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A NUMERICAL COMPUTATION OF THE ROOTS OF HURWITZ q-EULER ZETA FUNCTIONS

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ABSTRACT. In [2], we defined the q-Euler numbers $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$. By using q-Euler numbers $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$, q-Euler zeta function $\zeta_q(s)$and Hurwitz q-Euler zeta functions $\zeta_q(s,x)$ are defined. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of $\zeta_q(s,x))$ in complex plane. Finally, we investigate the roots of Hurwitz q-Euler zeta functions $\zeta_q(s,x)$.

Key words. q-Euler polynomials, q-Euler zeta function, Hurwitz q-Euler zeta functions

1. INTRODUCTION

Many mathematicians have studied Euler numbers, Euler polynomials, q-Euler numbers, and q-Euler polynomials (see [1,2,3,4]). Euler numbers, Euler polynomials, q-Euler numbers, and q-Euler polynomials numbers posses many interesting properties and arising in many areas of mathematics and physics. In [2], we observed the behavior of complex roots of the q-Euler polynomials $E_{n,q}(x)$, using numerical investigation. By means of numerical experiments, we demonstrated a remarkably regular structure of the complex roots of the q-Euler polynomials $E_{n,q}(x)$. In this paper, we introduce q-Euler zeta function $\zeta_q(s)$ and Hurwitz q-Euler zeta functions $\zeta_q(s,x)$. In order to study the q-Euler zeta function $\zeta_q(s)$ and Hurwitz q-Euler zeta functions $\zeta_q(s,x)$, we must understand the structure of the q-Euler zeta function $\zeta_q(s)$ and Hurwitz q-Euler zeta functions $\zeta_q(s,x)$. Therefore, using computer, a realistic study for the q-Euler zeta function $\zeta_q(s)$ and Hurwitz q-Euler zeta functions $\zeta_q(s,x)$ is very interesting. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of q-Euler zeta function $\zeta_q(s)$ and Hurwitz q-Euler zeta functions $\zeta_q(s,x)$ in complex plane.

The outline of this paper is as follows. We introduce the q-Euler polynomials $E_{n,q}(x)$ and q-Euler numbers $E_{n,q}$. We investigate some interesting results which are related to the q-Euler numbers $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$. In Section 2, we define q-Euler zeta function $\zeta_q(s)$ and Hurwitz q-Euler zeta functions $\zeta_q(s,x)$.

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We derive the existence of a specific interpolation function which interpolate the q-Euler numbers $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$ at negative integer. In section 3, we describe the beautiful zeros of Hurwitz q-Euler zeta functions $\zeta_q(s,x)$ using a numerical investigation. Finally, we investigate the roots of the Hurwitz q-Euler zeta functions $\zeta_q(s,x)$.

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers

First, we introduce the classical Euler numbers E_n and Euler polynomials $E_n(x)$. The Euler numbers E_n are defined by the generating function:

$$F(t) = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \text{ cf. [1,4,5]} \tag{1.1}$$

where we use the technique method notation by replacing E^n by $E_n (n \geq 0)$ symbolically. For $x \in \mathbb{R}$, we consider the Euler polynomials $E_n(x)$ as follows:

$$F(x,t) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.2}$$

Note that $E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k x^{n-k}$. In the special case $x = 0$, we define $E_n(0) = E_n$.

Let q be a complex number with $|q| < 1$. By the meaning of (1.1) and (1.2), we defined the q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ as follows (see [2]):

$$F_q(t) = \frac{2}{qe^t+1} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}, \tag{1.3}$$

$$F_q(t,x) = \frac{2}{qe^t+1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{1.4}$$

The following elementary properties of the q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ are readily derived form (1.3) and (1.4)(see, for details, [2]). We, therefore, choose to omit details involved.

