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AN ALMOST SECOND ORDER FITTED MESH NUMERICAL METHOD FOR A SINGULARLY PERTURBED DELAY PARABOLIC PARTIAL DIFFERENTIAL EQUATION

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ABSTRACT. In this paper we develop a numerical method for solving a singularly perturbed delay parabolic partial differential equation. The proposed method consists of Crank-Nicolson finite difference method constructed on a mesh of Shishkin type and hence referred to as a fitted mesh finite difference method. We analyzed the method for stability and convergence and found that it is unconditionally stable and converges with order $\mathcal{O}\left(N_t^{-2} + N_x^{-2}\ln^2 N_x\right)$ where N_t and N_x are the numbers of subintervals in the t and x directions, respectively. The performance of the method is illustrated through numerical experiments.

Key Words. Singular perturbations; Delay parabolic partial differential equation; Fitted mesh finite difference methods; Stability; Convergence

AMS Subject Classification (2000): 65L50, 65M06, 65M12, 65M15, 65M99

1. INTRODUCTION

We consider a singularly perturbed delay parabolic partial differential equation (SPDPPDE) of the form

$$\frac{\partial u(t,x)}{\partial t} - \varepsilon \frac{\partial^2 u(t,x)}{\partial x^2} + a(t,x)u(t,x) = f(t,x) - b(x)u(t-\tau,x) \tag{1.1}$$

$$(t,x) \in \overline{\Omega} \equiv [0,T] \times [0,1]$$

with the initial data

$$u(t,x) = u_0(t,x), \quad (t,x) \in [-\tau, 0] \times (0,1) \tag{1.2}$$

and boundary conditions

$$u(t,x) = \Gamma_L(t), (t,x) \in \Pi_L \tag{1.3}$$

and

$$u(t,x) = \Gamma_R(t), (t,x) \in \Pi_R, \tag{1.4}$$

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where $0 < \varepsilon \leq 1$ is the singular perturbation parameter and $\tau > 0$ is the delay parameter. The functions $a(t,x) \geq 0$, $b(t,x) \geq \beta \geq 0$, $f(t,x)$, $u_0(t,x)$, $\Gamma_L(t)$ and $\Gamma_R(t)$ are bounded and sufficiently smooth functions and Π_L and Π_R denote $[0,T]\times\{0\}$ and $[0,T]\times\{1\}$, respectively, are the left and right boundaries of the domain $\overline{\Omega}$. The terminal time $T > 0$ is assumed to satisfy $T = K\tau$ where K is a positive integer, whereas the initial function $u_0(t,x)$ is assumed to satisfy the compatibility conditions [12]:

$$\begin{aligned} u_0(0,0) &= \Gamma_L(0), \\ u_0(0,1) &= \Gamma_R(0), \\ \frac{\partial u_0(0,0)}{\partial t} &= \varepsilon \frac{\partial^2 u_0(0,0)}{\partial x^2} - b(0)u(-\tau,0) + f(0,0), \end{aligned}$$

and

$$\frac{\partial u_0(0,1)}{\partial t} = \varepsilon \frac{\partial^2 u_0(0,1)}{\partial x^2} - b(1)u(-\tau,1) + f(0,1).$$

Under the above assumptions and conditions, problem (1.1) with the initial data (1.2) and the boundary conditions (1.3) and (1.4) has a unique solution [1]. For the occurrence and applications of such problems, the readers are referred to the standard text by Murray [8] and some of the references therein.

Both fitted operator finite difference methods (FOFDMs) and fitted mesh finite difference methods (FMFDMs), nowadays, are widely being used for singularly perturbed problems. While FOFDMs (see, e.g., [9, 10, 11] and references therein) can provide a difference operator that reflects the dynamics of the solution on a uniform mesh, they sometimes suffer from the drawback that their construction is not always straightforward. In fact not many FOFDMs which are constructed for singularly perturbed two-point boundary value problems can easily be extended for singularly perturbed PDEs. The FMFDMs on the other hand are getting popularity because of their ease in the construction for multi-dimensional problems. See for example [6, 13, 14] and the references therein. To this end, in this paper we design and analyze a FMFDM for a SPDPPDE described in (1.1)–(1.4). This problem has been solved earlier by Ansari et al. in [1]. Unlike the work in [1], the proposed approach has better convergence properties. Moreover, by adding some novel proofs for the *a priori* estimates, we strengthen the mathematical theory related to such problems.

The rest of the paper is organized as follows. In Section 2, we derive estimates for the bounds on the solution $u(t,x)$ and its derivatives. Section 3 deals with the construction of the FMFDM which is analyzed in Section 4. In Section 5, we illustrate the performance of this method through a test example and compare the results with those obtained by a standard finite difference method. These results are discussed in Section 6 where we also provide some concluding remarks and scope for future works.

2. QUALITATIVE PROPERTIES OF THE SOLUTION

In this section we find estimates for the bounds on the solution $u(t,x)$ and its partial derivatives using method of steps [2].

Let us assume that the function $u(t,x) \in C^{3+\alpha,4+\beta}(\overline{\Omega})$ where $0 < \alpha, \beta < 1$.

Let $T_\ell = [(\ell-1)\tau, \ell\tau]$ and let $\Omega_\ell = T_\ell \times (0,1)$ for $\ell = 0, \ldots, K$. Also, let $u_\ell(t,x)$ be the restriction of $u(t,x)$ on Ω_ℓ, that is,

$$u_\ell(t,x) = u(t,x)|_{(t,x)\in\overline{\Omega}_\ell}, \quad \ell = 1, \ldots, K.$$

Let $(\Pi_L)_\ell$ and $(\Pi_R)_\ell$ be the sets $T_\ell \times \{0\}$ and $T_\ell \times \{1\}$, respectively, and let $\partial\Omega_\ell = \{(\ell-1)\tau\} \times [0,1]$.

