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MODELING, ANALYSIS AND TIMETABLE DESIGN OF A HELICOPTER MAINTENANCE PROCESS BASED ON TIMED EVENT PETRI NETS AND MAX-PLUS ALGEBRA

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ABSTRACT. In this paper an algorithm for computing a generalized eigenmode of reducible regular matrices over the max-plus algebra is applied to a helicopter maintenance process. A timed event Petri net model is constructed from the state transition dynamics table that characterizes the transport system. A max-plus recurrence equation, with a reducible and regular matrix, is associated to the timed event Petri net. Next, given the reducible and regular matrix, the problem consists in giving an algorithm which will tell us how to compute its generalized eigenmode over the max plus algebra. The solution to the problem is achieved by studying some type of recurrence equations. In fact, by transforming the reducible regular matrix into its normal form, and considering a very specific recurrence equation, an explicit mathematical characterization is obtained, upon which the algorithm is constructed. The generalized eigenmode obtained sets a timetable for the helicopter maintenance process.

Key Words Max-Plus Algebra, Reducible Matrices, Eigenmode, Recurrent Equations, Algorithm, Helicopter Maintenance Process.

1. INTRODUCTION

In this paper an algorithm for computing a generalized eigenmode of reducible regular matrices over the max-plus algebra is applied to a helicopter maintenance process. A timed event Petri net model is constructed from the state transition dynamics table that characterizes the process. A max-plus recurrence equation, with a reducible and regular matrix, is associated to the timed event Petri net model. Next, given the reducible and regular matrix of finite size i.e., a matrix such that in each one of its rows has at least one finite element and whose communication graph is not strongly connected, the problem consists in giving an algorithm which will tell us how to compute its generalized eigenmode over the max plus algebra, which indeed has an idempotent semiring, or also called dioid, mathematical structure. The notion of generalized eigenmode, as its name says, results to be a genaralization of the notion of eigenvalues and eigenvectors for the case when the matrix under study

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is irreducible i.e., has a communication graph which is strongly connected. The solution to the problem is achieved by studying some type of recurrence equations. In fact, by transforming the reducible regular matrix into a block upper triangular form, called normal form, and considering a very specific recurrence equation, an explicit mathematical characterization is obtained, upon which the algorithm is constructed. The generalized eigenmode obtained sets a timetable for the helicopter maintenance process. Another alternative algorithm for computing a generalized eigenmode of a reducible and regular matrix, is Howard's algorithm which is based on a policy iteration improvement procedure which in numerical examples has proven to be very efficient. The paper is organized as follows. In section 2, the concept of max-plus algebra is defined, its algebraic structure is also described. Matrices and graphs are presented, the spectral theory of matrices is discussed, finally the problem of solving linear equations is addressed. Section 3, starts by introducing the concept of generalized eigenmode. Once this has been done, it continues by discussing, how to compute the generalized eigenmode for recurrence equations for the cases of irreducible and reducible matrices, Mth order recurrence equations are also treated. In section 4, the algorithm is formally presented. In section 5, max-plus recurrence equations for timed event Petri nets are introduced. Section 6, presents the helicopter maintenance process. Finally, in section 7, some conclusions are given.

2. **MAX-PLUS ALGEBRAS** [1, 2]

2.1. **Basic Definitions. NOTATION:** $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ is the set of real numbers, $\epsilon = -\infty$, $e = 0$, $\mathbb{R}_{max} = \mathbb{R} \cup \{\epsilon\}$, $\underline{n} = 1, 2, ..., n$ Let $a, b \in \mathbb{R}_{max}$ and define the operations $\oplus$ and $\otimes$ by: $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$.

Definition 2.1. The set $\mathbb{R}_{max}$ with the two operations $\oplus$ and $\otimes$ is called a max-plus algebra and is denoted by $\Re_{\max} = (\mathbb{R}_{max}, \oplus, \otimes, \epsilon, e)$.

Definition 2.2. A semiring is a nonempty set R endowed with two operations $\oplus_R$, $\otimes_R$, and two elements ϵ_R and e_R such that: $\oplus_R$ is associative and commutative with zero element ϵ_R, $\otimes_R$ is associative, distributes over $\oplus_R$, and has unit element e_R, $\in_R$is absorbing for $\otimes_R$ i.e., $a \otimes_R \epsilon = \epsilon_R \otimes a = a$, $\forall a \in R$..