Proposition 1.1. For any positive integer n, the formula of q-polynomials

$$E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} E_{k,q} x^{n-k}.$$

Proposition 1.2 (Integral formula).

$$\int_a^b E_{n-1,q}(x)dx = \frac{1}{n}(E_{n,q}(b) - E_{n,q}(a)).$$

Proposition 1.3 (Addition theorem).

$$E_{n,q}(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_{k,q}(x) y^{n-k}.$$

Proposition 1.4 (Difference equation).

$$qE_{n,q}(x+1) + E_{n,q}(x) = 2x^n.$$

2. THE ANALOGUE OF THE EULER ZETA FUNCTION

By using q-Euler numbers and polynomials, q-Euler zeta function and Hurwitz q-Euler zeta functions are defined. These functions interpolate the q-Euler numbers and q-Euler polynomials, respectively. In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. From (1.3), we note that

$$\frac{d^k}{dt^k} F_q(t)\bigg|_{t=0} = 2\sum_{n=0}^{\infty}(-1)^n q^n n^k, (k \in \mathbb{N}).$$

By using the above equation, we are now ready to define q-Euler zeta functions.

Definition 2.1. Let $s \in \mathbb{C}$.

$$\zeta_q(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n^s}. \tag{2.1}$$

Note that $\zeta_q(s)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_q(s)$ and $E_{k,q}$ is given by the following theorem.

Theorem 2.2. For $k \in \mathbb{N}$, we have

$$\zeta_q(-k) = E_{k,q}.$$

Observe that $\zeta_{q,}(s)$ function interpolates $E_{k,q}$ numbers at non-negative integers. By using (1.4), we note that

$$\frac{d^k}{dt^k} F_q(t,x)\bigg|_{t=0} = 2\sum_{n=0}^{\infty}(-1)^n q^n (n+x)^k, (k \in \mathbb{N}), \tag{2.2}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}\right)\bigg|_{t=0} = E_{k,q}(x), \text{ for } k \in \mathbb{N}. \tag{2.3}$$

By (2.2) and (2.3), we are now ready to define the Hurwitz q-Euler zeta functions.

Definition 2.3. Let $s \in \mathbb{C}$.

$$\zeta_q(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(n+x)^s}. \tag{2.4}$$

Note that $\zeta_q(s,x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_q(s,x)$ and $E_{k,q}^{(h)}(x)$ is given by the following theorem.

Theorem 2.4. For $k \in \mathbb{N}$, we have

$$\zeta_q(-k,x) = E_{k,q}(x).$$

Observe that $\zeta_q(-k,x)$ function interpolates $E_{k,q}(x)$ numbers at non-negative integers.

3. ZEROS OF THE HURWITZ q-EULER ZETA FUNCTIONS

In this section, we show a plot of $\zeta_q(s,x)$, $q=1/2$, $-2 \le s \le 2$, $-1/2 \le x \le 1/2$ (Figs. 1-2). For $k=1,\dots,10$, we can draw a plot of the $\zeta_q(-k,x)$, respectively. This shows the ten plots combined into one. We display the shape of $\zeta_q(-k,x)$, $q=1/2$, $-2 \le x \le 2$ for any positive integer k (Fig. 3).

