Neural, Parallel, and Scientific Computations 18 (2010) 75-88

HIGHER ORDER NUMERICAL METHODS FOR SINGULARLY PERTURBED ELLIPTIC PROBLEMS

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ABSTRACT. We consider a family of singularly perturbed elliptic problems in two dimensions. A novel fitted operator finite difference method developed is proposed to solve this problems. Through a rigorous convergence analysis, we show that the method is second order convergent in both variables. Further attempts are made to improve the order of convergence via some convergence acceleration techniques, namely the Richardson extrapolation. In turn, we achieve fourth order accurate results. Error analysis after extrapolation is also presented. Furthermore, some numerical results confirming the theoretical estimates are provided. We also compare our results with those obtained in the literature (see, e.g., [R. Lin, Discontinuous discretization for least-squares formulation of singularly perturbed reaction-diffusion problems in one and two dimensions, *SIAM J. Numer. Anal.* **47(1)** 89–108.] and noticed that the error obtained by our approach is exponentially smaller than the one obtained by their approach.

Key Words Singular perturbations; elliptic partial differential equation, boundary layers; fitted operator finite difference methods; convergence analysis; error bounds; Richardson extrapolation.

AMS Subject Classification (2000): 35B25, 65B05, 65N06, 65N12, 65N15.

1. INTRODUCTION

We consider the problem

$$Lu := -\varepsilon\Delta u + b(x,y)u = f(x,y), \quad \text{in } \Omega = (0,1)\times(0,1), \tag{1.1}$$

$$u = 0, \quad \text{on } \partial\Omega. \tag{1.2}$$

where $\varepsilon \in (0,1]$ and b and f are sufficiently smooth functions in Ω. It is assumed that $b(x,y) \geq \alpha^2 > 0$, in Ω. Also, we impose the following compatibility conditions [11, 13] which guarantee that the solution $u(x,y)$ to problem (1.1)–(1.2) is a member of $C^4(\Omega)\cap C^2(\overline{\Omega})$, where $\overline{\Omega} = \Omega \cup \partial\Omega$:

$$f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0.$$

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Received September 13, 2009 1061-5369 $15.00

While singularly perturbed two-point boundary value problems are well studied from different angles, their higher dimensional counterparts are not tackled sufficiently. There were some attempts made to extend the approaches developed for singularly perturbed ordinary differential equation but the success was very limited. On the other hand, some researchers tried to solve these higher dimensional problems directly, for example, Li [6] proposed a quasi-optimal finite element method, Lin [8] solved the above problem using Local Discontinuous Galerkin method via least-squares formulation, O'Riordan and Stynes [11] designed a globally convergent finite element method, and so on. Some other notable works include [2, 4, 9, 13, 14, 15, 16].

A careful reading of the work by Kadalbajoo and Patidar [5] indicates that there are no extensions of any fitted operator methods developed for singularly perturbed ODEs that can solve the singularly perturbed PDEs, in particular the elliptic ones. To fill this gap, the first aim of this paper is to extend a FOFDM (which is developed for singularly perturbed ODEs) to solve the elliptic singular perturbation problem. Then, in order to achieve a higher order convergence, we perform the Richardson extrapolation.

The rest of this paper is organized as follows. In Section 2, we presents some qualitative features of the solution and its derivatives. Section 3 is concerned with the construction and analysis of the numerical method. Section 4 deals with the extrapolation of the method developed in Section 3. Numerical results to support the theory are provided in Section 5. Some concluding remarks are provided in Section 6.

2. BOUNDS ON THE SOLUTION AND ITS DERIVATIVES

Lemma 2.1 ([3] Continuous maximum principle)**.** *Let $\xi(x,y)$ be any sufficiently smooth function such that $\xi(x,y) \geq 0$ on $\partial\Omega$. Then $L\xi(x,y) \geq 0$ on Ω implies that $\xi(x,y) \geq 0$, $\forall (x,y) \in \overline{\Omega} = \partial\Omega \cup \Omega$.*

Proof. Let (x^*, y^*) be such that

$$\xi(x^*, y^*) = \min_{(x,y)\in\overline{\Omega}} \xi(x,y)$$

and assume that $\xi(x^*, y^*) < 0$. Clearly, $(x^*, y^*) \notin \partial\Omega$. We have

$$\frac{\partial}{\partial x}\xi(x,y)\Big|_{(x^*,y^*)} = 0,$$

$$\frac{\partial}{\partial y}\xi(x,y)\Big|_{(x^*,y^*)} = 0,$$

$$\frac{\partial^2}{\partial x^2}\xi(x,y)\Big|_{(x^*,y^*)} \geq 0$$

and

$$\frac{\partial^2}{\partial y^2}\xi(x,y)\Big|_{(x^*,y^*)} \geq 0.$$

Therefore,

$$L\xi(x^*, y^*) = -\varepsilon\Delta\xi(x^*, y^*) + b(x^*, y^*)\xi(x^*, y^*) < 0,$$

which is a contradiction. □

The following lemmas provide bounds on the solution of the problem (1.1)–(1.1) as well as those of its derivatives [7]. A suitable choice of barrier functions [6, 17] may be made in the proofs. One may also note in the following that C will denote a generic positive constant which is independent of ε.