Such a semiring is denoted by $\Re = (R, \oplus_R, \otimes_R, \epsilon, e)$. In addition if $\otimes_R$ is commutative then R is called a commutative semiring , and if $\oplus_R$ is such that $a \oplus_R a = a$, $\forall a \in R$ then it is called idempotent.

Theorem 2.3. *The max-plus algebra $\Re_{\max} = (\mathbb{R}_{max}, \oplus, \otimes, \epsilon, e)$ has the algebraic structure of a commutative and idempotent semiring.*

2.2. **Matrices and Graphs.** Let $\mathbb{R}_{max}^{n\times n}$ be the set of $n\times n$ matrices with coefficients in $\mathbb{R}_{max}$ with the following operations: The sum of matrices $A, B \in \mathbb{R}_{max}^{n\times n}$, denoted $A\oplus B$ is defined by: $(A\oplus B)_{ij} = a_{ij}\oplus b_{ij} = \max\,(a_{ij}, b_{ij})$ for i and $j \in \underline{n}$. The product of matrices $A\in\mathbb{R}_{max}^{n\times l}, B\in\mathbb{R}_{max}^{l\times n}$, denoted $A\otimes B$ is defined by: $(A\otimes B)_{ik} = \bigotimes_{j=1}^{l} a_{ij}\otimes b_{jk} = \max_{j\in\underline{l}}\,\{a_{ij}+b_{jk}\}$ for i and $k\in\underline{n}$. Let $\mathcal{E}\in\mathbb{R}_{max}^{n\times n}$ denote the matrix with all its elements equal to ϵ and denote by $E\in\mathbb{R}_{max}^{n\times n}$ the matrix which has its diagonal elements equal to e and all the other elements equal to ϵ. Then, the following result can be stated.

Theorem 2.4. *The 5-tuple* $\Re_{\max}^{n\times n} = (\mathbb{R}_{max}^{n\times n}, \oplus, \otimes, \mathcal{E}, E)$ *has the algebraic structure of a noncommutative idempotent semiring.*

Definition 2.5. Let $A\in\mathbb{R}_{max}^{n\times n}$ and $k\in\mathbb{N}$ then the k-th power of A denoted by $A^{\otimes k}$ is defined by: $A^{\otimes k} = \underbrace{A\otimes A\otimes\cdots\otimes A}_{k-\text{times}}$, where $A^{\otimes 0}$ is set equal to E.

Definition 2.6. A matrix $A\in\mathbb{R}_{max}^{n\times n}$ is said to be regular if A contains at least one element distinct from ϵ in each row.

Definition 2.7. Let N be a finite and non-empty set and consider $D\subseteq N\times N$. The pair $G=(N,D)$ is called a directed graph, where N is the set of elements called nodes and D is the set of ordered pairs of nodes called arcs. A directed graph $G=(N,D)$ is called a weighted graph if a weight $w(i,j)\in\mathbb{R}$ is associated with any arc $(i,j)\in D$.

Let $A\in\mathbb{R}_{max}^{n\times n}$ be any matrix, a graph $G(A)$, called the communication graph of A, can be associated as follows. Define $N(A)=\underline{n}$ and a pair $(i,j)\in\underline{n}\times\underline{n}$ will be a member of $D(A)\Leftrightarrow a_{ji}\neq\epsilon$, where $D(A)$ denotes the set of arcs of $G(A)$.

Definition 2.8. A path from node i to node j is a sequence of arcs $p=\{(i_k,j_k)\in D(A)\}_{k\in\underline{m}}$ such that $i=i_1, j_k=i_{k+1}$, for $k<m$ and $j_m=j$. The path p consists of the nodes $i=i_1,i_2,...,i_m,j_m=j$ with length m denoted by $\mid p\mid_1 = m$. In the case when $i=j$ the path is said to be a circuit. A circuit is said to be elementary if nodes i_k and i_l are different for $k\neq l$. A circuit consisting of one arc is called a self-loop.

