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DYNAMICS OF A FISHING MODEL

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ABSTRACT. In this paper, the authors give sufficient conditions for the existence and global attractivity of a positive periodic solution of the first order nonlinear differential equation

$$N'(t) = -a(t)N(t) + b(t)\frac{N(t)}{1 + \left(\frac{N(t)}{p(t)}\right)^{\gamma}},$$

where the coefficients are periodic functions. This equation is used to model fish populations where $N(t)$ is the population size at time t. Some examples showing the independence of the results are included.

Key Words. Periodic solution, attractivity, population model, fishing model

1. INTRODUCTION

Periodicity plays an important role in problems associated with real world applications such as those involving ecosystem dynamics and environmental variability. There have been considerable contributions to the literature in recent years on the existence and global attractivity of periodic solutions of differential equation models of such phenomena; see, for example [4, 7, 9, 10]. Here, we are interested in investigating the existence and global attractivity of a positive periodic solution of a first order differential equation for a fishing model.

Differential equations of the form

$$N'(t) = [R(t, N) - M(t, N)]N - F(t)N, \tag{1}$$

where R, $M : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ are continuous functions, are often used as population models, and in particular for models of fish populations (see, for example, [3, 6]). Here, $N = N(t)$ denotes the population size at time t, $R(t, N)$ is the birth rate, $M(t, N)$ is the mortality rate, and $F(t)$ denotes the harvesting rate. Seasonal effects including such things as weather, food supply, mating habits, and seasonal harvesting are often incorporated into these kind of models in the form of

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periodic coefficients. A common choice for the function R is what is often referred to as Hill's function (see [1, 2, 3, 6])

$$R(t, N) = \frac{b}{1 + \left(\frac{N}{p}\right)^{\gamma}}, \tag{2}$$

where b and p are positive constants. The parameter $\gamma > 0$ affects how quickly density dependence sets in.

Berezansky and Idels [2] considered the delay differential equation model

$$N'(t) = -a(t)N(t) + \frac{b(t)}{1 + \left(\frac{N(\theta(t))}{p(t)}\right)^{\gamma}} N(t), \tag{3}$$

and they studied the existence of a periodic solution and its stability properties. In particular, they proved the existence of a positive periodic solution of (3) that is a global attractor for all positive solutions. They also considered the case of equation (3) with proportional coefficients. In another nice paper, Berezansky, Braverman, and Idels [3], studied the model

$$N'(t) = -a(t)N(t) + \frac{b(t)N(t)}{1 + N^{\gamma}(t)} - r(t)N(\theta(t)), \tag{4}$$

and also gave sufficient conditions for the existence of a positive bounded solution.

Here, we will consider the equation

$$N'(t) = -a(t)N(t) + b(t)\frac{N(t)}{1 + \left(\frac{N(t)}{p(t)}\right)^{\gamma}}, \tag{5}$$

where a, b, $p : [0, \infty) \to [0, \infty)$ are positive continuous periodic functions with period T. We too will determine conditions, different from those in [2] and [3], under which this equation has a positive periodic solution. We will also give conditions under which this periodic solution is a global attractor as defined in the following definition.

Definition 1.1. Suppose that $x(t)$ and $\overline{x}(t)$ are two positive solutions of equation (5) on $[0, \infty)$. The solution $\overline{x}(t)$ is said to be asymptotically attractive to $x(t)$ if

$$\lim_{t \to \infty} [x(t) - \overline{x}(t)] = 0.$$

Furthermore, $\overline{x}(t)$ is called globally attractive if $\overline{x}(t)$ is asymptotically attractive to all positive solutions of (5).

In the next section of this paper, we prove the existence of a positive periodic solution $\overline{N}(t)$ of equation (5) and give sufficient conditions for the global attractivity of this solution. The final section contains some examples showing that our various results are independent of each other.

For convenience, we introduce the following notations. For any continuous T-periodic function $h : [0, \infty) \to \mathbb{R}$, let

$$h^* = \max_{0 \le t \le T} h(t) \quad \text{and} \quad h_* = \min_{0 \le t \le T} h(t).$$

Throughout the paper, we assume that $\gamma > 1$.

2. EXISTENCE AND GLOBAL ATTRACTIVITY OF A PERIODIC SOLUTION

Our first lemma shows that, given a positive initial condition, solutions of equation (5) are positive and bounded.