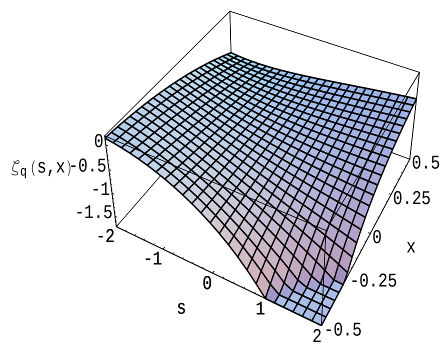

FIGURE 1. Plot of $\zeta_q(s,x)$

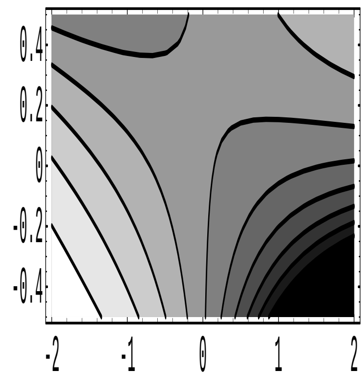

FIGURE 2. Contour plot

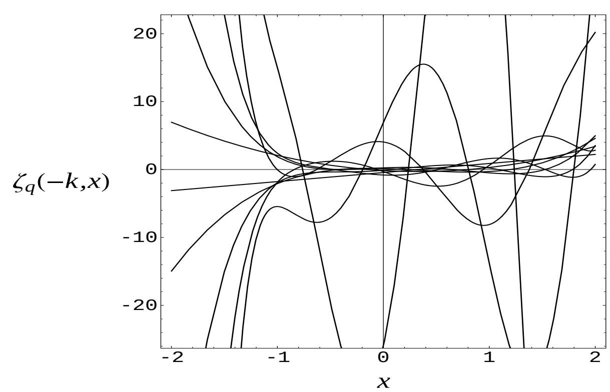

FIGURE 3. Curve of $\zeta_{1/2}(-k,x)$

Next, we investigate the zeros of $\zeta_q(-k,x)$, $q=1/2$, $k=5,10,15,20$, $x \in \mathbb{C}$ (Fig. 4). We display the zeros of $\zeta_q(-k,x)$, $q=1/10,1/20,1/30,1/40$, $k=20$, $x \in \mathbb{C}$ (Fig. 5) and $\zeta_q(-k,x)$, $q=-1/10,-1/20,-1/30,-1/40$, $k=20$, $x \in \mathbb{C}$ (Fig. 6).

Stacks of zeros of $\zeta_{1/2}(-k,x)$ for $1 \le k \le 25$ from a 3-D structure are presented. (Fig. 7). Our numerical results for numbers of real and complex zeros of $\zeta_q(-k,x)$, $x \in \mathbb{C}$ are displayed (Table 1).