Let us denote by $P(i,j;m)$ the set of all paths from node i to node j of length $m\geq 1$ and for any arc $(i,j)\in D(A)$ let its weight be given by a_{ij} then the weight of a path $p\in P(i,j;m)$ denoted by $\mid p\mid_w$ is defined to be the sum of the weights of all the arcs that belong to the path. The average weight of a path p is given by $\mid p\mid_w / \mid p\mid_1$. Given two paths, as for example, $p=((i_1,i_2),(i_2,i_3))$ and $q=((i_3,i_4),((i_4,i_5)$ in $G(A)$ the concatenation of paths $\circ : G(A)\times G(A)\to G(A)$ is defined as $p\circ q = ((i_1,i_2),(i_2,i_3),(i_3,i_4),(i_4,i_5))$. The communication graph $G(A)$ and powers of matrix A are closely related as it is shown in the next theorem.

Theorem 2.9. *Let* $A \in \mathbb{R}^{n\times n}_{max}$*, then* $\forall k \geq 1$*:* $[A^{\otimes k}]_{ji} = max\{\mid p \mid_w : p \in P(i,j;k)\}$*, where* $[A^{\otimes k}]_{ji} = \epsilon$ *in the case when* $P(i,j;k)$ *is empty i.e., no path of length* k *from node* i *to node* j *exists in* $G(A)$*.*

Definition 2.10. Let $A \in \mathbb{R}^{n\times n}_{max}$ then define the matrix $A^+ \in \mathbb{R}^{n\times n}_{max}$ as: $A^+ = \bigoplus_{k=1}^{\infty} A^{\otimes k}$. Where the element $[A^+]_{ji}$ gives the maximal weight of any path from j to i. If in addition one wants to add the possibility of staying at a node then one must include matrix E in the definition of matrix A^+ giving rise to its Kleene star representation defined by: $A^* = \bigoplus_{k=0}^{\infty} A^{\otimes k}$.

Lemma 2.11. *Let* $A \in \mathbb{R}^{n\times n}_{max}$ *be such that any circuit in* $G(A)$ *has average circuit weight less than or equal to* ϵ*. Then it holds that:* $A^* = \bigoplus_{k=0}^{n-1} A^{\otimes k}$*.*

Proof. [4]. □

Definition 2.12. Let $G = (N, D)$ be a graph and $i, j \in N$, node j is reachable from node i, denoted as iRj, if there exists a path from i to j. A graph G is said to be strongly connected if $\forall i, j \in N, jRi$. A matrix $A \in \mathbb{R}^{n\times n}_{max}$ is called irreducible if its communication graph is strongly connected, when this is not the case matrix A is called reducible.

Definition 2.13. Let $G = (N, D)$ be a not strongly connected graph and $i, j \in N$, node j communicates with node i, denoted as iCj, if either $i = j$ or iRj and jRi.

The relation iCj defines an equivalence relation in the set of nodes, and therefore a partition of N into a disjoint union of subsets, the equivalence classes, $N_1, N_2, ..., N_q$ such that $N = N_1 \cup N_2 \cup ... \cup N_q$ or $N = \bigcup_{i\in N} [i]$; $[i] = \{j \in N : iCj\}$.

Given the above partition, it is possible to focus on subgraphs of G denoted by $G_r = (N_r, D_r); r \in \underline{q}$ where D_r denotes the subset of arcs, which belong to D, that have both the begin node and end node in N_r. If $D_r \neq \varnothing$, the subgraph $G_r = (N_r, D_r)$ is known as a maximal strongly connected subgraph of G.

Definition 2.14. The reduced graph $\widetilde{G} = (\widetilde{N}, \widetilde{D})$ of G is defined by setting $\widetilde{N} = \{[i_1], [i_2], ... [i_q]\}$ and $([i_r], [i_s]) \in \widetilde{D}$ if $r \neq s$ and there exists an arc $(k, l) \in D$ for some $k \in [i_r]$ and $l \in [i_s]$.