In Ω_ℓ problem (1.1)–(1.4) is transformed to a sequence of K singularly-perturbed parabolic partial differential equations given by

$$\frac{\partial u_\ell(t,x)}{\partial t} - \varepsilon\frac{\partial^2 u_\ell(t,x)}{\partial x^2} + a_\ell(t,x)u_\ell(t,x) = f_\ell(t,x) - b(x)u_{\tau,\ell}(t,x), \quad (t,x) \in \overline{\Omega_\ell}, \tag{2.1}$$

with the initial condition

$$u_\ell((\ell-1)\tau, x) = u_{\ell-1}((\ell-1)\tau, x), x \in [0,1] \tag{2.2}$$

and boundary conditions

$$u_\ell(t,0) = \Gamma_L(t), t \in T_\ell \tag{2.3}$$

and

$$u_\ell(t,1) = \Gamma_R(t), t \in T_\ell, \tag{2.4}$$

for $\ell = 1, \ldots, K$.

The function $u_{\tau,\ell}(t,x)$ is given by

$$u_{\tau,\ell}(t,x) = u_{\ell-1}(t-\tau, x), \text{ for } (t,x) \in \overline{\Omega}_\ell.$$

In the presentation below, C_ℓ and C will denote positive constants that are always independent of ε (and the mesh step sizes used in the later sections).

Following lemma presents bounds on the solution function $u(t,x)$:

Lemma 2.1. *If the initial function $u_0(t,x)$ is bounded by a constant at $t = 0$, then there exists a positive constant C such that $|u(t,x)| \le C$ for all $(t,x) \in \overline{\Omega}$.*

Proof. The solution function $u(t,x)$ satisfies the compatibility conditions at the two corners $(0,0)$ and $(0,1)$, so does the function $u_1(t,x)$. This guarantees that

$$|u_1(t,x) - u_0(0,x)| \le M_1 t,$$

where M_1 is a positive constant that is independent of ε. Hence,

$$|u_1(t,x)| - |u_0(0,x)| \le |u_1(t,x) - u_0(0,x)| \le M_1 t \le M_1\tau \Rightarrow |u_1(t,x)| \le C_1,$$

where C_1 is a constant. This proves that $u_1(t,x)$ is bounded by C_1 in Ω_1.

In Ω_ℓ, $\ell = 2,\dots,K$, the continuity of $u(t,x)$ implies that

$$u_\ell((\ell-1)\tau, x) = u_{\ell-1}((\ell-1)\tau, x), \quad x \in [0,1].$$

Then by using a similar argument as the above, we have

$$|u_\ell(t,x)| \le C_\ell, \quad \ell = 1,\dots,K.$$

Let $C = \max_\ell\{C_\ell\}$, $\ell = 1,\dots,K$, then

$$|u(t,x)| \le C,$$

which completes the proof. □

Now, we prove that problem (1.1)–(1.4) satisfies a continuous maximum principle.

Lemma 2.2 (Continuous Maximum principle)**.** *Let $\Phi(t,x)$ be a sufficiently smooth function satisfying $\Phi(t,x) \ge 0$ on $\partial\Omega$, then $L_\varepsilon\Phi(t,x) \ge 0$ in $\overline{\Omega}$ implies $\Phi(t,x) \ge 0$ for all $(t,x) \in \overline{\Omega}$.*

Proof. To begin with, let us define the differential operator L_ε in (1.1) by

$$L_\varepsilon \equiv \frac{\partial}{\partial t} - \varepsilon\frac{\partial^2}{\partial x^2} + a(t,x).$$

First we prove that the lemma is satisfied in Ω_1 and then we generalize the proof for Ω_ℓ.

In Ω_1, we assume that the function $\Phi(t,x)$ takes its minimum value at a point (t_1^*, x_1^*) and this minimum is negative, i.e.,

$$\Phi(t_1^*, x_1^*) = \min_{(t,x)\in\overline{\Omega}_1} \Phi(t,x) < 0,$$

then

$$\frac{\partial\Phi(t_1^*, x_1^*)}{\partial t} = \frac{\partial\Phi(t_1^*, x_1^*)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial^2\Phi(t_1^*, x_1^*)}{\partial x^2} > 0.$$

Hence,

$$L_\varepsilon\Phi(t_1^*, x_1^*) = -\varepsilon\Phi_{xx}(t_1^*, x_1^*) + a(t_1^*, x_1^*)\Phi(t_1^*, x_1^*) < 0,$$

which is a contradiction and therefore,

$$\Phi(t,x) \ge 0 \quad \text{for all } (t,x) \in \overline{\Omega}_1.$$

This implies that $\Phi(\tau, x) \ge 0$.

Similarly, by using the result $\Phi(\tau,x) \ge 0$ along with $\Phi(t,0) \ge 0$, $\Phi(t,1) \ge 0$, $t \in T_2$ and $L_\varepsilon\Phi(t,x) \ge 0 \in \overline{\Omega}_2$ we obtain

$$\Phi(t,x) \ge 0 \quad \text{for all } (t,x) \in \overline{\Omega}_2,$$

and in general, given that $\Phi((\ell-1)\tau, x) \geq 0$ along with $\Phi(t,0) \geq 0$, $\Phi(t,1) \geq 0$, $t \in T_\ell$ and $L_\varepsilon \Phi(t,x) \geq 0$ in $\overline{\Omega}_\ell$ gives the result that

$$\Phi(t,x) \geq 0 \quad \text{for all } (t,x) \in \overline{\Omega}_\ell.$$

Proceeding in this manner, finally we get that

$$\Phi(t,x) \geq 0 \quad \text{for all } (t,x) \in \cup_{\ell=1}^{K} \overline{\Omega}_\ell = \overline{\Omega}. \qquad \square$$

The following theorem gives the bounds on the derivatives of the solution.

Theorem 2.3. *Let* $b(x) \in C^{4+\beta}([0,1])$, $f(t,x) \in C^{3+\alpha,4+\beta}(\overline{\Omega})$, $u_0(t,x) \in C^{3+\alpha,4+\beta}(\overline{\Omega})$, $\Gamma_L, \Gamma_R \in C^{3+\alpha}([0,T])$ *and* $u(t,x) \in C^{3,4}(\overline{\Omega})$, *where* $\alpha, \beta \in (0,1)$. *Then, we have*

$$\left| \frac{\partial^{i+j} u(t,x)}{\partial t^i \partial x^j} \right| \leq C \left(1 + \varepsilon^{1-j/2} + \varepsilon^{-j/2} \left(e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}} \right) \right), \tag{2.5}$$

for all the integers i *and* j *such that* $0 \leq 2i + j \leq 6$.