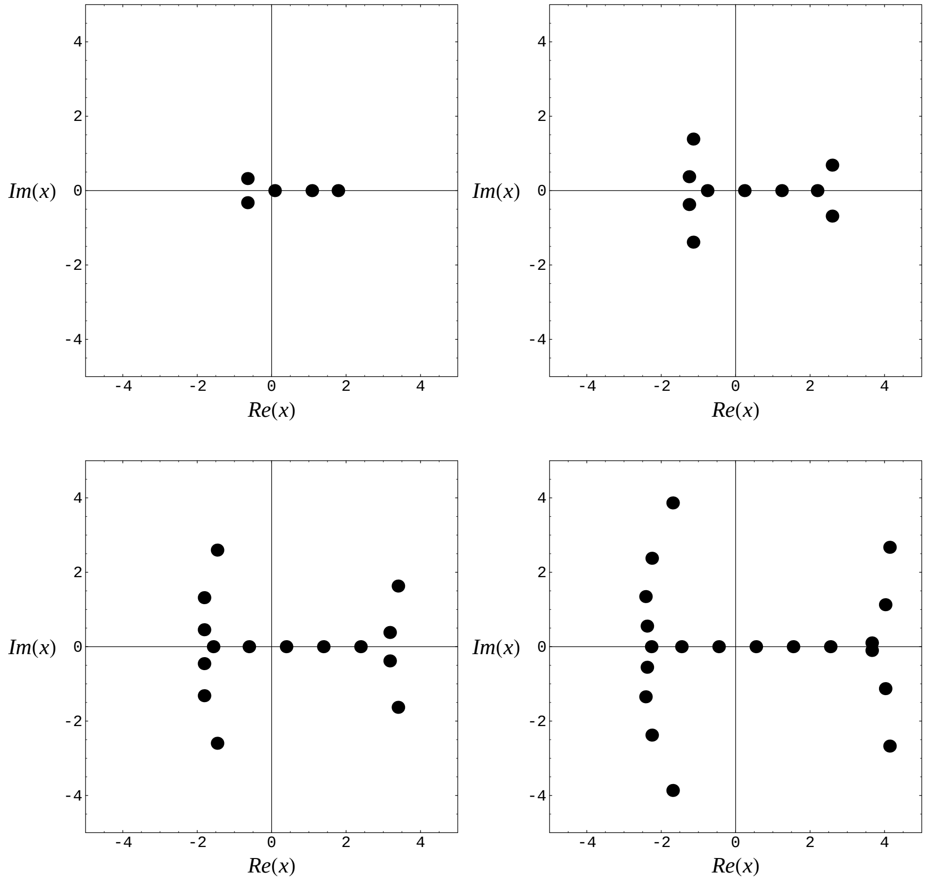

FIGURE 4. Zeros of $\zeta_{1/2}(-k,x), k = 5, 10, 15, 20$

Table 1. Numbers of real and complex zeros of $\zeta_q(-k,x)$

degree n	$q = 1/2$		$q = -1/2$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	0	2
3	3	0	1	2
4	2	2	0	4
5	3	2	1	4
6	4	2	0	6
7	3	4	1	5
8	4	4	0	8
9	3	6	1	8
10	4	6	0	10
11	5	6	1	10
12	6	6	0	12
13	5	8	1	12
14	4	10	0	14

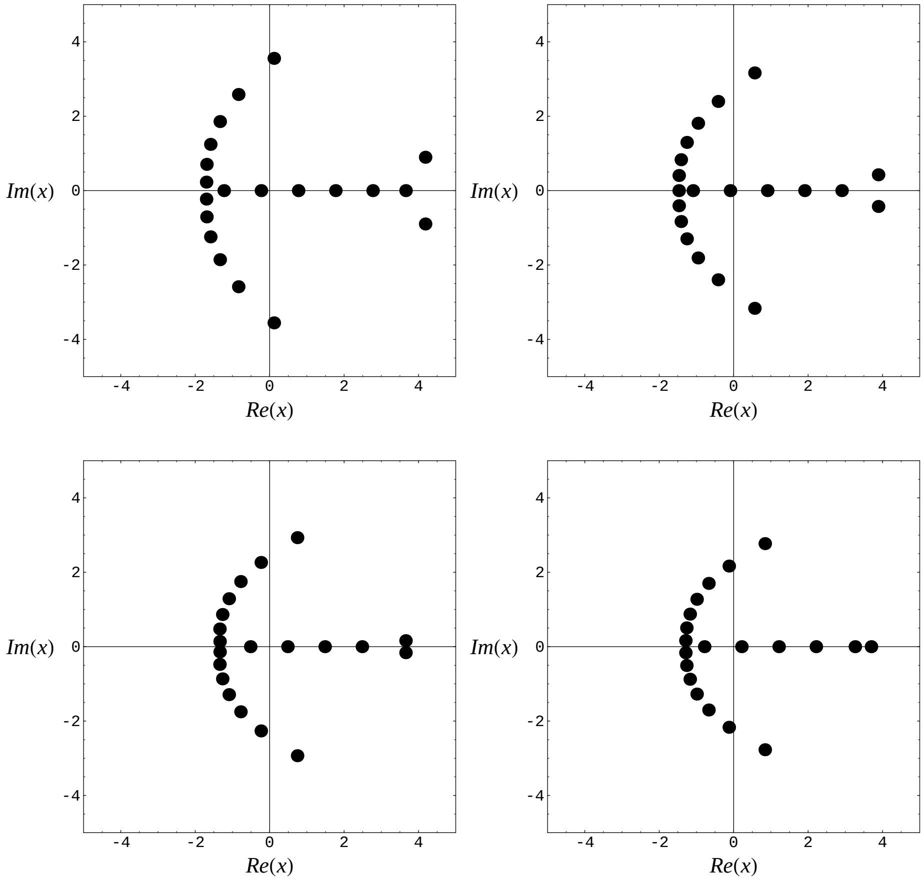

FIGURE 5. Zeros of $\zeta_q(-k, x), q = 1/10, 1/20, 1/30, 1/40$

Table 2. Approximate solutions of $\zeta_q(-k, x) = 0, q = 1/2, x \in \mathbb{R}$