Let A_{rr} denote the matrix by restricting A to the nodes in $[i_r]$ $\forall r \in \underline{q}$ i.e., $[A_{rr}]_{kl} = a_{kl}$ $\forall k, l \in [i_r]$. Then $\forall r \in \underline{q}$ either A_{rr} is irreducible or is equal to ϵ. Therefore since by construction the reduced graph does not contain any circuits, the original reducible matrix A after a possible relabeling of the nodes in $G(A)$, can be written as:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1q} \\ \mathcal{E} & A_{22} & \cdots & \cdots & A_{2q} \\ \mathcal{E} & \mathcal{E} & A_{33} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \cdots & \mathcal{E} & A_{qq} \end{pmatrix} \tag{2.1}$$

with matrices A_{sr} $1 \leq s < r \leq q$, where each finite entry in A_{sr} corresponds to an arc from a node in $[i_r]$ to a node in $[i_s]$.

Definition 2.15. Let $A \in \mathbb{R}^{n\times n}_{max}$ be a reducible matrix then, the block upper triangular given by (2.1) is said to be a normal form of matrix A.

2.2.1. **Spectral Theory.**

Definition 2.16. Let $A \in \mathbb{R}^{n\times n}_{max}$ be a matrix. If $\mu \in R_{max}$ is a scalar and $v \in R^n_{max}$ is a vector that contains at least one finite element such that: $A \otimes v = \mu \otimes v$ then, μ is called an eigenvalue and v an eigenvector.

Remark 2.17. Notice that the eigenvalue can be equal to ϵ and is not necessarily unique. Eigenvectors are certainly not unique indeed.

Let $C(A)$ denote the set of all elementary circuits in $G(A)$ and write: $\lambda = \max\limits_{p\in C(A)} \frac{|p|_w}{|p|_1}$ for the maximal average circuit weight. Notice that since $C(A)$ is a finite set, the maximum is attained (which is always the case when matrix A is irreducible). In case $C(A) = \emptyset$ define $\lambda = \epsilon$.

Definition 2.18. A circuit $p \in G(A)$ is said to be critical if its average weight is maximal. The critical graph of A, denoted by $G^c(A) = (N^c(A), D^c(A))$, is the graph consisting of those nodes and arcs that belong to critical circuits in $G(A)$.

Theorem 2.19. *If $A \in \mathbb{R}^{n\times n}_{max}$ is irreducible, then there exists one and only one finite eigenvalue (with possible several eigenvectors). This eigenvalue is equal to the maximal average weight of circuits in $G(A)$ $\lambda(A) = \max\limits_{p\in C(A)} \frac{|p|_w}{|p|_1}$*

Proof. [4]. □

2.2.2. **Linear Equations.**

Theorem 2.20. *Let $A \in \mathbb{R}^{n\times n}_{max}$ and $b \in \mathbb{R}^n_{max}$. If the communication graph $G(A)$ has maximal average circuit weight less than or equal to e, then $x = A^* \otimes b$ solves the equation $x = (A \otimes x) \oplus b$. Moreover, if the circuit weights in $G(a)$ are negative then, the solution is unique.*

Proof. ([4]). □

3. GENERALIZED EIGENMODE AND RECURRENCE EQUATIONS [2]

Definition 3.1. Let $A \in \mathbb{R}_{max}^{n\times n}$ be a regular matrix, a pair of vectors $(\eta, v) \in \mathbb{R}^n \times \mathbb{R}^n$ is called a generalized eigenmode of A if for all $k \geq 0$: $A \oplus (k\times\eta + v) = (k+1)\times\eta + v$

Remark 3.2. It is important to underline that the second vector v in a generalized eigenmode is not unique.