Proof. To find estimates for the bounds on the solution function $u(t,x)$ and its partial derivatives, we consider the stretched variable $\tilde{x} = x/\sqrt{\varepsilon}$ which transforms problem (1.1)–(1.4) into the following delayed parabolic partial differential equation

$$\frac{\partial \tilde{u}}{\partial t} - \frac{\partial \tilde{u}}{\partial \tilde{x}^2} + \tilde{a}(t,\tilde{x})\tilde{u} = \tilde{f} - \tilde{b}(\tilde{x})\tilde{u}(t-\tau,\tilde{x}) \tag{2.6}$$
$$(t,\tilde{x}) \in \widetilde{\Omega} = [0,T] \times [0, 1/\sqrt{\varepsilon}],$$

with the initial data

$$\tilde{u}(t,\tilde{x}) = u_0(t,\tilde{x}), \quad (t,\tilde{x}) \in [-\tau, 0] \times \left[0, \frac{1}{\sqrt{\varepsilon}}\right] \tag{2.7}$$

and boundary conditions

$$\tilde{u}(t,0) = \Gamma_L(t) \tag{2.8}$$

and

$$\tilde{u}\left(t, \frac{1}{\sqrt{\varepsilon}}\right) = \Gamma_R(t) \tag{2.9}$$

which by the method of steps can be transformed to a sequence of K parabolic partial differential equations of the form

$$\frac{\partial \tilde{u}_\ell}{\partial t} - \frac{\partial \tilde{u}_\ell}{\partial \tilde{x}^2} + \tilde{a}(t,\tilde{x})\tilde{u}_\ell = \tilde{f}_\ell - \tilde{b}(\tilde{x})\tilde{u}_\ell(t-\tau,\tilde{x}) \tag{2.10}$$
$$(t,\tilde{x}) \in \widetilde{\Omega}_\ell \equiv T_\ell \times \left[0, \frac{1}{\sqrt{\varepsilon}}\right],$$

with the initial data

$$\tilde{u}_\ell(t-\tau,\tilde{x}) = \tilde{u}_{\ell-1}(t-\tau,\tilde{x}), \quad (t,\tilde{x}) \in T_\ell \times \left[0, \frac{1}{\sqrt{\varepsilon}}\right] \tag{2.11}$$

and boundary conditions

$$\tilde{u}_\ell(t,0) = \Gamma_L(t), \quad t \in T_\ell \tag{2.12}$$

and

$$\tilde{u}_\ell\left(t, \frac{1}{\sqrt{\varepsilon}}\right) = \Gamma_R(t), \quad t \in T_\ell, \tag{2.13}$$

for $\ell = 1, \ldots, K$.

As is mentioned in [7] that problem (2.10)–(2.13) defined on $\widetilde{\Omega}_\ell$ is independent of ε, hence, the solution $\tilde{u}_\ell(t, \tilde{x})$ and its partial derivatives with respect to both t and $\tilde{x}$ must satisfy

$$\left|\frac{\partial^{i+j}\tilde{u}_\ell(t,\tilde{x})}{\partial t^i \partial \tilde{x}^j}\right| \le \widetilde{C}_\ell, \tag{2.14}$$

for all the non-negative integers i and j such that $2i + j \le 6$. In terms of the upstretched variable, (2.14) is reduced to

$$\left|\frac{\partial^{i+j}u_\ell(t,x)}{\partial t^i \partial x^j}\right| \le C_\ell \varepsilon^{-j/2}, \quad 0 \le 2i + j \le 6. \tag{2.15}$$

This implies that

$$\left|\frac{\partial^{i+j}u(t,\tilde{x})}{\partial t^i \partial x^j}\right| \le C \varepsilon^{-j/2},$$

for all the non-negative integers i and j such that $2i + j \le 6$.

The above bounds do not show the explicit dependence on the boundary layer solutions. Therefore, to obtain stronger estimates for the bounds on the solution function $u(t,x)$ and its partial derivatives, using the standard approaches, e.g., these given in [6, 7] for singular perturbation problems.

We decompose the solution $u(t,x)$ into its smooth and singular components $v(t,x)$ and $w(t,x)$ respectively, that is,

$$u(t,x) = v(t,x) + w(t,x),$$

where the function $v(t,x)$ satisfies

$$\frac{\partial v(t,x)}{\partial t} - \varepsilon \frac{\partial^2 v(t,x)}{\partial x^2} = f(t,x) - b(x)v(t-\tau,x), \quad (t,x) \in \Omega, \tag{2.16}$$

$$v(0,x) = u_0(0,x), \quad x \in (0,1), \tag{2.17}$$

and the values of the function $v(t,x)$ at $x = 0$ and $x = 1$ are to be specified later such that the bounds on the first two partial derivatives of v with respect to x are independent of ε. The two terms asymptotic expansion for the smooth component $v(t,x)$ is

$$v(t,x) = v_0(t,x) + \varepsilon v_1(t,x),$$

where the function $v_0(t,x)$ satisfies the reduced problem

$$\frac{\partial v_0(t,x)}{\partial t} = f(t,x) - b(x)v_0(t-\tau,x), \quad (t,x) \in \overline{\Omega}, \tag{2.18}$$

$$v_0(0,x) = u_0(0,x), \quad x \in (0,1), \tag{2.19}$$

whereas the function $v_1(t,x)$ satisfies

$$\frac{\partial v_1(t,x)}{\partial t} - \varepsilon\frac{\partial^2 v_1(t,x)}{\partial x^2} = -b(x)v_1(t-\tau,x) + \frac{\partial^2 v_0(t,x)}{\partial x^2}, \quad (t,x) \in \overline{\Omega}$$
$$v_1(t,x) = 0, \quad \text{for } (t,x) \in \partial\Omega.$$