Lemma 2.2. *Let $u(x, y)$ be the solution of problem* (1.1)–(1.2)*. Then we have*

(a). $|u(x, y)| \le C\left(1 - e^{-\alpha x/\sqrt{\varepsilon}}\right)$ on $\bar{\Omega}$,
(b). $|u(x, y)| \le C\left(1 - e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right)$ on $\bar{\Omega}$,
(c). $|u(x, y)| \le C\left(1 - e^{-\alpha y/\sqrt{\varepsilon}}\right)$ on $\bar{\Omega}$,
(d). $|u(x, y)| \le C\left(1 - e^{-\alpha(1-y)/\sqrt{\varepsilon}}\right)$ on $\bar{\Omega}$.

Proof. (a). Using the barrier function

$$\phi(x, y) = C(1 - e^{(-\alpha x/\sqrt{\varepsilon})}),$$

we see that

$$\begin{aligned} L(\phi \pm u) &= -\varepsilon\Delta(\phi \pm u) + b(\phi \pm u), \\ &= C\alpha^2 e^{(-\alpha x/\sqrt{\varepsilon})} + bC(1 - e^{(-\alpha x/\sqrt{\varepsilon})}) \pm f, \\ &= C(\alpha^2 - b)\left(e^{(-\alpha x/\sqrt{\varepsilon})} - 1\right) + C\alpha^2 \pm f. \end{aligned}$$

Since

$$(\alpha^2 - b)\left(e^{(-\alpha x/\sqrt{\varepsilon})} - 1\right) \ge 0,$$

we have

$$L(\phi \pm u) \ge C\alpha^2 \pm f \ge 0.$$

Using the maximum principle (Lemma 2.1) and the fact that $(\phi \pm u)|_{\partial\Omega} \ge 0$, we get $|u| \le \phi$.

The proof of part (b), (c) and (d) is done in a similar way by choosing the barrier functions

$$\phi(x, y) = \left(1 - e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right),$$

$$\phi(x, y) = \left(1 - e^{-\alpha y/\sqrt{\varepsilon}}\right)$$

and

$$\phi(x, y) = \left(1 - e^{-\alpha(1-y)/\sqrt{\varepsilon}}\right),$$

respectively. □

Now we have

Lemma 2.3. *Let $u(x, y)$ be the solution of problem* (1.1)–(1.2)*. Then*

(a). $|u_x(x,y)| \le C\varepsilon^{-1/2}$ *on* $\partial\Omega$,
(b). $|u_y(x,y)| \le C\varepsilon^{-1/2}$ *on* $\partial\Omega$.

Proof. Using Lemma 2.2, we have

$$|u_x(0,y)| = \left|\lim_{x\to 0^+} \frac{u(x,y)-u(0,y)}{x}\right| \le \lim_{x\to 0^+} \frac{C(1-e^{(-\alpha x/\sqrt{\varepsilon})})}{x} = C\frac{\alpha}{\varepsilon^{1/2}} \le C\varepsilon^{-1/2}.$$

Applying the estimate in part (b) of Lemma 2.2, we get the estimate for $u_x(1,y)$. Differentiating the given boundary conditions $u(x,y)=0$ at $y=0$ and $y=1$ with respect to x gives us $u_x(x,0)=u_x(x,1)=0$. Similarly,

$$|u_y(x,0)| = \left|\lim_{y\to 0^+} \frac{u(x,y)-u(x,0)}{y}\right| \le \lim_{y\to 0^+} \frac{C(1-e^{(-\alpha y/\sqrt{\varepsilon})})}{y} \le C\varepsilon^{-1/2}.$$

We get the estimate of $u_y(x,1)$ by applying the estimate in part (d) of Lemma 2.2. Differentiating the given boundary conditions $u(x,y)=0$ at $x=0$ and $x=1$ with respect to y we get $u_y(0,y)=u_y(1,y)=0$. This completes the proof. □

Lemma 2.4. *Let* $u(x,y)$ *be the solution of problem* (1.1)–(1.2). *Then we have*

(a). $|u_x(x,y)| \le C\left(1-\varepsilon^{-1/2}e^{-\alpha x/\sqrt{\varepsilon}}+\varepsilon^{-1/2}e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right)$ *on bar*Ω,
(b). $|u_y(x,y)| \le C\left(1-\varepsilon^{-1/2}e^{-\alpha y/\sqrt{\varepsilon}}+\varepsilon^{-1/2}e^{-\alpha(1-y)/\sqrt{\varepsilon}}\right)$ *on* $\bar{\Omega}$.