Theorem 3.3. *Consider the inhomogeneous recurrence equation*

$$x(k+1) = A \otimes x(k) \oplus \bigoplus_{j=1}^{m} B_j \otimes u_j(k),\ k \geq 0 \tag{3.1}$$

with $A \in \mathbb{R}_{max}^{n\times n}$ irreducible with eigenvalue $\lambda = \lambda(A)$, or $A \in \mathbb{R}_{max}$ $A = \epsilon$ with $\lambda = \epsilon$, $\{B_j\}_{j=1}^m \in \mathbb{R}_{max}^{n\times m_j}$ for some appropriate $m_j \geq 1$ matrices different from $\mathbb{E}$, $u_j(k) \in \mathbb{R}^{m_j}$ such that $u_j(k) = w_j(k) \otimes \tau_j^{\otimes k}$, $k \geq 0$, with $\tau_j \in \mathbb{R}$ and $w_j \in \mathbb{R}^{m_j}$. Denote $\tau = \bigoplus_{j\in \underline{m}} \tau_j$. Then, there exists an integer $K \geq 0$ and a vector $v \in \mathbb{R}^n$ such that the sequence $x(k) = v \otimes \mu^{\otimes k}$ with $\mu = \lambda \otimes \tau$ satisfies equation (3.1) for all $k \geq K$.

Proof. [4]. □

Remark 3.4. Notice that in theorem (3.3) equation (3.1) is satisfied for all $k \geq K$. However, in the case where it is possible to reinitialize the sequences $u_j(k) = w_j(k) \otimes \tau_j^{\otimes k}$, $k \geq 0$, by redefining the vectors w_j for $j \in \underline{m}$ then, it is possible to satisfy equation (3.1) $\forall k \geq 0$. Indeed, just set $v = v \otimes \mu^{\otimes K}, w_j(k) = w_j(k) \otimes \tau_j^{\otimes K}$, $j \in \underline{m}$. Then, the new sequences $x(k) = v \otimes \mu^{\otimes k}, u_j(k) = w_j(k) \otimes \tau_j^{\otimes k}$ $j \in \underline{m}$ solve our problem $\forall k \geq 0$.

Now, let us consider the recurrence equation: $x(k+1) = A \otimes x(k)$, $k \geq 0$ with A reducible and regular. Recalling what was presented in sub-section (2.2) (see also definition (2.15)), and using that matrix A is regular, it follows that matrix A can always be rewritten in its normal form (2.1) with the conditions that A_{qq} is irreducible, that for $i \in \underline{q-1}$ either A_{ii} is an irreducible matrix or is equal to ϵ, and that the A_{ij} matrices are different from $\mathcal{E}$ for $i, j = i+1; i \in \underline{q}$. Let the vector $x(k)$ be partitioned according to the normal form given by equation (2.1) as: $x(k) = (x_1(k), x_2(k), ..., x_q(k))$ where $x_i(k), i \in \underline{q}$ are vectors of suitable size. Therefore the recurrence equation given by equation $x(k+1) = A \otimes x(k)$, $k \geq 0$ can be written as:

$$x(k+1) = A_{ii} \otimes x_i(k) \oplus \bigoplus_{j=1+1}^{q} A_{ij} \otimes x_j(k), i \in \underline{q},\ k \geq 0 \tag{3.2}$$

Theorem 3.5. *Consider the recurrence equation given by equation (3.2). Assume that* A_{qq} *is irreducible and that for* $i \in \underline{q-1}$ *either* A_{ii} *is an irreducible matrix or is equal to* ϵ*. Assume also, that the* A_{ij} *matrices are different from* $\mathcal{E}$ *for* $i, j = i+1; i \in \underline{q}$*. Then, there exist finite vectors* $v_1, v_2, ..., v_q$ *of suitable size and scalars* $\xi_1, \xi_2, ..., \xi_q \in \mathbb{R}$ *such that the sequences:* $x_i(k) = v_i \otimes \xi_i^{\otimes k}, i \in \underline{q}$ *satisfy equation (3.2) for all* $k \geq 0$*. The scalars* $\xi_1, \xi_2, ..., \xi_q \in \mathbb{R}$ *are determined by:* $\xi_i = \bigoplus_{j \in H_i} \xi_j \oplus \lambda_i$*, where* $H_i = \{j \in \underline{q} : j > i, A_{ij} \neq \mathcal{E}\}$.

Proof. [4]. □

Corollary 3.6. *Let* $A \in \mathbb{R}_{max}^{n \times n}$ *be a reducible and regular matrix, then there exist a pair of vectors* $(\eta, v) \in \mathbb{R}^n \times \mathbb{R}^n$*, a generalized eigenmode, such that for all* $k \geq 0$*:* $A \oplus (k \times \eta + v) = (k+1) \times \eta + v$

Proof. [4]. □

The result provided by corollary (3.6) plays a fundamental role in the proposed algorithm for reducible matrices, as will be seen in the next section.