On the other hand, the singular component $w(t,x)$ solves the problem

$$\frac{\partial w(t,x)}{\partial t} - \varepsilon\frac{\partial^2 w(t,x)}{\partial x^2} = -b(x)w(t-\tau,x), \quad (t,x) \in \Omega \tag{2.20}$$
$$w(0,x) = 0, \tag{2.21}$$
$$w(t,0) = u(t,0) - v(t,0), \tag{2.22}$$
$$w(t,1) = u(t,1) - v(t,1) \tag{2.23}$$

and is further decomposed into the left boundary layer solution $w_L(t,x)$ and the right boundary layer solution $w_R(t,x)$ respectively. The component w_L satisfies

$$\frac{\partial w_L(t,x)}{\partial t} - \varepsilon\frac{\partial^2 w_L(t,x)}{\partial x^2} = -b(x)w_L(t-\tau,x), \quad (t,x) \in \overline{\Omega}, \tag{2.24}$$
$$w_L(t,x) = 0, \quad \text{for } (t,x) \in [-\tau,0] \times [0,1], \tag{2.25}$$
$$w_L(t,0) = \Gamma_L(t) - v_0(t,0), \quad \text{for } (t,x) \in [0,T] \times \{0\}, \tag{2.26}$$
$$w_L(t,1) = 0, \quad \text{for } t \in ([0,T] \tag{2.27}$$

and the component w_R satisfies

$$\frac{\partial w_R(t,x)}{\partial t} - \varepsilon\frac{\partial^2 w_R(t,x)}{\partial x^2} = -b(x)w_R(t-\tau,x), \quad (t,x) \in \overline{\Omega}, \tag{2.28}$$
$$w_R(t,x) = 0, \quad \text{for } (t,x) \in [-\tau,0] \times [0,1], \tag{2.29}$$
$$w_R(t,0) = 0, \quad \text{for } (t,x) \in ([0,T], \tag{2.30}$$
$$w_R(t,1) = \Gamma_R(t) - v_0(t,1), \quad \text{for } (t,x) \in [0,T] \times \{1\}. \tag{2.31}$$

We find estimates for each component that belongs to either the smooth component v or the singular component w.

The method of steps applied in this case, suggests that the function $v_0(t,x)$ should be written as a union of functions $(v_0)_\ell(t,x)$ each is defined on $\overline{\Omega}_\ell$ and satisfies a problem of the form

$$\frac{\partial (v_0)_\ell(t,x)}{\partial t} = f_\ell(t,x) - b(x)(v_0)_\ell(t-\tau,x), \quad (v_0)_0(0,x) = u_0(0,x), \quad (t,x) \in \Omega_\ell.$$

Since each function $(v_0)_\ell$ is independent of ε, then for some constant C_ℓ the following estimate is satisfied

$$\left|\frac{\partial^{i+j}(v_0)_\ell}{\partial t^i \partial x^j}\right| \le C_\ell.$$

By taking $C = \max_{\ell}\{C_\ell\}$, $\ell = 1, \ldots, K$, the following estimates for the bounds on $v_0(t,x)$ and its partial derivatives is obtained

$$\left|\frac{\partial^{i+j} v_0}{\partial t^i \partial x^j}\right| \leq C, \tag{2.32}$$

for all the integers i and j such that $0 \leq 2i + j \leq 6$.

Using the above procedure and the fact that the equation in $v_1(t,x)$ has the same form as that for $u(t,x)$, we obtain

$$\left|\frac{\partial^{i+j} v_1}{\partial t^i \partial x^j}\right| \leq C\varepsilon^{-\frac{j}{2}}. \tag{2.33}$$

By using the estimates (2.32) and (2.33), we proved the following lemma.

Lemma 2.4. *The partial derivatives of $v(t,x)$ satisfy*

$$\left|\frac{\partial^{i+j} v}{\partial t^i \partial x^j}\right| \leq C\left(1 + \varepsilon^{1-\frac{j}{2}}\right). \tag{2.34}$$

for all the integers i and j such that $0 \leq 2i + j \leq 6$

In the following two lemmas we give bounds on $w_L(t,x)$ and $w_R(t,x)$ proof of which follows the barrier function approach described in [3] and [5]. □

Lemma 2.5. *The partial derivatives of $w_L(t,x)$ satisfy*

$$\left|\frac{\partial^{i+j} w_L}{\partial t^i \partial x^j}\right| \leq C\varepsilon^{-\frac{j}{2}} e^{-\frac{x}{\sqrt{\varepsilon}}}, \quad (t,x) \in \overline{\Omega}. \tag{2.35}$$

for all the integers i and j such that $0 \leq 2i + j \leq 6$.

Proof. We transform problem (2.24)–(2.27) to a sequence of K singularly perturbed parabolic partial differential equations of the form

$$\frac{\partial (w_L)_\ell(t,x)}{\partial t} - \varepsilon \frac{\partial^2 (w_L)_\ell(t,x)}{\partial x^2} = -b(x)(w_L)_\ell(t-\tau,x), \quad (t,x) \in \overline{\Omega}_\ell, \tag{2.36}$$

$$(w_L)_\ell(t,0) = \Gamma_L(t) - (v_0)_\ell(t,0), \quad \text{for } (t,x) \in T_\ell \times \{0\}, \tag{2.37}$$

$$(w_L)_\ell(t,x) = 0, \quad \text{for } (t,x) \in (T_\ell \times \{1\}) \cup (\{0\} \times [0,1]). \tag{2.38}$$

In each Ω_ℓ we define a barrier function

$$\Phi_\ell^{\pm}(t,x) = C_\ell\, e^{-\frac{x}{\sqrt{\varepsilon}}} \pm (w_L)_\ell(t,x).$$