degree n	x
1	0.33333
2	−0.13807, 0.8047
3	−0.42060, 0.22004, 1.2006
4	0.6547, 1.5273
5	0.08542, 1.0854, 1.7866
6	−0.4719, 0.5160, 1.528, 1.958
7	−0.8424, −0.05293, 0.9471
8	−1.0017, −0.6275, 0.3779, 1.378
9	0.19123, 0.8088, 1.805
10	−0.7586, 0.23965, 1.2396, 2.1965.085502

We observe a remarkably regular structure of the complex roots of the $\zeta_q(-k, x)$. We hope to verify a remarkably regular structure of the complex roots of the $\zeta_q(-k, x)$

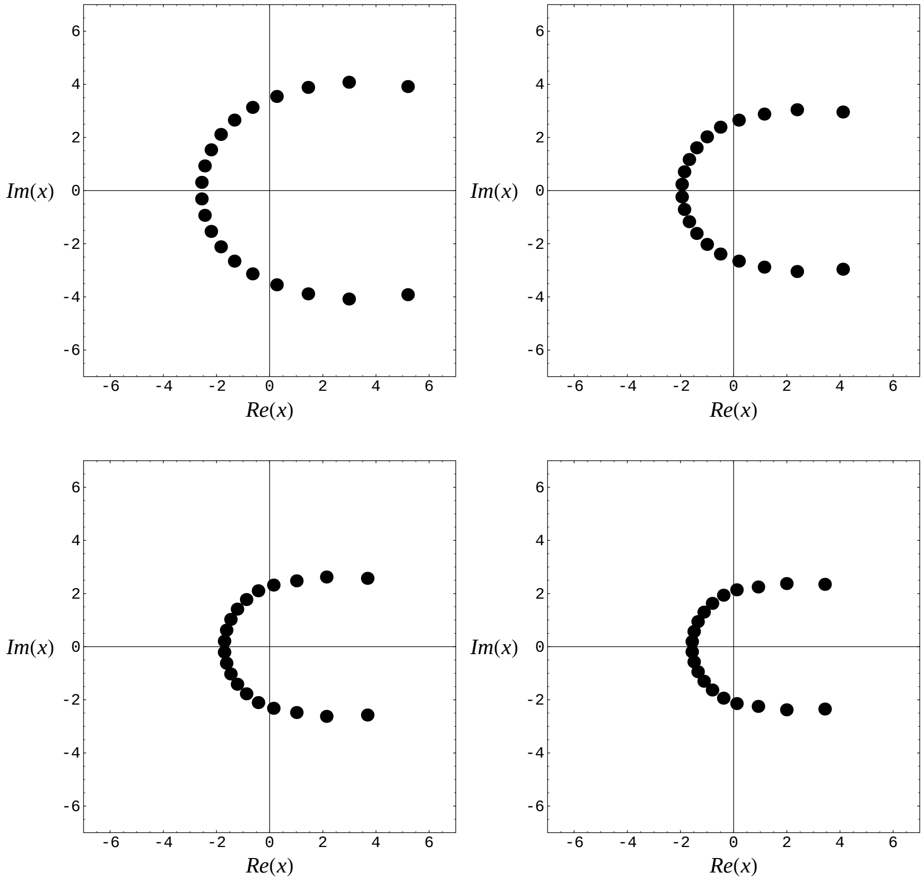

FIGURE 6. Zeros of $\zeta_q(-k, x), q = -1/10, -1/20, -1/30, -1/40$

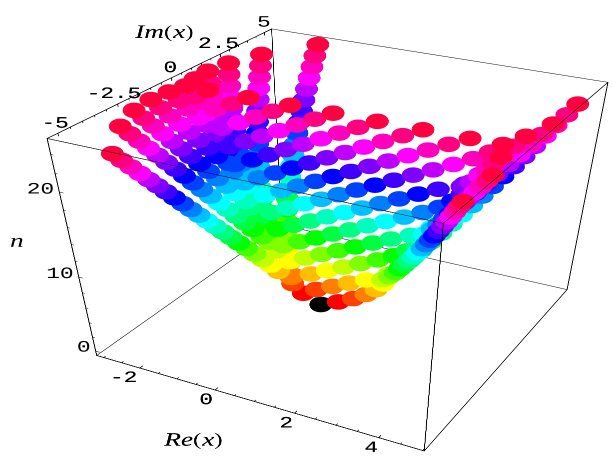

FIGURE 7. Stacks of zeros of $\zeta_{1/2}(-k, x)$

(Table 1). Next, we calculated an approximate solution satisfying $\zeta_q(-k, x) = 0$, $x \in \mathbb{C}$. The results are given in Table 2, Table 3, and Table 4. Finally, we shall consider the more general problems. Prove that $\zeta_q(-k, x)) = 0$ has n distinct solutions. Find the numbers of complex zeros $C_{\zeta_q(-k,x)}$ of $\zeta_q(-k, x), Im(x) \neq 0$. The number of real zeros $R_{\zeta_q(-k,x)}$ lying on the real plane $Im(x) = 0$ is then $R_{\zeta_q(-k,x)} = n - C_{\zeta_q(-k,x)}$, where $C_{\zeta_q(-k,x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{\zeta_q(-k,x)}$ and $C_{\zeta_q(-k,x)}$. We prove that $\zeta_q(-k,x)$, $x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry. (Figs. 4–6). For related topics the interested reader is referred to [2,3,4,5,6,7,8,9].