Definition 3.7. Let $A_m \in \mathbb{R}_{max}^{n \times n}$ for $0 \leq m \leq M$ and $x(m) \in \mathbb{R}_{max}^n$ for $-M \leq m \leq -1$; $M \geq 0$. Then, the recurrence equation: $x(k) = \bigoplus_{m=0}^{M} A_m \otimes x(k-m)$; $k \geq 0$ is called an Mth order recurrence equation.

Theorem 3.8. *The* M*th order recurrence equation, given by equation* $x(k) = \bigoplus_{m=0}^{M} A_m \otimes x(k-m)$*;* $k \geq 0$*, can be transformed into a first order recurrence equation* $x(k+1) = A \otimes x(k)$*;* $k \geq 0$ *provided that* A_0 *has circuit weights less than or equal to zero.*

Proof. Since by hypothesis, A_0 has circuit weights less than or equal to zero, lemma (2.11) allows A_0 to be written as $A_0^* = \bigoplus_{i=0}^{n-1} A_0^{\otimes i}$. Setting $b(k) = \bigoplus_{m=1}^{M} A_m \otimes x(k-m)$ the original equation reduces to $x(k) = A_0 \otimes x(k) \oplus b(k)$ which by theorem (2.20) can be rewritten as $x(k) = A_0^* \otimes b(k)$. Finally, defining $\hat{x}(k) = (x^T(k-1), x^T(k-2), ..., x^T(k-M))^T$ and,

$$\hat{A} = \begin{pmatrix} A_0^* \otimes A_1 & A_0^* \otimes A_2 & \cdots & \cdots & A_0^* \otimes A_M \\ E & \mathcal{E} & \cdots & \cdots & \mathcal{E} \\ \mathcal{E} & E & \ddots & & \mathcal{E} \\ \vdots & & \ddots & & \vdots \\ \mathcal{E} & \mathcal{E} & \cdots & E & \mathcal{E} \end{pmatrix}$$

we get that $\hat{x}(k+1) = \hat{A} \otimes \hat{x}(k)$; $k \geq 0$ as desired. □

4. AN ALGORITHM FOR COMPUTING GENERALIZED EIGENMODES OF REDUCIBLE MATRICES

This section proposes an algorithm for computing a generalized eigenmode for reducible matrices. The main idea of the algorithm was inspired by [2].

Algorithm: 1 Take $A \in \mathbb{R}^{n\times n}_{max}$ a reducible and regular matrix. **2** Using the material presented in (2.2) bring it to the normal form and write it in the form of system (3.2). **3** Consider the last equation of system (3.2) i.e., the nth equation, and compute its eigenvalue λ_n with associated eigenvector v_n, set $\xi_n = \lambda_n$ and $j = n$. **4** Consider the above next $(j-1)$th equation , and compute the eigenvalue of matrix $A_{(j-1)(j-1)}$, called it λ_{j-1}. **5** Is $\lambda_{j-1} > \xi_j$, if this is the case go to 6 if not, go to 7. **6** Set $\xi_{j-1} = \lambda_{j-1}$ and compute v_{j-1} according to the first case of the proof of theorem (3.3). Go to 8. **7** Set $\xi_{j-1} = \xi_j$ and compute v_{j-1} according to the second case of the proof of theorem (3.3). **8** Decrease j by one. Is $j \neq 1$ go back to 4 if not finish. At the end the algorithm provides one pair of vectors $\eta = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, $v = (v_1, v_2, ..., v_q) \in \mathbb{R}^n$ which result to be a generalized eigenmode of matrix $A \in \mathbb{R}^{n\times n}_{max}$.

Remark 4.1. Theorem (2.19) can be used for computing the eigenvalues of the irreducible matrices $\{A_{ii}\,;\ i \in \underline{n}\}$. In addition, the power algorithm ([2]) results of great help for computing the eigenvector in case it comes from the solution of equation $A \otimes v = \mu \otimes v$.