It is clear that $\Phi_\ell^{\pm}(t,x) \geq 0$ for all $(t,x) \in \partial\Omega_\ell$ and is satisfying

$$L_\varepsilon \Phi_\ell^{\pm}(t,x) \geq 0,$$

for all $(t,x) \in \overline{\Omega}_\ell$. Then by Lemma 2.2, we have

$$\Phi_\ell^{\pm}(t,x) \geq 0, \quad \text{for all } (t,x) \in \overline{\Omega}_\ell,$$

which implies that

$$|(w_L)_\ell(t,x)| \le C_\ell e^{-\frac{x}{\sqrt{\varepsilon}}}, \quad (t,x) \in \overline{\Omega}_\ell.$$

By taking $C = \max_\ell \{C_\ell\},\ \ell = 1,\dots,K$ we obtain the estimates

$$|w_L(t,x)| \le C e^{-\frac{x}{\sqrt{\varepsilon}}}, \quad (t,x) \in \overline{\Omega}. \tag{2.39}$$

Now the problem in w_L also satisfies a continuous maximum principle and therefore, by using the transformation $\tilde{x} = x/\sqrt{\varepsilon}$ for problem (2.36)–(2.38) and the same technique that was used for finding bounds on the transformed problem (2.6)–(2.9), we obtain

$$\left|\frac{\partial^{i+j} w_L}{\partial t^i \partial x^j}\right| \le C\,|w_L(t,x)| \le C\varepsilon^{-\frac{j}{2}} e^{-\frac{x}{\sqrt{\varepsilon}}}. \tag{2.40}$$

□

Lemma 2.6. *The partial derivatives of $w_L(t,x)$ satisfy*

$$\left|\frac{\partial^{i+j} w_R}{\partial t^i \partial x^j}\right| \le C\varepsilon^{-\frac{j}{2}} e^{-\frac{1-x}{\sqrt{\varepsilon}}}, \quad (t,x) \in \overline{\Omega}, \tag{2.41}$$

for all the integers i and j such that $0 \le 2i + j \le 6$.

Proof. Analogous to the proof of Lemma 2.5.

From the two lemmas 2.5 and 2.6 we see that

Lemma 2.7. *The partial derivatives of $w(t,x)$ satisfy*

$$\left|\frac{\partial^{i+j} w}{\partial t^i \partial x^j}\right| \le C\varepsilon^{-\frac{j}{2}} \left(e^{-\frac{x}{\sqrt{\varepsilon}}} + e^{-\frac{1-x}{\sqrt{\varepsilon}}}\right), \quad (t,x) \in \overline{\Omega}. \tag{2.42}$$

for all the integers i and j such that $0 \le 2i + j \le 6$.

Proof. The proof is accomplished by using the decomposition $w = w_L + w_R$ and the estimates (2.35) and (2.41).

Finally, the proof of the theorem is completed by using the estimates in Lemma 2.4 and 2.7. □

The above bounds on the solution will be used later in the analysis of the numerical method.

3. CONSTRUCTION OF THE FITTED MESH METHOD

Let N_x be a positive integer and let

$$\sigma = \min\{0.25, 2\sqrt{\varepsilon}\ln N_x\}$$

be the transition point. Let $N_x^\sigma = N_x/4$. To generate the Shishkin mesh we divide each of the subintervals $[0,\sigma]$ and $[1-\sigma,1]$ into N_x^σ subintervals through the points

$x_0, \ldots, x_{N_x^\sigma}$ and $x_{3N_x^\sigma}, \ldots, x_{N_x}$, respectively, whereas the subinterval $[\sigma, 1-\sigma]$ is divided into $2N_x^\sigma$ subintervals through the points $x_{N_x^\sigma}, \ldots, x_{3N_x^\sigma}$. The associated step size $h_m = x_{m+1} - x_m$ is then given by

$$h_m = \begin{cases} 4\sigma/Nx, & \text{if } m \in \{0, \ldots, N_x^\sigma - 1\} \\ 2(1-2\sigma)/N_x, & \text{if } m \in \{N_x^\sigma, \ldots, 3N_x^\sigma\} \\ 4\sigma/Nx, & \text{if } m \in \{3N_x^\sigma + 1, \ldots, N_x\}. \end{cases}$$

Let N_t be any positive integer and $k = T/N_t$. We divide the space $[0, T]$ into N_t subintervals through the points $t_0 = 0, \ldots, t_{N_t} = T$ where $t_{n+1} = t_n + k$. We assume that $T = K\tau$ for some positive integer K and that N_t is chosen in such a way that $\tau = t_s = sk$ for some positive integer s.

Let Ω^{N_t} denotes $\{t_n : n = 0, \ldots, N_t\}$, $\Omega_\sigma^{N_x}$ denotes $\{x_m : m = 0, \ldots, N_x\}$, where $N_x \geq 4$ and N denotes (N_t, N_x), then the fitted piecewise uniform mesh Ω_σ^N is then given by the following tensor product grid

$$\Omega_\sigma^N = \Omega^{N_t} \times \Omega_\sigma^{N_x}.$$

Let U_m^n be the numerical approximation of $u(t_n, x_m)$, $D_x^+ U_m^n$, $D_x^- U_m^n$ and δ_x^2 be the forward, backward and central difference operators defined as

$$D_x^+ U_m^n = \frac{U_{m+1}^n - U_m^n}{x_{m+1} - x_m},$$

$$D_x^- U_m^n = \frac{U_m^n - U_{m-1}^n}{x_m - x_{m-1}}$$

and

$$\delta_x^2 U_m^n = \frac{(D_x^+ - D_x^-) U_m^n}{x_{m+1} - x_{m-1}}.$$

Furthermore, the approximations of the functions $a(t, x)$ and $f(t, x)$ at a local grid point (t_n, x_m) are denoted by a_m^n and f_m^n, respectively, whereas the value of the function $b(x)$ at x_m is denoted by b_m.