Table 3. Approximate solutions of $\zeta_q(-k,x) = 0, q = 1/5, x \in \mathbb{R}$

degree n	x
1	0.16667
2	$-0.20601, \quad 0.53934$
3	$-0.3009, \quad -0.10220, \quad 0.9031$
4	$0.24627, \quad 1.2384$
5	$-0.3717, \quad 0.5951, \quad 1.5439$
6	$-0.6447,, \quad -0.05472, \quad 0.9453, \quad 1.8195$
7	$0.29446, \quad 1.2945, \quad 2.0634$
8	$-0.3562, \quad 0.6437, \quad 1.6473, \quad 2.2696$
9	$-0.863, -0.007003, \quad 0.9930, \quad 2.025, \quad 2.416$
10	$-1.019, \quad -0.660, \quad 0.34230, \quad 1.3423$

Table 4. Approximate solutions of $\zeta_q(-k,x) = 0, q = -1/2, x \in \mathbb{C}$

degree n	x
1	-1.0000
2	$-1.0000 - 1.4142i, \quad -1.0000 + 1.4142i$
3	$-1.8846, \quad -0.5577 - 2.5665i, \quad -0.5577 + 2.5665i$
4	$-2.076 - 1.256i, \quad -2.076 + 1.256i, \quad 0.0756 - 3.5686i,$ $0.0756 + 3.5686i$
5	$-2.739, \quad 1.951 - 2.402i, \quad -1.951 + 2.402i, 0.820 - 4.468i,$ $0.820 + 4.468i$
6	$-2.999 - 1.188i, \quad -2.999 + 1.188i, \quad -1.640 - 3.461i,$ $-1.640 + 3.461i, \quad 1.640 - 5.290i, \quad 1.640 + 5.290$
7	$-3.58, \quad -3.019 - 2.313i, \quad -3.019 + 2.313i, \quad -1.206 - 4.451i,$ $-1.206 + 4.451i, \quad 2.514 - 6.052i, \quad 2.514 + 6.052i$
8	$-3.88 - 1.15i, \quad -3.88 + 1.15i, \quad -2.876 - 3.382i, \quad -2.876 + 3.382i$ $-0.680 - 5.384i, \quad -0.680 + 5.384i, \quad 3.431 - 6.766i, \quad 3.431 + 6.766i$

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