Proof. By choosing the barrier function

$$\phi(x,y) = C\left(1-\varepsilon^{-1/2}e^{-\alpha x/\sqrt{\varepsilon}}+\varepsilon^{-1/2}e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right),$$

we obtain

$$L(\phi \pm u_x) \ge bC \pm (f_x - b_x u) \ge 0,$$

and since $(\phi \pm u_x)|_{\partial\Omega} \ge 0$, the proof is completed by making use of the maximum principle (Lemma 2.1).

The proof for the estimate in part (b) can be constructed analogously using the barrier function

$$\phi(x,y) = C\left(1-\varepsilon^{-1/2}e^{-\alpha y/\sqrt{\varepsilon}}+\varepsilon^{-1/2}e^{-\alpha(1-y)/\sqrt{\varepsilon}}\right).$$

□

Now, the following results for the bounds on the second derivatives hold:

Lemma 2.5. *Let* $u(x,y)$ *be the solution of problem* (1.1)–(1.2). *Then we have*

(a). $|u_{xx}(x,y)| \le C\varepsilon^{-1}$ *on* $\partial\Omega$,
(b). $|u_{yy}(x,y)| \le C\varepsilon^{-1}$ *on* $\partial\Omega$.
(c). $|u_{xx}(x,y)| \le C\left(1+\varepsilon^{-1}e^{-\alpha x/\sqrt{\varepsilon}}+\varepsilon^{-1}e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right)$ *on* $\bar{\Omega}$,
(d). $|u_{yy}(x,y)| \le C\left(1+\varepsilon^{-1}e^{-\alpha y/\sqrt{\varepsilon}}+\varepsilon^{-1}e^{-\alpha(1-y)/\sqrt{\varepsilon}}\right)$ *on* $\bar{\Omega}$.

Proof. (a). At $y = 0$ and $y = 1$, we have $u(x, y) = 0$. Therefore, $u_{xx} = 0$ at $y = 0$ and $y = 1$. Also, the fact that $u = u_{yy} = 0$ at $x = 0$ and $x = 1$ leads to $u_{xx} = 0$ at $x = 0$ and $x = 1$.
(b). The proof can follow similar lines as in part (a).
(c). Using the barrier function

$$\phi(x, y) = C\left(1 + \varepsilon^{-1}e^{-\alpha x/\sqrt{\varepsilon}} + \varepsilon^{-1}e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right),$$

we see that

$$\begin{aligned} L(\phi \pm u_{xx}) &= bC + C(b - \alpha^2)\left(\varepsilon^{-1}e^{-\alpha x/\sqrt{\varepsilon}} + \varepsilon^{-1}e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right) \pm (-\varepsilon\Delta u_{xx} + bu_{xx}) \\ &= bC + C(b - \alpha^2)\left(\varepsilon^{-1}e^{-\alpha x/\sqrt{\varepsilon}} + \varepsilon^{-1}e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right) \\ &\quad \pm (f_{xx} - 2b_x u_x - b_{xx}u) \\ &\geq bC \pm (f_{xx} - 2b_x u_x - b_{xx}u). \end{aligned}$$

It follows that, for C sufficiently large, $L(\phi \pm u_{xx}) \geq 0$. Since $(\phi \pm u_{xx})|_{\partial\Omega} \geq 0$, the continuous maximum principle (Lemma 2.1) concludes the proof.
(d). The proof follows the same lines as in part (c) with the barrier function

$$\phi(x, y) = C\left(1 + \varepsilon^{-1}e^{-\alpha y/\sqrt{\varepsilon}} + \varepsilon^{-1}e^{-\alpha(1-y)/\sqrt{\varepsilon}}\right).$$

□

3. CONSTRUCTION AND ANALYSIS OF THE FITTED OPERATOR FINITE DIFFERENCE METHOD

Let n and m be positive integers.

We consider the following partitions of the interval $[0, 1]$:

$$x_0 = 0, \quad x_i = x_0 + ih, \quad i = 1(1)n, \quad h = x_i - x_{i-1}, \quad x_n = 1.$$

$$y_0 = 0, \quad y_j = y_0 + jk, \quad j = 1(1)m, \quad k = y_j - y_{j-1}, \quad y_m = 1.$$

The tensor product of these two partitions gives the mesh grid

$$\mu_{(n,m)} = \{(x_i, y_j), \quad i = 0(1)n, \quad j = 0(1)m\}.$$

In the rest of this paper, we adopt the notation $W_i^j = W(x_i, y_j)$ and denote the approximations of the u_i^j at the grid points (x_i, y_j) by the unknowns v_i^j.

Using the theory of difference equations for problems in one dimension, we construct the following FOFDM (looking at one dimension at a time):

$$-\varepsilon\left[\frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\phi_i^j)_h^2} + \frac{v_i^{j+1} - 2v_i^j + v_i^{j-1}}{(\phi_i^j)_k^2}\right] + b_i^j v_i^j = f_i^j, \tag{3.1}$$

with the discrete boundary conditions

$$v_i^0 = v_0^j = v_i^m = v_n^j = 0 \quad i = 0(1)n, \quad j = 0(1)m, \tag{3.2}$$

where

$$(\phi_i^j)_h \equiv \phi_i^j(h,\varepsilon) := \frac{2}{\rho_i^j}\sinh\left(\frac{\rho_i^j h}{2}\right) \tag{3.3}$$

and

$$(\phi_i^j)_k \equiv \phi_i^j(k,\varepsilon) := \frac{2}{\rho_i^j}\sinh\left(\frac{\rho_i^j k}{2}\right), \tag{3.4}$$

with $\rho_i^j = \sqrt{b_i^j/\varepsilon}$.