Lemma 2.1. *Every solution of* (5) *having a positive initial condition is positive and satisfies the property*

$$\limsup_{t \to \infty} N(t) \le K \tag{6}$$

where

$$K = \frac{b^*}{a_*} p_* \left(\frac{p^*}{p_*}\right)^{\gamma} \left(\frac{1}{\gamma}\right) (\gamma - 1)^{\frac{\gamma-1}{\gamma}}. \tag{7}$$

Proof. Let $N(t)$ be a solution of (5) with $N(0) = N_0 > 0$; then

$$N(t) = N_0 \exp \int_0^t \left[-a(s) + \frac{b(s)}{1 + \left(\frac{N(s)}{p(s)}\right)^{\gamma}} \right] ds.$$

Consequently, $N(t)$ is defined on $[0, \infty)$ and $N(t) > 0$ for $t \ge 0$.

Next, we claim that $N(t)$ is bounded. Setting

$$\widetilde{g}(N) = \frac{p^{*\gamma} N(t)}{p^{*\gamma} + N^{\gamma}(t)},$$

we see that

$$\widetilde{g}(N) \le \overline{g} = \frac{p^{*\gamma} \mu}{p^{*\gamma} + \mu^{\gamma}} \quad \text{where} \quad \mu = p^* \left(\frac{1}{\gamma - 1}\right)^{\frac{1}{\gamma}}. \tag{8}$$

From (5), we obtain

$$N'(t) \le -a(t) N(t) + b^* \overline{g}. \tag{9}$$

To prove (6), note that from (8) and (9), we have

$$\begin{aligned} N(t) &\le N_0 e^{-\int_0^t a(s)\, ds} + \int_0^t b^* \overline{g} e^{-\int_s^t a(u)\, du}\, ds \\ &\le N_0 e^{-a_* t} + b^* \overline{g} \int_0^t e^{-a_*(t-s)}\, ds \\ &\le N_0 e^{-a_* t} + \frac{b^* \overline{g}}{a_*} \left(1 - e^{-a_* t}\right). \end{aligned}$$

This in turn implies that

$$\limsup_{t\to\infty} N(t) \le \frac{b^*\overline{g}}{a_*} = K,$$

where K is given in (7). This completes the proof of the lemma. □

Remark 2.2. Note that from the proof of Lemma 2.1, we have

$$N(t) < N_0 + \frac{b^*\overline{g}}{a_*} = N_0 + K$$

for all $t \ge 0$.

The following theorem gives a sufficient condition for the existence of a positive periodic solution of (5). We shall use a method of proof similar to that used by Graef et al. in [5].

Theorem 2.3. *If $b_* > a^*$, then* (5) *has at least one positive periodic solution.*

Proof. From Lemma 2.1, it follows that every solution of (5) with a positive initial condition is positive and bounded. Furthermore, the function $\widetilde{g}(N) = \frac{p^{*\gamma}N(t)}{p^{*\gamma}+N^\gamma(t)}$ is decreasing in $(p^*(\frac{1}{\gamma-1})^{\frac{1}{\gamma}}, \infty)$. Consider the function

$$f(N) = -aN + b\frac{N}{1+\left(\frac{N}{p^*}\right)^\gamma},$$

where a and b are constants. Clearly, the function attains its maximum

$$f_{\max} = \frac{p^*}{(2a)^{\frac{1}{\gamma}}}\left[(4ab\gamma + (\gamma-1)^2b^2)^{\frac{1}{2}} - (2a + (\gamma-1)b)\right]^{\frac{1}{\gamma}}$$

$$\times\left[-a + \frac{(4ab\gamma + (\gamma-1)^2b^2)^{\frac{1}{2}} + (\gamma-1)b}{2\gamma}\right]$$

at

$$N = \hat{N} = \frac{p^*}{(2a)^{\frac{1}{\gamma}}}\left[(4ab\gamma + (\gamma-1)^2b^2)^{\frac{1}{2}} - (2a + (\gamma-1)b)\right]^{\frac{1}{\gamma}}.$$

Now, $\hat{N} > 0$, and $f_{\max} > 0$ for $b > a$, and since $f(N) \to -\infty$ as $N \to \infty$, it follows that there exists an $\alpha \in (\hat{N}, \infty)$ such that $f(\alpha) = 0$. Thus, $f_1(N) = 0$ and $f_2(N) = 0$ has roots N_1 and N_2 respectively, where

$$f_1(N) = \frac{b_*N}{1+\left(\frac{N}{p^*}\right)^\gamma} - a^*N$$

and

$$f_2(N) = \frac{b^*N}{1+\left(\frac{N}{p^*}\right)^\gamma} - a_*N.$$

Furthermore, a simple calculation shows that

$$\max\left\{\hat{N},\, p^*\left(\frac{1}{\gamma-1}\right)^{\frac{1}{\gamma}}\right\} < N_1 < N_2.$$