Our fitted mesh finite difference method (FMFDM) is then consists of the Crank-Nicolson discretization for problem (1.1)–(1.4) on the Shishkin mesh (described above) and reads as

$$\begin{aligned} D_t^+ U_m^n - \frac{\varepsilon}{2}\left(\delta_x^2 U_m^n + \delta_x^2 U_m^{n+1}\right) + \frac{1}{2}(a_m^n U_m^n + a_m^{n+1} U_m^{n+1}) &= \frac{1}{2}\left(f_m^n + f_m^{n+1}\right) \\ &\quad -\frac{1}{2}\left(b_m H_m^n + b_m H_m^{n+1}\right), \end{aligned} \tag{3.1}$$

along with the initial data

$$U_m^0 = u_0(0, x_m) \tag{3.2}$$

and boundary conditions

$$U_0^n = \Gamma_L(t_n, 0) \tag{3.3}$$

and

$$U_{N_x}^n = \Gamma_R(t_n, 1). \tag{3.4}$$

The term H_m^n in (3.1) is called the history term and is given by

$$H_m^n = \begin{cases} u_0(t_n - \tau, x_m), & \text{if } t_n < \tau, \\ U_m^{n-s}, & \text{if } t_n \geq \tau. \end{cases} \tag{3.5}$$

Expanding (3.1), we obtain

$$\begin{aligned} &\frac{U_m^{n+1} - U_m^n}{k} - \frac{\varepsilon}{2} \frac{\frac{U_{m+1}^{n+1} - U_m^{n+1}}{h_m} - \frac{U_m^{n+1} - U_{m-1}^{n+1}}{h_{m-1}} + \frac{U_{m+1}^n - U_m^n}{h_m} - \frac{U_m^n - U_{m-1}^n}{h_{m-1}}}{\frac{h_m + h_{m-1}}{2}} \\ &\qquad + \frac{1}{2}\left(a_m^n U_m^n + a_m^{n+1} U_m^{n+1}\right) = \frac{1}{2}\left((f_m^n + f_m^{n+1}) - b_m(H_m^n + H_m^{n+1})\right) \\ &\qquad m = 1, \ldots, N_x - 1; \quad n = 0, \ldots, N_t - 1, \end{aligned}$$

which can be simplified to

$$\begin{aligned} &-\frac{\varepsilon}{h_{m-1}(h_m + h_{m-1})} U_{m-1}^{n+1} + \left(\frac{1}{k} + \frac{\varepsilon}{h_m h_{m-1}} + \frac{a_m^{n+1}}{2}\right) U_m^{n+1} - \frac{\varepsilon}{h_m(h_m + h_{m-1})} U_{m+1}^{n+1} \\ &= \frac{\varepsilon}{h_{m-1}(h_m + h_{m-1})} U_{m-1}^n + \left(\frac{1}{k} - \frac{\varepsilon}{h_m h_{m-1}} - \frac{a_m^n}{2}\right) U_m^n + \frac{\varepsilon}{h_m(h_m + h_{m-1})} U_{m+1}^n \\ &\qquad + \frac{1}{2}\left((f_m^n + f_m^{n+1}) - b_m\left(H_m^n + H_m^{n+1}\right)\right). \end{aligned} \tag{3.6}$$

Equation (3.6) can further be written as a linear system of the form

$$T_L U^{n+1} = T_R U^n + \frac{1}{2}\left((f^n + f^{n+1}) - b \star \left(H^n + H^{n+1}\right) + \left(g^n + g^{n+1}\right)\right), \tag{3.7}$$

for $n = 1, \ldots, N_t - 1$, where $\star$ denotes the componentwise multiplication of the two vectors and T_L and T_R are two tridiagonal matrices given by

$$T_L = \text{Tri}\left(-\frac{\varepsilon}{h_{m-1}(h_m + h_{m-1})}, \frac{1}{k} + \frac{\varepsilon}{h_m h_{m-1}} + \frac{a_m^{n+1}}{2}, -\frac{\varepsilon}{h_m(h_m + h_{m-1})}\right),$$

and

$$\begin{aligned} &T_R = \text{Tri}\left(\frac{\varepsilon}{h_{m-1}(h_m + h_{m-1})}, \frac{1}{k} - \frac{\varepsilon}{h_m h_{m-1}} - \frac{a_m^n}{2}, \frac{\varepsilon}{h_m(h_m + h_{m-1})}\right) \\ &m = 1, \ldots, N_x. \end{aligned}$$

Furthermore, the vector g^n is given by

$$g^n = \left[\frac{\varepsilon(U_0^n + U_0^{n+1})}{h_0(h_1 + h_0)}, 0, \ldots, 0, \frac{\varepsilon(U_{N_x}^n + U_{N_x}^{n+1})}{h_{N_x-1}(h_{N_x-2} + h_{N_x-1})}\right]^T \in \mathbb{R}^{N_x - 1}.$$

The numerical solution is obtained by solving (3.7) along with (3.2), (3.3), (3.4) and (3.5).

4. CONVERGENCE OF THE METHOD

The convergence analysis presented in this section is based on some of approaches used in [7].

Let Φ_m^n be any mesh function on Ω_σ^N and from (3.1) we define the discrete operator L_ε^N at (t_n, x_m) as

$$L_\varepsilon^N \Phi_m^n \equiv D^+\Phi_m^n - \frac{\varepsilon}{2}\left(\delta_x^2\Phi_m^n + \delta_x^2\Phi_m^{n+1}\right) + \frac{1}{2}\left(a_m^n\Phi_m^n + a_m^{n+1}\Phi_m^{n+1}\right).$$

We show that the following discrete maximum principle is satisfied.

Lemma 4.1. *Assume that $\Phi_m^n \geq 0$ on the boundaries of Ω_σ^N. Then $L_\varepsilon^N\Phi_m^n \geq 0$ on Ω_σ^N implies that $\Phi_m^n \geq 0$ on Ω_σ^N.*

Proof. Assume that $\Phi_m^n < 0$ for some n, m, and its minimum denoted by Φ^* is achieved at a point (t_{n^*}, x_{m^*}). Then $D^+\Phi^* = 0$ and $\delta_x^2\Phi^* > 0$.