Note that

$$\phi_i^j(h,\varepsilon) = h + \mathcal{O}\left(\frac{h^3}{\varepsilon}\right), \quad \text{and} \quad \phi_i^j(k,\varepsilon) = k + \mathcal{O}\left(\frac{k^3}{\varepsilon}\right).$$

For the sake of simplicity, we assume that $h = k$, and hence the common denominator will be $(\phi_i^j)^2 (= (\phi_i^j)_h^2) = (\phi_i^j)_k^2)$. Thus equation (3.1) becomes

$$-\varepsilon\left[\frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\phi_i^j)^2} + \frac{v_i^{j+1} - 2v_i^j + v_i^{j-1}}{(\phi_i^j)^2}\right] + b_i^j v_i^j = f_i^j, \tag{3.5}$$

which we rewrite as

$$-\frac{\varepsilon}{(\phi_i^j)^2}\left[v_{i+1}^j + v_{i-1}^j + v_i^{j+1} + v_i^{j-1} - 4v_i^j\right] + b_i^j v_i^j = f_i^j. \tag{3.6}$$

The method consisting of (3.6) along with (3.2) is termed as the fitted operator finite difference method (FOFDM).

One should note that, in the above we have considered $h = k$ merely for the sake of simplicity. However, in the analysis below, we keep the general set up.

In the discussion below, M may denote a different positive constant but is always independent of ε and the step-sizes h and k.

Following lemmas play a primordial role in the analysis of the method developed above.

Lemma 3.1 (Discrete maximum principle)**.** *Let $\{\xi_i^j\}$ be any mesh function satisfying $\xi_i^0 \geq 0$, $\xi_i^m \geq 0$, $i = 1(1)n-1$; $\xi_0^j \geq 0$, $\xi_n^j \geq 0$, $i = 1(1)m-1$; $\xi_0^0 \geq 0$, $\xi_n^0 \geq 0$, $\xi_0^m \geq 0$, $\xi_m^n \geq 0$ and $L_h^k \xi_i^j \geq 0$, $i = 1(1)n-1$; $j = 1(1)m-1$. Then $\xi_i^j \geq 0$, $\forall i = 0(1)n$, $j = 0(1)m$.*

Proof. Let (s,t) be indices such that

$$\xi_s^t = \min_{(i,j)} \xi_i^j, \quad \forall \ (i,j) \in \{0,1,\ldots,n\} \times \{0,1,\ldots,m\}.$$

Assume that $\xi_s^t < 0$. It is clear that

$$(s,t) \in \{1,2,\ldots,n-1\} \times \{1,2,\ldots,m-1\},$$

or else, $\xi_s^t \geq 0$.

We observe that

$$\xi_{s+1}^t - \xi_s^t > 0, \quad \xi_{s-1}^t - \xi_s^t > 0, \quad \xi_s^{t+1} - \xi_s^t > 0, \quad \text{and} \quad \xi_s^t - \xi_s^{t-1} > 0.$$

Therefore

$$L_h^k \xi_s^t < 0,$$

which is a contradiction. □

Lemma 3.2. *If Z_i^j is any mesh function such that $Z_i^j = 0$ on $(\partial\Omega)_i^j$, then there exists a constant C such that*

$$|Z_l^s| \le \frac{1}{\alpha^2} \max_{1\le i\le n-1;1\le j\le m-1} |L_h^k Z_i^j|, \quad \text{for } 0 \le l \le n;\ 0 \le s \le m.$$

Proof. Let

$$M = \frac{1}{\alpha^2} \max_{1\le i\le n-1;1\le j\le m-1} |L_h^k Z_i^j|$$

and $(\Psi^{\pm})_i^j$ be the mesh function defined by

$$(\Psi^{\pm})_i^j = M \pm Z_i^j.$$

It is clear that $(\Psi^{\pm})_i^0 = (\Psi^{\pm})_i^m = (\Psi^{\pm})_0^j = (\Psi^{\pm})_n^j = M > 0$. Also, for $1 \le i \le n-1$ and $1 \le j \le m-1$, we have

$$\begin{aligned}
L_h^k(\Psi^{\pm})_i^j &= -\varepsilon \left[\frac{M \pm Z_{i+1}^j - 2(M \pm Z_i^j) + M \pm Z_{i-1}^j}{(\Phi_i^j)_h^2} \right. \\
&\quad \left. + \frac{M \pm Z_i^{j+1} - 2(M \pm Z_i^j) + M \pm Z_i^{j-1}}{(\Phi_i^j)_k^2} \right] + b_i^j (M \pm Z_i^j) \\
&= M b_i^j \pm L_h^k Z_i^j \\
&= \frac{b_i^j}{\alpha^2} \max |L_h^k Z_i^j| \pm L_h^k Z_i^j.
\end{aligned}$$