Let $N(t) = N(t, 0, \alpha)$, $\alpha \geq \max\{\hat{N}, p^*(\frac{1}{\gamma-1})^{\frac{1}{\gamma}}\}$, be the unique solution of (5) through $(0, \alpha)$. We claim that $N(t) \in [N_1, N_2]$ for every $\alpha \in [N_1, N_2]$ and $t \geq 0$. Suppose this is not the case; say, let

$$t_1 = \inf\{t > 0 : N(t) > N_2\}.$$

Then there exists a $t_2 \geq t_1$ such that $N(t_2) > N_2$ and $N'(t_2) > 0$. Hence, from (5) we obtain

$$\begin{aligned} 0 < N'(t_2) &= -a(t_2)N(t_2) + \frac{b(t_2)N(t_2)}{1 + \left(\frac{N(t_2)}{p(t_2)}\right)^{\gamma}} \\ &\leq -a_* N_2 + \frac{b^* N_2}{1 + \left(\frac{N_2}{p^*}\right)^{\gamma}} = f_2(N_2) = 0, \end{aligned}$$

which is a contradiction. Consequently, $N(t) \leq N_2$. By a similar argument, we can show that $N(t) \geq N_1$. Thus, in particular,

$$N_T = N(T, 0, \alpha) \in [N_1, N_2].$$

Next, we define a mapping $F : [N_1, N_2] \to [N_1, N_2]$ as follows: for each $\alpha \in [N_1, N_2]$, let $F(\alpha) = N_T$. Since the solution $N(t, 0, \alpha)$ depends continuously on the initial value α, the mapping F is continuous and maps the interval $[N_1, N_2]$ into $[N_1, N_2]$. By Brouwer's fixed point theorem, F has a fixed point $\overline{N}$. Thus, the unique solution $\overline{N} = N(t, 0, \alpha)$ is periodic with period T. This completes the proof of the theorem. □

We want to show that $\overline{N}(t)$ is an attractor to all other positive solutions of (5). To prove our theorems, we need the following lemma. An indirect proof of this lemma can be found in [8]. We will present a proof here for the sake of completeness.

Lemma 2.4. *Let $z \in C^1([0, \infty), R)$ and $\sigma \in R$. If $z \in L^2[\sigma, \infty)$ and $z'(t)$ is bounded, then $z(t) \to 0$ as $t \to \infty$.*

Proof. Clearly the statement holds for any nonoscillatory function on $[0, \infty)$, so let $z(t)$ be an oscillatory function on $[0, \infty)$. Since $z \in L^2[0, \infty)$, $\int_0^\infty z^2(t)dt < \infty$. This in turn implies that $\liminf_{t\to\infty} z(t) = 0$. To complete the proof of the theorem, it remains to show that $\limsup_{t\to\infty} z(t) = 0$. Suppose that $\limsup_{t\to\infty} z(t) \neq 0$. Then there exist an $\epsilon > 0$ and a sequence $\{t_n\}_{n=1}^\infty$ such that $t_n \to \infty$ as $n \to \infty$ and $z(t_n) > 2\epsilon$ for large n. Since $\liminf_{t\to\infty} z(t) = 0$, there exists a sequence $\{t_n^*\}_{n=1}^\infty$ such that $t_n^* \to \infty$ as $n \to \infty$ and $z(t_n^*) \to 0$ as $n \to \infty$. Thus, $z(t_n^*) < \epsilon$ for $n \geq N_\epsilon$

for some positive integer N_ϵ. It is possible to extract sequences $\{s_n\}_{n=1}^\infty$ and $\{\sigma_n\}_{n=1}^\infty$ such that $\sigma_n < s_n < \sigma_{n+1}$, $\sigma_n \to \infty$ as $n \to \infty$, $z(s_n) > 2\epsilon$, and $z(\sigma_n) < \epsilon$ for $n \geq N_\epsilon$. Since $z(t)$ is continuous, there exist sequences $\{\tau_n\}_{n=1}^\infty$ and $\{\tau_n^*\}_{n=1}^\infty$ such that $\sigma_n < \tau_n^* < \tau_n < s_n$ with $z(\tau_n^*) = \epsilon$ and $z(\tau_n) = 2\epsilon$. It is clear that the intervals (τ_n^*, τ_n) are disjoint. Then,

$$\sum_{n=1}^\infty (\tau_n - \tau_n^*)\epsilon^2 \leq \sum_{n=1}^\infty \int_{\tau_n^*}^{\tau_n} z^2(t)\, dt \leq \int_0^\infty z^2(t)\, dt < \infty,$$

which implies that $\lim_{n\to\infty}(\tau_n - \tau_n^*) = 0$. By the Mean Value Theorem, the calculation

$$z'(\xi_n) = \frac{z(\tau_n) - z(\tau_n^*)}{\tau_n - \tau_n^*}, \quad \tau_n^* < \xi_n < \tau_n,$$

implies that $z'(\xi_n) \to \infty$ as $n \to \infty$, which contradicts the hypothesis of the lemma. Hence, our claim holds, that is, $\limsup_{t\to\infty} z(t) = 0$. Consequently, $z(t) \to 0$ as $t \to \infty$, and the lemma is proved. □