5. MAX-PLUS RECURRENCE EQUATIONS FOR TIMED EVENT PETRI NETS [1, 2, 3]

Definition 5.1. A Petri net is a 5-tuple, $PN = \{P, T, F, W, M_0\}$ where: $P = \{p_1, p_2, ..., p_m\}$is a finite set of places, $T = \{t_1, t_2, ..., t_n\}$ is a finite set of transitions, $F \subset (P \times T) \cup (T \times P)$ is a set of arcs, $W : F \to N_1^+$ is a weight function, M_0: $P \to N$ is the initial marking, $P \cap T = \varnothing$ and $P \cup T \neq \varnothing$.

A Petri net structure without any specific initial marking is denoted by N. A Petri net with the given initial marking is denoted by (N, M_0). Notice that if $W(p, t) = \alpha$ (or $W(t, p) = \beta$) then, this is often represented graphically by α, (β) arcs from p to t (t to p) each with no numeric label. A Petri net is called an event Petri net when every $p_i \in P$ has one input and one output transition. Let $M_k(p_i)$ denote the marking (i.e., the number of tokens) at place $p_i \in P$ at time k and let $M_k = [M_k(p_1), ..., M_k(p_m)]^T$ denote the marking (state) of PN at time k. A transition $t_j \in T$ is said to be enabled at time k if $M_k(p_i) \geq W(p_i, t_j)$ for all $p_i \in P$ such that $(p_i, t_j) \in F$. It is assumed that at each time k there exists at least one transition to fire. If a transition is enabled then, it can fire. If an enabled transition $t_j \in T$ fires at time k then, the next marking for $p_i \in P$ is given by $M_{k+1}(p_i)$.

Definition 5.2. The clock structure associated with a place $p_i \in P$ is a set $\mathbf{V} = \{V_i : p_i \in P\ \}$ of clock sequences $V_i = \{v_{i,1}, v_{i,2}, ...\}$, $v_{i,k} \in R^+$, $k = 1, 2, ...$

The positive number $v_{i,k}$, associated to $p_i \in P$, called holding time, represents the time that a token must spend in this place until its outputs enabled transitions $v_{i,1}, v_{i,2}, ...$, fire. Some places may have a zero holding time while others not. Thus, we partition P into subsets P_0 and P_h, where P_0 is the set of places with zero holding time, and P_h is the set of places that have some holding time.

Definition 5.3. A timed Petri net is a 6-tuple $TPN = \{P, T, F, W, M_0, \mathbf{V}\}$ where $\{P, T, F, W, M_0\}$ are as before, and $\mathbf{V} = \{V_i : p_i \in P\ \}$ is a clock structure. A timed Petri net is a timed event petri net when every $p_i \in P$ has one input and one output transition, in which case the associated clock structure set of a place $p_i \in P$ reduces to one element $V_i = \{v_i\}$

With any timed event Petri net, matrices $A_0, A_1, ..., A_M \in \mathbb{N}^n \times \mathbb{N}^n$ can be defined by setting $[A_m]_{jl} = a_{jl}$, where a_{jl} is the largest of the holding times with respect to all places between transitions t_l and t_j with m tokens, for $m = 0, 1, ..., M$, with M equal to the maximum number of tokens with respect to all places. Let $x_i(k)$ denote the kth time that transition t_i fires, then the vector $x(k) = (x_1(k), x_2(k), ...x_m(k))^T$, called the state of the system, satisfies the Mth order recurrence equation: $x(k) = \bigoplus_{m=0}^{M} A_m \otimes x(k-m);\ k \geq 0$ Now, assuming that all the hypothesis of theorem (3.8) are satisfied, and setting $\hat{x}(k) = (x^T(k), x^T(k-1), ..., x^T(k-M+1))^T$, equation $x(k) = \bigoplus_{m=0}^{M} A_m \otimes x(k-m);\ k \geq 0$ can be expressed as: $\hat{x}(k+1) = \hat{A} \otimes \hat{x}(k);\ k \geq 0$, which is known as the standard autonomous equation.