Since $b_i^j \ge \alpha^2$, we have

$$L_h^k(\Psi^{\pm})_i^j \ge 0.$$

Then, by the discrete maximimum principle (Lemma 3.1), we obtain

$$(\Psi^{\pm})_i^j \ge 0 \quad \text{for } 0 \le i \le n, \quad 0 \le j \le m.$$

□

3.1. **Error estimate before extrapolation.** From (3.1), we see that the local truncation error of the FOFDM is

$$\begin{aligned} L_h^k(u_i^j - v_i^j) &= \left\{-\varepsilon(\Delta u)_i^j + b_i^j u_i^j\right\} \\ &\quad - \left\{-\varepsilon\left[\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\phi_i^j)_h^2} + \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\phi_i^j)_k^2}\right] + b_i^j u_i^j\right\} \\ &= -\varepsilon(u_{xx})_i^j - \varepsilon(u_{yy})_i^j + \frac{\varepsilon}{(\phi_i^j)_h^2}\left[h^2(u_{xx})_i^j + \frac{h^4}{12}(u_{xxxx})_i^j + \cdots\right] \\ &\quad + \frac{\varepsilon}{(\phi_i^j)_k^2}\left[k^2(u_{yy})_i^j + \frac{k^4}{12}(u_{yyyy})_i^j + \cdots\right] \\ &= -\varepsilon(u_{xx})_i^j - \varepsilon(u_{yy})_i^j \\ &\quad + \left(\frac{\varepsilon}{h^2} - \frac{b_i^j}{12} + \frac{h^2(b_i^j)^2}{240\varepsilon} + \cdots\right)\left[h^2(u_{xx})_i^j + \frac{h^4}{12}(u_{xxxx})_i^j + \cdots\right] \\ &\quad + \left(\frac{\varepsilon}{k^2} - \frac{b_i^j}{12} + \frac{k^2(b_i^j)^2}{240\varepsilon} + \cdots\right)\left[h^2(u_{yy})_i^j + \frac{k^4}{12}(u_{yyyy})_i^j \cdots\right]. \end{aligned}$$

This implies that

$$\begin{aligned} L_h^k(u_i^j - v_i^j) &= \frac{\varepsilon h^2}{12}(u_{xxxx})_i^j - \frac{h^2(b_i^j)^2}{12}(u_{xx})_i^j - \frac{h^4(b_i^j)}{144}(u_{xxxx})_i^j + \frac{h^4(b_i^j)^2}{240\varepsilon}(u_{xx})_i^j \\ &\quad + \frac{\varepsilon k^2}{12}(u_{yyyy})_i^j - \frac{k^2(b_i^j)^2}{12}(u_{yy})_i^j - \frac{k^4(b_i^j)}{144}(u_{yyyy})_i^j + \frac{k^4(b_i^j)^2}{240\varepsilon}(u_{yy})_i^j + \cdots \end{aligned} \tag{3.7}$$

Using Lemma 2.5, we obtain

$$|L_h^k(u_i^j - v_i^j)| \leq M\left[h^2\left(1 + \frac{h^2}{\varepsilon}\right) + k^2\left(1 + \frac{k^2}{\varepsilon}\right)\right].$$

Then by Lemma 3.2, we have

$$\max_{0\leq i\leq n}\max_{0\leq j\leq m}|u_i^j - v_i^j| \leq M\left[h^2\left(1 + \frac{h^2}{\varepsilon}\right) + k^2\left(1 + \frac{k^2}{\varepsilon}\right)\right]. \tag{3.8}$$

Note that, if $h = k$, then we have the estimate

$$\max_{0\leq i\leq n}\max_{0\leq j\leq m}|u_i^j - v_i^j| \leq Mh^2\left(1 + \frac{h^2}{\varepsilon}\right). \tag{3.9}$$

4. EXTRAPOLATION ON THE FITTED OPERATOR FINITE DIFFERENCE METHOD

4.1. **Extrapolation formula.** Let $\mu_{(2n,2m)} = \{(\bar{x}_i, \bar{y}_j)\}$ be the mesh with $\bar{x}_0 = 0$, $\bar{x}_n = 1$, $\bar{y}_0 = 0$, $\bar{y}_m = 1$, and $\bar{x}_i - \bar{x}_{i-1} = \bar{h} = h/2$, $i = 1(1)2n$, and $\bar{y}_j - \bar{y}_{j-1} = \bar{k} = k/2$, $j = 1(1)2m$, and $\bar{v}_i^j$ denote the numerical solution computed on the mesh $\mu_{(2n,2m)}$.