Theorem 2.5. *Assume that*

(H$_1$) $\quad b^* p^{*\gamma}(\gamma - 1)^2 < 4a_*(p^{*\gamma} + p_*^\gamma(\gamma - 1))$

holds. Then $\overline{N}(t)$ is a global attractor to all other positive solutions of (5)*. That is, every positive solution $N(t)$ of* (5) *satisfies*

$$\lim_{t\to\infty}[N(t) - \overline{N}(t)] = 0.$$

Proof. Setting $Z(t) = N(t) - \overline{N}(t)$, we obtain

$$Z'(t) = -a(t)Z(t) + b(t)\left[\frac{N(t)}{1 + \left(\frac{N(t)}{p(t)}\right)^\gamma} - \frac{\overline{N}(t)}{1 + \left(\frac{\overline{N}(t)}{p(t)}\right)^\gamma}\right]. \tag{10}$$

Equation (10) is equivalent to

$$\left(\frac{1}{2}Z^2(t)\right)' = -a(t)Z^2(t) + Z(t)b(t)p^\gamma(t)\left[\frac{N(t)}{p^\gamma(t) + N^\gamma(t)} - \frac{\overline{N}(t)}{p^\gamma(t) + \overline{N}^\gamma(t)}\right]. \tag{11}$$

Let $F(t,\theta) = \dfrac{\theta}{p^\gamma(t) + \theta^\gamma}$; then

$$\frac{\partial}{\partial\theta}F(t,\theta) = \frac{p^\gamma(t) + (1-\gamma)\theta^\gamma}{(p^\gamma(t) + \theta^\gamma)^2} < F_1(t,\theta) = \frac{p^{*\gamma} + (1-\gamma)\theta^\gamma}{(p^\gamma(t) + \theta^\gamma)^2},$$

where θ lies between $N(t)$ and $\overline{N}(t)$. Since $F_1(t,\theta) < 0$ for $\theta > \dfrac{p^*}{(\gamma-1)^{\frac{1}{\gamma}}}$, set

$$G_1(t,\theta) = -F_1(t,\theta) = \frac{(\gamma-1)\theta^\gamma - p^{*\gamma}}{(p^\gamma(t) + \theta^\gamma)^2}.$$

Note that $G_1(t,\theta) > 0$ for $\theta > \dfrac{p^*}{(\gamma-1)^{\frac{1}{\gamma}}}$. Furthermore,

$$G_1(t,\theta) < G(\theta) = \frac{(\gamma-1)\theta^\gamma - p^{*\gamma}}{(p_*^\gamma + \theta^\gamma)^2}.$$

A simple calculation shows that $G(\theta)$ attains its maximum value $\dfrac{(\gamma-1)^2}{4(p_*^\gamma(\gamma-1)+p^{*\gamma})}$ at $\theta = \left[\dfrac{2p^{*\gamma} + (\gamma-1)p_*^\gamma}{\gamma-1}\right]^{\frac{1}{\gamma}}$.

Applying the Mean Value Theorem, (11) yields

$$\begin{aligned}\left(\frac{1}{2}Z^2(t)\right)' &\le -a(t)Z^2(t) + Z(t)b(t)p^\gamma(t)|N(t) - \overline{N}(t)|G(\theta)\\ &\le -a(t)Z^2(t) + b(t)p^\gamma(t)G(\theta)Z^2(t)\\ &\le -\left[a_* - b^*p^{*\gamma}\frac{(\gamma-1)^2}{4(p_*^\gamma(\gamma-1)+p^{*\gamma})}\right]Z^2(t)\end{aligned}$$

that is,

$$\left(\frac{1}{2}Z^2(t)\right)' \le -\mu Z^2(t), \tag{12}$$

where

$$\mu = a_* - b^*p^{*\gamma}\frac{(\gamma-1)^2}{4(p_*^\gamma(\gamma-1)+p^{*\gamma})} > 0$$

by (H_1). Integrating inequality (12) from t_1 to t with $t_1 \ge \dfrac{p^*}{(\gamma-1)^{\frac{1}{\gamma}}}$, we obtain

$$\mu\int_{t_1}^{t} Z^2(s)\,ds \le \frac{1}{2}Z^2(t_1) - \frac{1}{2}Z^2(t) \le \frac{1}{2}Z^2(t_1).$$