Now we can choose N_t big enough in order to have either $\Phi_{m^*}^{n^*+1} < 0$ or $\left|\Phi_{m^*}^{n^*}\right| > \left|\Phi_{m^*}^{n^*+1}\right|$ and $\delta_x^2\Phi_{m^*}^{n^*+1} \geq 0$. Then

$$L_\varepsilon^N\Phi_{m^*}^{n^*} < 0,$$

which is a contradiction. Thus $\Phi_m^n \geq 0$ at any mesh point (t_n, x_m).

We also note that the above mesh function satisfies the stability estimate provided in the following lemma.

Lemma 4.2. *Let Φ be any mesh function satisfying $\Phi_m^n = 0$ on $\partial\Omega_\sigma^N$ and $\bar{a} = \min\limits_{m,n}\{a_m^n\}$, $m = 0, \ldots, N_x$ and $n = 0, \ldots, N_t$. Then*

$$\begin{cases} |\Phi_m^n| \leq (1+T)\max\left|L_\varepsilon^N\Phi_m^n\right|, & \text{if } \bar{a} = 0 \\ |\Phi_m^n| \leq \dfrac{1+T}{\bar{a}}\max\left|L_\varepsilon^N\Phi_m^n\right|, & \text{if } \bar{a} > 0 \end{cases}$$

Proof. Let $\widetilde{M}$ denotes $\max\limits_{m,n}\left|L_\varepsilon^N\Phi_m^n\right|$. We define a barrier function $(\Psi_m^n)^\pm$ as

$$(\Psi_m^n)^\pm = \begin{cases} (1+t)\widetilde{M} \pm \Phi_m^n, & \text{if } \bar{a} = 0 \\ \dfrac{1+T}{\bar{a}}\widetilde{M} \pm \Phi_m^n, & \text{if } \bar{a} > 0 \end{cases}$$

Since $\Phi_m^n = 0$ on $\partial\Omega_\sigma^N$ and $\widetilde{M} > 0$ on $\partial\Omega_\sigma^N$, then on $\partial\Omega_\sigma^N$ we have

$$(\Psi_m^n)^\pm = \begin{cases} (1+t)\widetilde{M}, & \text{if } \bar{a} = 0 \\ \dfrac{1+T}{\bar{a}}\widetilde{M}, & \text{if } \bar{a} > 0 \end{cases} \geq \begin{cases} \widetilde{M}, & \text{if } \bar{a} = 0 \\ \dfrac{1+T}{\bar{a}}\widetilde{M}, & \text{if } \bar{a} > 0 \end{cases} \geq 0$$

Now,

$$L_\varepsilon^N(\Psi_m^n)^\pm = \begin{cases} \widetilde{M} \pm L_\varepsilon^N\Phi_m^n, & \text{if } \bar{a} = 0 \\ \dfrac{(1+T)}{2\bar{a}}\widetilde{M}\left(a_m^n + a_m^{n+1}\right) \pm L_\varepsilon^N\Phi_m^n \geq (1+T)\widetilde{M} \pm L_\varepsilon\Phi_m^n & \text{if } \bar{a} > 0 \end{cases} \geq 0$$

on Ω_σ^N.

Using the discrete maximum principle, we have $(\Psi_m^n)^{\pm} \geq 0$ on Ω_σ^N. The proof is then completed by noticing that $0 \leq t \leq T$.

Now, we find an error estimate in approximating the exact solution $u(t_n, x_m)$ by the numerical solution U_m^n using the FMFDM. To simplify the notations, we denote the quantity $f(t_n, x_m) - b_m H_m^n$ by G_m^n and the values of a mesh function Φ at the boundaries of Ω by $\Phi(\partial\Omega_\sigma^N)$. That is,

$$\Phi(\partial\Omega_\sigma^N) = \Phi(t_n, x_m), \quad (t_n, x_m) \in \partial\Omega_\sigma^N.$$

We decompose the numerical solution U into its smooth and singular components V and W respectively, that is,

$$U = V + W,$$

where the smooth component V satisfies

$$L_\varepsilon V_m^n = \frac{1}{2}\left(G_m^n + G_m^{n+1}\right), \quad V\left(\partial\Omega_\sigma^N\right) = v\left(\partial\Omega_\sigma^N\right)$$

and the singular component W satisfies

$$L_\varepsilon W_m^n = 0, \quad W\left(\partial\Omega_\sigma^N\right) = u\left(\partial\Omega_\sigma^N\right) - v\left(\partial\Omega_\sigma^N\right).$$

The error at the point t_n, x_m is then given by

$$u(t_n, x_m) - U_m^n = v(t_n, x_m) - V_m^n + w(t_n, x_m) - W_m^n,$$

which by the triangle inequality implies that

$$|u(t_n, x_m) - U_m^n| = |v(t_n, x_m) - V_m^n| + |w(t_n, x_m) - W_m^n| \tag{4.1}$$

Thus,

$$\begin{aligned}
L_\varepsilon^N(V_m^n - v(t_n, x_m)) &= L_\varepsilon^N V_m^n - L_\varepsilon^N v(t_n, x_m) \\
&= \frac{1}{2}\left(G_m^n + G_m^{n+1}\right) - L_\varepsilon^N(v(t_n, x_m)) \\
&= \frac{1}{2}\left(G_m^n + G_m^{n+1}\right) - \left(D^+ - \frac{\partial}{\partial t}\right)v(t_n, x_m) \\
&\quad + \varepsilon\left(\frac{\delta_x^2 v(t_n, x_m) + \delta_x^2 v(t_{n+1}, x_m)}{2} - \frac{\partial^2}{\partial x^2}v(t_n, x_m)\right) \\
&= \frac{1}{2}\left(G_m^n + G_m^{n+1}\right) - \frac{N_t^{-2}}{12}\left(\varepsilon v_{xxttt}(\xi, x_m) + (av)_{ttt}(\xi, x_m) + f_{ttt}(\xi, x_m)\right) \\
&\quad + \begin{cases} \varepsilon\dfrac{h_{m+1} - h_m}{3}v_{xxx}(t_n, \zeta), & \text{if } x_m = \sigma \text{ or } x_m = 1 - \sigma \\ -\varepsilon\dfrac{h_{m+1}^2 - h_m h_{m+1} + h_m^2}{12}v_{xxxx}(t_n, \zeta), & \text{otherwise} \end{cases}
\end{aligned}$$