On one hand, we have from (3.8),

$$u_i^j - v_i^j = M\left[h^2\left(1 + \frac{h^2}{\varepsilon}\right) + k^2\left(1 + \frac{k^2}{\varepsilon}\right)\right] + R_n^m(x_i, y_j),$$

$$1 \leq i \leq n-1, \quad 1 \leq j \leq m-1.$$

On the other hand, we have

$$\bar{u}_i^j - \bar{v}_i^j = M\left[\bar{h}^2\left(1+\frac{\bar{h}^2}{\varepsilon}\right) + \bar{k}^2\left(1+\frac{\bar{k}^2}{\varepsilon}\right)\right] + R_{2n}^{2m}(x_i, y_j),$$

$$1 \leq i \leq 2n-1, \quad 1 \leq j \leq 2m-1.$$

In the above expressions, both remainders R_n^m and R_{2n}^{2m} are $\mathcal{O}(h^4+k^4)$. It follows that

$$4(\bar{u}_i^j - \bar{v}_i^j) - (u_i^j - v_i^j) = 4R_{2n}^{2m}(x_i, y_j) - R_n^m(x_i, y_j) = \mathcal{O}(h^4+k^4), \quad (x_i, y_j) \in \mu_{(n,m)}.$$

Hence,

$$u_i^j - \frac{4\bar{v}_i^j - v_i^j}{3} = \mathcal{O}(h^4+k^4), \quad \forall (x_i, y_j) \in \mu_{(n,m)}.$$

We therefore set

$$(v_i^j)^{ext} := \frac{4\bar{v}_i^j - v_i^j}{3}$$

as the numerical approximation of u after extrapolation at the grid point (x_i, y_j).

4.2. **Analysis of the extrapolation process.** The local truncation error after extrapolation is

$$\bar{L}_h^k\left(u_i^j - (v_i^j)^{ext}\right) = \frac{4}{3}L_{\bar{h}}^{\bar{k}}(u_i^j - \bar{v}_i^j) - \frac{1}{3}L_h^k(u_i^j - v_i^j). \tag{4.1}$$

While $L_h^k(u_i^j - v_i^j)$ is given by equation (3.7), $L_{\bar{h}}^{\bar{k}}(u_i^j - \bar{v}_i^j)$ is obtained from $L_h^k(u_i^j - v_i^j)$ by substituting h and k by $\bar{h}$ and $\bar{k}$, respectively. It follows that

$$\begin{aligned}
\bar{L}_h^k\left(u_i^j - (v_i^j)^{ext}\right) = \frac{4}{3}\Bigg[&\frac{\varepsilon\bar{h}^2}{12}(u_{xxxx})_i^j - \frac{\bar{h}^2(b_i^j)^2}{12}(u_{xx})_i^j - \frac{\bar{h}^4(b_i^j)}{144}(u_{xxxx})_i^j \\
&+\frac{\bar{h}^4(b_i^j)^2}{240\varepsilon}(u_{xx})_i^j + \frac{\varepsilon\bar{k}^2}{12}(u_{yyyy})_i^j - \frac{\bar{k}^2(b_i^j)^2}{12}(u_{yy})_i^j \\
&-\frac{\bar{k}^4(b_i^j)}{144}(u_{yyyy})_i^j + \frac{\bar{k}^4(b_i^j)^2}{240\varepsilon}(u_{yy})_i^j + \cdots\Bigg] \\
&-\frac{1}{3}\Bigg[\frac{\varepsilon h^2}{12}(u_{xxxx})_i^j - \frac{h^2(b_i^j)^2}{12}(u_{xx})_i^j - \frac{h^4(b_i^j)}{144}(u_{xxxx})_i^j \\
&+\frac{h^4(b_i^j)^2}{240\varepsilon}(u_{xx})_i^j + \frac{\varepsilon k^2}{12}(u_{yyyy})_i^j - \frac{k^2(b_i^j)^2}{12}(u_{yy})_i^j \\
&-\frac{k^4(b_i^j)}{144}(u_{yyyy})_i^j + \frac{k^4(b_i^j)^2}{240\varepsilon}(u_{yy})_i^j + \cdots\Bigg].
\end{aligned} \tag{4.2}$$

Simplifying above, we obtain

$$\begin{aligned}\bar{L}_h^k\left(u_i^j-(v_i^j)^{ext}\right) &= \frac{b_i^j h^4}{576}(u_{xxxx})_i^j-\frac{(b_i^j)^2h^4}{960\varepsilon}(u_{xx})_i^j\\ &\quad+\frac{b_i^j k^4}{576}(u_{xxxx})_i^j-\frac{(b_i^j)^2k^4}{960\varepsilon}(u_{xx})_i^j+\cdots . \end{aligned}\tag{4.3}$$

Using Lemma 2.5 and its analogues for fourth order derivative terms, we obtain

$$\left|\bar{L}_h^k\left(u_i^j-(v_i^j)^{ext}\right)\right|\le M(h^4+k^4)\left(1+\frac{1}{\varepsilon}\right).\tag{4.4}$$