This shows $\int_{t_1}^\infty Z^2(s)\,ds < \infty$, i.e., $Z \in L^2[t_1,\infty)$. Furthermore, the boundedness of $N(t)$ and $\overline{N}(t)$ imply that $Z'(t)$ is bounded. Hence, by Lemma 2.4, $Z(t) \to 0$ as $t \to \infty$, that is, $N(t) \to \overline{N}(t)$ as $t \to \infty$. This completes the proof of the theorem. □

Our next theorem gives another sufficient condition for $\overline{N}(t)$ to be a global attractor to all other positive solutions of (5). As we shall see in the examples below, this condition is independent of the one given in Theorem 2.5 above.

Theorem 2.6. *Suppose that*

$$a_* > b^*p^{*\gamma}\left[\frac{(\gamma-1)K^\gamma}{p_*^{2\gamma}} - \frac{p^{*\gamma}}{(p^{*\gamma}+K^\gamma)^2}\right] \tag{H$_2$}$$

holds. Then $\overline{N}(t)$ is a global attractor to all other positive solutions of (5).

Proof. Letting $Z(t) = N(t) - \overline{N}(t)$ and proceeding as in the proof of the Theorem 2.5, we again obtain (11). Applying the Mean Value Theorem yields

$$\left(\frac{1}{2}Z^2(t)\right)' \le -a(t)Z^2(t) + Z(t)b(t)p^\gamma(t)|N(t) - \overline{N}(t)|G_1(t,\theta). \tag{13}$$

We then have

$$\begin{aligned} G_1(t,\theta) &= \frac{(\gamma-1)\theta^\gamma - p^{*\gamma}}{(p^\gamma(t)+\theta^\gamma)^2} = \frac{(\gamma-1)\theta^\gamma}{(p^\gamma(t)+\theta^\gamma)^2} - \frac{p^{*\gamma}}{(p^\gamma(t)+\theta^\gamma)^2} \\ &< \frac{(\gamma-1)\theta^\gamma}{p^{2\gamma}(t)} - \frac{p^{*\gamma}}{(p^\gamma(t)+\theta^\gamma)^2} \\ &\le \frac{(\gamma-1)K^\gamma}{p_*^{2\gamma}} - \frac{p^{*\gamma}}{(p^{*\gamma}+K^\gamma)^2}. \end{aligned}$$

Since $G_1(t,\theta) > 0$ for $\theta > \dfrac{p^*}{(\gamma-1)^{\frac{1}{\gamma}}}$, we have $\dfrac{(\gamma-1)K^\gamma}{p_*^{2\gamma}} - \dfrac{p^{*\gamma}}{(p^{*\gamma}+K^\gamma)^2} > 0$ for $\theta > \dfrac{p^*}{(\gamma-1)^{\frac{1}{\gamma}}}$. Hence, from (13), we obtain

$$\begin{aligned} \left(\frac{1}{2}Z^2(t)\right)' &\le \left[-a(t) + b(t)p^\gamma(t)\left(\frac{(\gamma-1)K^\gamma}{p_*^{2\gamma}} - \frac{p^{*\gamma}}{(p^{*\gamma}+K^\gamma)^2}\right)\right] Z^2(t) \\ &\le -\left[a_* - b^*p^{*\gamma}\left(\frac{(\gamma-1)K^\gamma}{p_*^{2\gamma}} - \frac{p^{*\gamma}}{(p^{*\gamma}+K^\gamma)^2}\right)\right] Z^2(t) \end{aligned}$$

or

$$\left(\frac{1}{2}Z^2(t)\right)' \le -\mu Z^2(t),$$

where $\mu = a_* - b^*p^{*\gamma}\left(\dfrac{(\gamma-1)K^\gamma}{p_*^{2\gamma}} - \dfrac{p^{*\gamma}}{(p^{*\gamma}+K^\gamma)^2}\right) > 0$ by (H_2).

The remainder of the proof is similar to the proof of Theorem 2.5. Lemma 2.4 again implies that $\overline{N}(t)$ is a global attractor to all other positive solutions of (5), and this completes the proof of the theorem. □

Remark 2.7. A somewhat easier condition to verify than (H_2) is

$$\left(\frac{b^*}{a_*}\right)^{\gamma+1}\left(\frac{p^*}{p_*}\right)^{\gamma(\gamma+1)}\left(\frac{\gamma-1}{\gamma}\right)^{\gamma} < 1. \tag{H_3}$$

It can be shown that this condition implies (H_2).