6. THE HELICOPTER MAINTENANCE PROCESS

Aerospace Ltd is an International owned company which gives maintenance service to broken helicopters due to engine failure. A faulty engine, which has to repaired, is first removed from the helicopter by the Engine Replacement Team (ERT). The ERT work is done at our headquarters located at the Airport, taking 6 hours to remove a faulty engine, and then sending it to the company's repair shop located 6 hours away from the airport where the Engine Repair Team restore the faulty engine. The time taken to repair a faulty engine is in the worst possible scenario of 8 hours. Once a faulty engine has been repaired is sent back to our headquarters where the ERT fits it back. This takes another 6 hours. The characteristics of the helicopter maintenance process are provided by the state transition dynamics summarized in table 1.

Table 1. State Transition Dynamics Table

Origin	Destination	Travel to Dest.	Waiting at Dest.	Depart
HQ	ERT	0	6	3
ERT	Repair	6	8	9
Repair	HQ	6	6	0

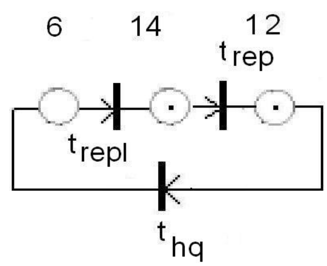

Figure 1. Petri net model

The timed event Petri net that models the state transition dynamics is shown in fig 1. The number attached to each place is its associated holding time H, which is equal to the sum of the travel time to destination plus the waiting time at the destination. The initial marking, or token distribution, at each place is computed in such a way that the corresponding timed Petri net can be executed with cycle time T, which in this case is equal to 24 hours and it is determined by the next formula: $M_0(p_i) = \lceil \frac{D_i - D_J + H_{p_i}}{T} \rceil$, where D_i represents the departure time at the origin station, D_j represents the departure time at the destination station, and H_{p_i} is the holding time associated to place p_i. From the timed Petri net model we obtain that: $A_0 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ 6 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$ and $A_1 = \begin{pmatrix} \varepsilon & \varepsilon & 12 \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 14 & \varepsilon \end{pmatrix}$ and making the required computations that: $A_0^* = \begin{pmatrix} 0 & \varepsilon & \varepsilon \\ 6 & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \end{pmatrix}$, which leads to: $\hat{A} = A_0^* \otimes A_1 = \begin{pmatrix} \varepsilon & \varepsilon & 12 \\ \varepsilon & \varepsilon & 18 \\ \varepsilon & 14 & \varepsilon \end{pmatrix}$, which is already in its normal form, with $A_{11} = \epsilon$, and $A_{22} = \begin{pmatrix} \varepsilon & 18 \\ 14 & \varepsilon \end{pmatrix}$. From A_{22} we get that $\lambda_2 = 16 = \xi_2$ and doing algebra that $v_2 = (7, 5)$. Now, since $A_{11} = \epsilon$ this implies that $\lambda_1 = 1 \leq \xi_2$ therefore $\xi_1 = \xi_2 = 16$ and $v_1 = 1$ is obtained as the solution of $12 \otimes v_{2_2} = 16 \otimes v_1$. Therefore, the pair $\eta = (16, 16, 16)$, $v = (1, 7, 5)$ results to be a generalized eigenmode, which describes the process and since it satisfies equation $\hat{x}(k+1) = \hat{A} \otimes \hat{x}(k)$; $k \geq 0$, it provides a possible timetable given by: $x(k) = k \times [16, 16, 16]^T + [1, 7, 5]^T, k \geq 0$. Moreover, since the maximum numerical value attained by the elements that form vector η, which in this case is 16, determines the highest frequency at which the process operates (or in other words the slowest one) is less than the cycle time $T = 24$, we can conclude that the process works properly.

7. CONCLUSIONS

This work gives and applies an algorithm for computing a generalized eigenmode of reducible regular matrices over the max-plus algebra to a helicopter maintenance process timed event Petri net model. Given a reducible regular matrix, the first step consists in, transforming it into its normal form. Once this has been done the following steps are constructed based on an explicit mathematical characterization, which comes out to be a consequence of considering a very specific recurrence equation. Finally, applying the algorithm a timetable for the helicopter maintenance process was obtained.

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