By Lemma 3.2, we obtain

$$\left|u_i^j-(v_i^j)^{ext}\right|\le M(h^4+k^4)\left(1+\frac{1}{\varepsilon}\right).\tag{4.5}$$

We summarize the results in the following theorem

Theorem 4.1. *Let $b(x,y)$ and $f(x,y)$ be sufficiently smooth functions in the problem* (1.1)–(1.2) *so that $u(x,y)\in C^4(\overline{\Omega})$. Then the numerical solutions v and v^{ext} obtained via the FOFDM* (3.1)–(3.2) *before and after extrapolation, respectively, satisfy the following estimates*

$$\max_{0\le i\le n}\max_{0\le j\le m}|u_i^j-v_i^j|\le M\left[h^2\left(1+\frac{h^2}{\varepsilon}\right)+k^2\left(1+\frac{k^2}{\varepsilon}\right)\right].\tag{4.6}$$

$$\max_{0\le i\le n}\max_{0\le j\le m}|u_i^j-(v_i^j)^{ext}|\le M(h^4+k^4)\left(1+\frac{1}{\varepsilon}\right).\tag{4.7}$$

5. NUMERICAL RESULTS

In this section, we give some numerical results for a test example corresponding to problem (1.1)–(1.2). In the implementation of the numerical method (3.1)–(3.2) before and after extrapolation, we assume that the step-sizes h and k in x- and y-directions, respectively, are equal.

Example 5.1. Consider problem (1.1)–(1.2) with $b=2$,

$$\begin{aligned}f(x,y)=&-\frac{e^{-x/\sqrt{\varepsilon}}+e^{-(1-x)/\sqrt{\varepsilon}}}{1+e^{-1/\sqrt{\varepsilon}}}-\frac{e^{-y/\sqrt{\varepsilon}}+e^{-(1-y)/\sqrt{\varepsilon}}}{1+e^{-1/\sqrt{\varepsilon}}}\\ &+2\left[1+\varepsilon\left(x(1-x)+y(1-y)+xy(1-x)(1-y)\right)\right].\end{aligned}$$

The exact solution is

$$\begin{aligned}u(x,y)=&\left(1-\frac{e^{-x/\sqrt{\varepsilon}}+e^{-(1-x)/\sqrt{\varepsilon}}}{1+e^{-1/\sqrt{\varepsilon}}}\right)\left(1-\frac{e^{-y/\sqrt{\varepsilon}}+e^{-(1-y)/\sqrt{\varepsilon}}}{1+e^{-1/\sqrt{\varepsilon}}}\right)\\ &+xy(1-x)(1-y).\end{aligned}$$

The maximum errors at all mesh points are calculated using the formulas

$$E_{\varepsilon,n} := \max_{0\le i,j\le m} |u_i^j - v_i^j|, \quad \text{before extrapolation}$$

and

$$E_{\varepsilon,n}^{ext} := \max_{0\le i,j\le m} |u_i^j - (v_i^j)^{ext}|, \quad \text{after extrapolation.}$$

The numerical rates of convergence are computed using the formula [1]

$$r_{\varepsilon,s} := \log_2(\widetilde{E}_{n_s}/\widetilde{E}_{2n_s}), \quad s = 1, 2, \ldots$$

where $\widetilde{E}$ stands for $E_{\varepsilon,n}$ and $E_{\varepsilon,n}^{ext}$, respectively. For the comparison purpose, we also consider the following example from [8]:

Example 5.2. Consider problem (1.1)–(1.2) with $b = 2$ where the author chooses f in such a way that the exact solution is

$$u(x,y) = y(1-y)\left(1 - e^{-x/\sqrt{\varepsilon}}\right)\left(1 - e^{-(1-x)/\sqrt{\varepsilon}}\right) + x(1-x)\left(1 - e^{-y/\sqrt{\varepsilon}}\right)\left(1 - e^{-(1-y)/\sqrt{\varepsilon}}\right).$$

The numerical solution (and the error) obtained for this example using the proposed fitted operator method is displayed in Figure 1.

TABLE 1. Maximum errors **before** extrapolation

ε n=512	n=8	n=16	n=32	n=64	n=128	n=256
2^{-1}	3.59E-04	8.98E-05	2.25E-05	5.58E-06	1.27E-06	2.00E-07
2^{-2}	8.98E-04	2.25E-04	5.64E-05	1.41E-05	3.41E-06	4.09E-07
2^{-3}	2.26E-03	5.68E-04	1.42E-04	3.56E-05	8.81E-06	1.86E-06
2^{-4}	4.52E-03	1.15E-03	2.89E-04	7.24E-05	1.80E-05	4.28E-06
2^{-5}	6.71E-03	1.76E-03	4.46E-04	1.12E-04	2.80E-05	6.91E-06
2^{-6}	1.10E-02	3.07E-03	7.88E-04	1.99E-04	4.97E-05	1.24E-05
2^{-7}	1.95E-02	5.76E-03	1.51E-03	3.83E-04	9.61E-05	2.40E-05
2^{-8}	2.65E-02	1.04E-02	2.91E-03	7.53E-04	1.90E-04	4.75E-05

6. CONCLUDING REMARKS AND FUTURE PLANS

This paper was concerned with singularly perturbed elliptic problems in two dimensions. Our aim was to design a fitted operator finite difference method for these problems and to investigate the effect of extrapolation on the convergence of this novel method. The method showed to be second order convergent. The extrapolation improves this convergence up to fourth order. Numerical results presented in tables 1-4 confirm the theoretical estimates given in (4.6)–(4.7).