3. EXAMPLES

The following examples illustrate our theorems as well as show the independence of the hypotheses.

Example 3.1. Consider the equation

$$N'(t) = -\left(1+\frac{\sin^2 t}{10}\right)N(t) + \frac{(1.5+\frac{\cos^2 t}{20})N(t)}{1+\left(\frac{N(t)}{2+\cos^2 t}\right)^3}, \quad t \ge 0.$$

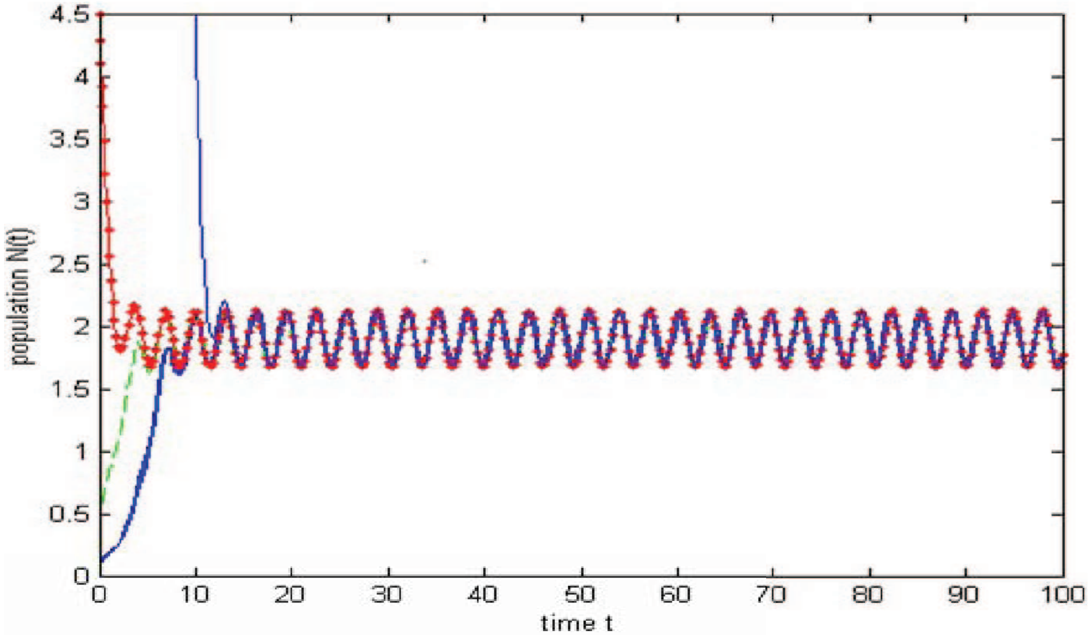

FIGURE 1. Simulation for Example 1

Here $a(t) = 1 + \frac{\sin^2 t}{10}$, $b(t) = 1.5 + \frac{\cos^2 t}{20}$, $p(t) = 2 + \cos^2 t$, and $\gamma = 3$. Clearly $b_* = 1.5 > 1.1 = a^*$. It is easy to see that condition (H_1) of Theorem 2.5 is satisfied. Therefore, this equation has a positive periodic solution that is a global attractor to all other positive solutions. On the other hand,

$$b^* p^{*\gamma} \left[\frac{(\gamma - 1)K^\gamma}{p_*^{2\gamma}} - \frac{p^{*\gamma}}{(p^{*\gamma} + K^\gamma)^2} \right] = 218.7 > 1 = a_*$$

implying that (H_2) fails to hold. Consequently, Theorem 2.6 cannot be applied to this example.

Example 3.2. Consider the equation

$$N'(t) = -\left(1.2 + \frac{\sin^2 t}{50}\right) N(t) + \frac{(1.3 + \frac{\cos^2 t}{10})N(t)}{1 + \left(\frac{N(t)}{0.5 + \frac{\sin^2 t}{1000}}\right)^6}, \quad t \geq 0.$$

Here $a(t) = 1.2 + \frac{\sin^2 t}{50}$, $b(t) = 1.3 + \frac{\cos^2 t}{10}$, $p(t) = 0.5 + \frac{\sin^2 t}{1000}$, and $\gamma = 6$. Now $b_* = 1.3 > 1.22 = a^*$ and

$$\left(\frac{b^*}{a_*}\right)^{\gamma+1} \left(\frac{p^*}{p_*}\right)^{\gamma(\gamma+1)} \left(\frac{\gamma - 1}{\gamma}\right)^\gamma = 0.918412032 < 1,$$

that is, (H_3) and hence (H_2) is satisfied. Thus, Theorem 2.6 can be applied, and so the equation has a positive periodic solution that is a global attractor to all other positive solutions. On the other hand, condition (H_1) fails to hold, which means that Theorem 2.5 cannot be applied to this example.

