pp. 281–293.