TABLE 2. Maximum errors **after** extrapolation

ε	n=8	n=16	n=32	n=64	n=128	n=256
2^{-1}	1.10E-07	2.69E-09	4.17E-08	1.71E-07	6.89E-07	2.76E-06
2^{-2}	8.69E-07	4.67E-08	3.25E-08	1.46E-07	5.90E-07	2.36E-06
2^{-3}	6.01E-06	3.74E-07	4.28E-09	1.12E-07	4.57E-07	1.83E-06
2^{-4}	2.96E-05	1.89E-06	1.01E-07	7.06E-08	3.15E-07	1.27E-06
2^{-5}	1.02E-04	6.77E-06	4.25E-07	2.41E-08	1.93E-07	7.82E-07
2^{-6}	3.51E-04	2.48E-05	1.60E-06	9.50E-08	1.07E-07	4.42E-07
2^{-7}	1.19E-03	9.31E-05	6.19E-06	3.92E-07	5.53E-08	2.37E-07
2^{-8}	3.18E-03	3.40E-04	2.40E-05	1.55E-06	9.60E-08	1.23E-07

TABLE 3. Rates of convergence **before** extrapolation, $n_k = 8, 16$

ε	r_1	r_2
2^{-2}	2.00	2.00
2^{-3}	1.99	2.00
2^{-4}	1.97	1.99
2^{-5}	1.93	1.98
2^{-6}	1.84	1.96
2^{-7}	1.76	1.93
2^{-8}	1.35	1.84

TABLE 4. Rates of convergence **after** extrapolation, $n_k = 8, 16$

ε	r_1	r_2
4.00		
2^{-2}	3.99	4.00
2^{-3}	3.98	4.00
2^{-4}	3.96	3.99
2^{-5}	3.91	3.98
2^{-6}	3.82	3.95
2^{-7}	3.68	3.91
2^{-8}	3.23	3.82

We have also compared our results with those seen in the literature. See for example Figure 1. In this figure, the numerical solution (and the error) obtained using the proposed fitted operator method is displayed. One can compare these errors with those obtained by Lin [8] (see, page 105, the right plot on their Fig 6).

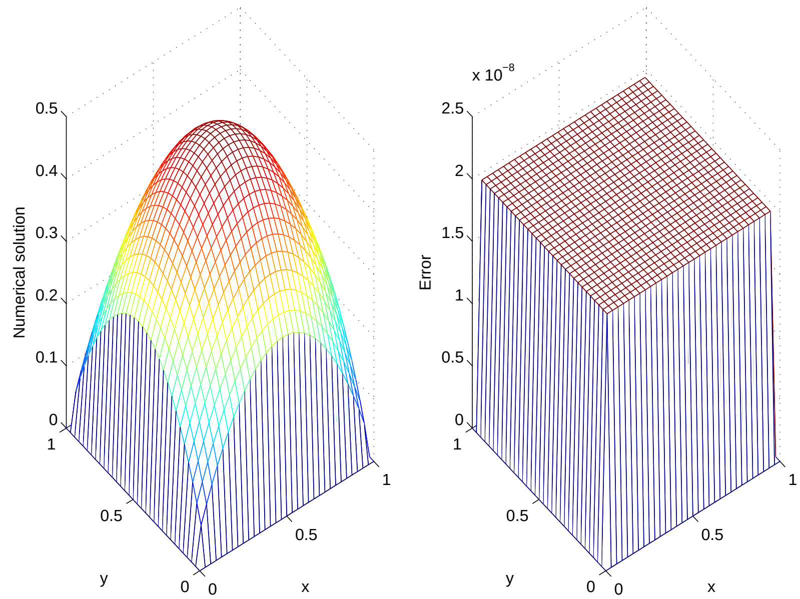

FIGURE 1. Numerical solution and errors for $h = 1/32$ by the proposed FOFDM before extrapolation. (The right figure can be compared with the one on page 105 in [8].)

TABLE 5. Comparison of the errors obtained by our method and those in [8] for $\varepsilon = 10^{-8}$ and $n = 32$

Maximum errors in [8]	Maximum errors obtained by our approach
$\approx 4 \times 10^{-2}$	$\approx 2 \times 10^{-8}$

As indicated in the Table 5 below, the error there is of the magnitude of 10^{-2} where as ours is of the magnitude of 10^{-8}.

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