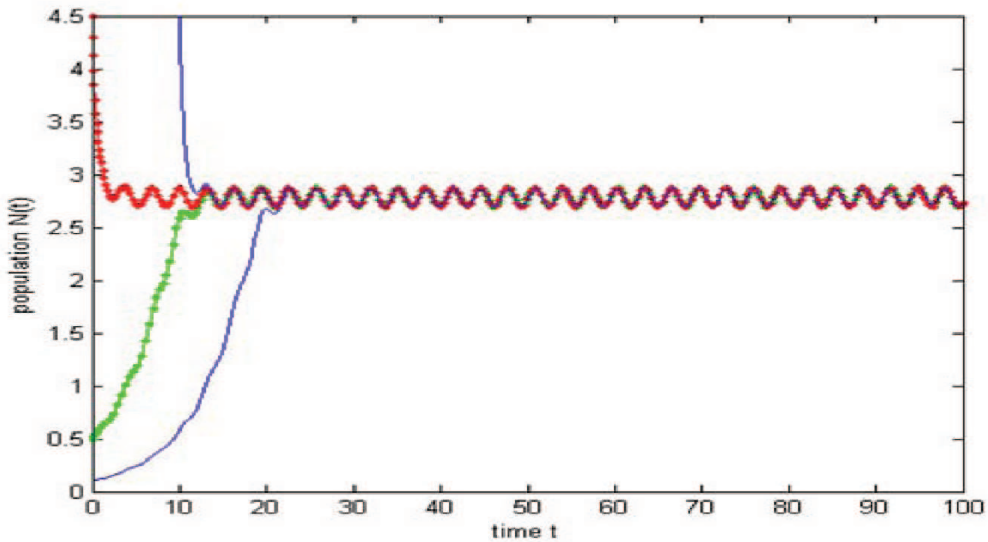

FIGURE 2. Simulation for Example 2

Example 3.3. Consider the equation

$$N'(t) = -\left(1.5 + \frac{\sin^2 t}{10}\right) N(t) + \frac{(1.7 + \frac{\cos^2 t}{20})N(t)}{1 + \left(\frac{N(t)}{4+\frac{\sin^2 t}{100}}\right)^6}, \quad t \geq 0.$$

Here $a(t) = 1.5 + \frac{\sin^2 t}{10}$, $b(t) = 1.7 + \frac{\cos^2 t}{20}$, $p(t) = 4 + \frac{\sin^2 t}{100}$, and $\gamma = 6$. Since $b_* = 1.7 > 1.6 = a^*$, $K = 3.0189$, and

$$b^* p^{*\gamma} \left[\frac{(\gamma - 1)K^\gamma}{p_*^{2\gamma}} - \frac{p^{*\gamma}}{(p^{*\gamma} + K^\gamma)^2}\right]$$
$$= (1.75)(4.01)^6 \left[\frac{5(3.0189)^6}{4^{12}} - \frac{(4.01)^6}{((4.01)^6 + (3.0189)^6)^2}\right]$$
$$< 0.42 < 1.5 = a_*,$$

condition (H_2) of Theorem 2.6 is satisfied. This shows that again we have a positive periodic solution that is a global attractor to all other positive solutions. However, a simple calculation shows that (H_1) fails to hold, so Theorem 2.5 cannot be applied to this example.

Our final example is a simple one, yet it shows that our results may hold when those in [2] do not.

Example 3.4. Consider the equation

$$N'(t) = -N(t) + \frac{2N(t)}{1 + N^2(t)}, \quad t \geq 0.$$

Here, $a(t) = 1 < b(t) = 2$, $p(t) = 1$, and $\gamma = 2$. It is easy to see that condition (H_1) of Theorem 2.5 is satisfied, so this equation has a positive periodic solution globally

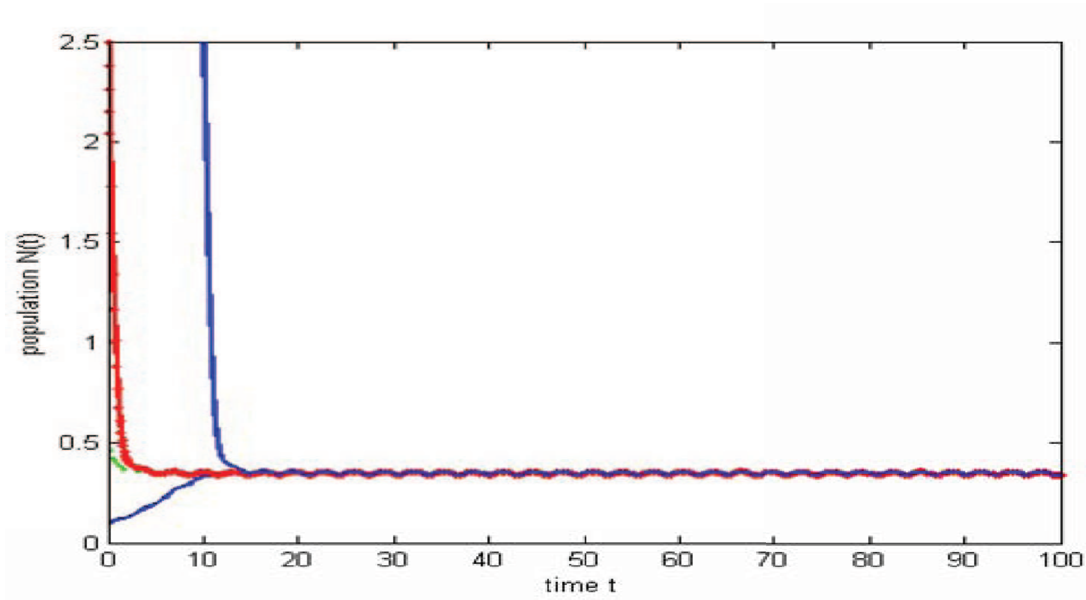

FIGURE 3. Simulation for Example 3

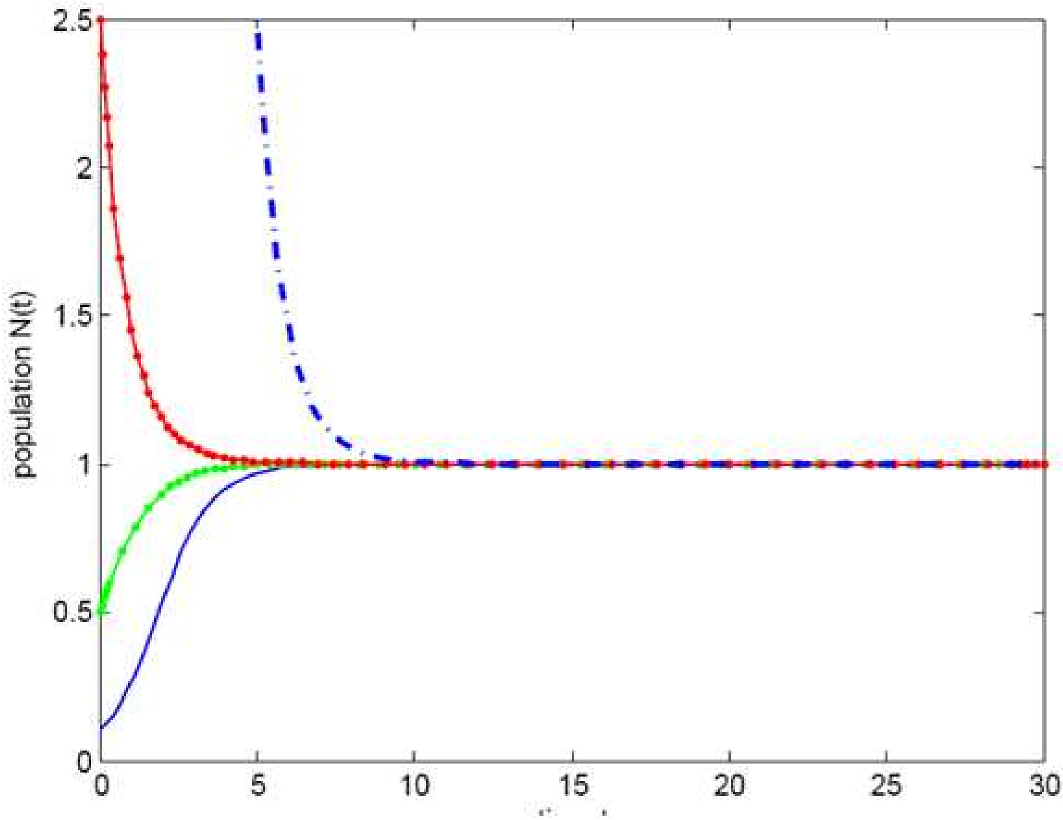

FIGURE 4. Simulation for Example 4

attracting all other positive solutions. In fact, this solution is $N(t) \equiv 1$. However, Theorem 2.1 in [2] does not apply to this example.

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