which implies that

$$
\begin{aligned}
|L_\varepsilon^N(V_m^n - v(t_n,x_m))| &\le \frac{N_t^{-2}}{12}\left(\varepsilon\,|v_{xxttt}| + |a_{ttt}|\,|v| + |a(t_n,x_m)|\,|v_{ttt}| + |f_{ttt}|\right)(\xi,x_m) \\
&\quad + \begin{cases} \varepsilon\left|\frac{h_m - h_{m-1}}{3}\right| |v_{xxx}(t_n,\zeta)|, & \text{if } x_m = \sigma \text{ or } x_m = 1-\sigma \\ \varepsilon\left|\frac{h_m^2 - h_m h_{m+1} + h_{m+1}^2}{12}\right| |v_{xxxx}(t_n,\zeta)| & \text{otherwise,} \end{cases} \\
&\le \begin{cases} \varepsilon\left|\frac{h_m - h_{m-1}}{3}\right| |v_{xxx}(t_n,\zeta)|, & \text{if } x_m = \sigma \text{ or } x_m = 1-\sigma \\ \varepsilon\left|\frac{h_m^2 - h_m h_{m+1} + h_{m+1}^2}{12}\right| |v_{xxxx}(t_n,\zeta)| & \text{otherwise,} \end{cases} \\
&\le \begin{cases} C\left(N_t^{-2} + N_x^{-1}\ln N_x\right), & \text{if } x_m = \sigma \text{ or } x_m = 1-\sigma \\ C\left(N_t^{-2} + N_x^{-2}\right), & \text{otherwise.} \end{cases} \qquad (4.2)
\end{aligned}
$$

Defining a barrier function

$$\phi(t_n,x_m) = C\left(\frac{\sigma}{\varepsilon}N_x^{-2}\theta(x_m) + (1+t_n)N_x^{-2} + t_n N_t^{-2}\right)$$

where

$$\theta(x) = \begin{cases} \frac{x}{\sigma}, & \text{if } 0 \le x \le \sigma \\ 1, & \text{if } \sigma \le x \le 1-\sigma \\ \dfrac{1-x}{\sigma}, & \text{if } 1-\sigma \le x \le 1 \end{cases}$$

and applying the discrete maximum principle (Lemma 4.2), we have

$$|V_m^n - v(t_n,x_m)| \le \begin{cases} C\left(N_t^{-2} + N_x^{-2}\ln^2 N_x\right), & \text{if } x_m = \sigma \text{ or } x_m = 1-\sigma \\ C\left(N_t^{-2} + N_x^{-2}\right), & \text{otherwise.} \end{cases} \qquad (4.3)$$

On the other hand, the singular component W is decomposed into its left boundary solution W_L and right boundary solution W_R, that is,

$$W = W_L + W_R$$

and hence the error can then be written as

$$W_m^n - w(t_n,x_m) = (W_L)_m^n - w_L(t_n,x_m) + (W_R)_m^n - w_R(t_n,x_m).$$

We estimate the errors $(W_L)^n_m - w_L(t_n, x_m)$ and $(W_R)^n_m - w_R(t_n, x_m)$, separately. We have

$$
\begin{aligned}
L^N_\varepsilon((W_L)^n_m - w_L(t_n, x_m)) &= -L^N_\varepsilon(w_L(t_n, x_m)) \\
&\leq -\left(D^+ - \frac{\partial}{\partial t}\right) w_L(t_n, x_m) \\
&\quad + \varepsilon\left(\frac{\delta^2_x w_L(t_n, x_m) + \delta^2_x w_L(t_{n+1}, x_m)}{2} - \frac{\partial^2}{\partial x^2} w_L(t_n, x_m)\right) \\
&= \frac{N_t^{-2}}{12}\left((w_L)_{xxttt} + (aw_L)_{ttt}\right)(\xi, x_m) \\
&\quad - \begin{cases} \varepsilon \dfrac{h_{m+1} - h_m}{3}(w_L)_{xxx}(t_n, \zeta), & \text{if } x_m = \sigma \text{ or } x_m = 1 - \sigma \\ -\varepsilon \dfrac{h^2_{m+1} - h_m h_{m+1} + h^2_m}{12}(w_L)_{xxxx}(t_n, \zeta), & \text{otherwise} \end{cases}
\end{aligned}
$$

By taking the absolute values of the two sides, applying the triangle inequality, using the estimates of the bounds on w_L from Lemma 2.5 and simplifying further, we obtain

$$
\left|L^N_\varepsilon((W_L)^n_m - w_L(t_n, x_m))\right| \leq C\left(N_t^{-2} + \left(N_x^{-1} \ln N_x\right)^2\right).
$$

Finally, applying Lemma 4.2, we get

$$
|(W_L)^n_m - w_L(t_n, x_m)| \leq C\left(N_t^{-2} + \left(N_x^{-1} \ln N_x\right)^2\right). \tag{4.4}
$$

Similarly, we can prove that

$$
|(W_R)^n_m - w_R(t_n, x_m)| \leq C\left(N_t^{-2} + \left(N_x^{-1} \ln N_x\right)^2\right). \tag{4.5}
$$

Combining (4.1), (4.3), (4.4) and (4.5), we have the following theorem.

Theorem 4.3. *The FMFDM* (3.1)–(3.4) *is convergent of order* $\mathcal{O}(N_t^{-2} + N_x^{-2} \ln^2 N_x)$ *in the sense that*

$$
\sup_{0<\varepsilon\leq 1} \max_{1\leq m,n\leq N-1} |u(t_n, x_m) - U^n_m| \leq C(N_t^{-2} + N_x^{-2} \ln^2 N_x).
$$

where U *is the numerical solution obtained by the FMFDM* (3.1)–(3.4) *and* N *is the total number of subintervals taken in either directions.*

5. NUMERICAL RESULTS

In this section we provide numerical results confirming the estimate given in Theorem 4.3. We also compare the results by applying the Crank-Nicolson's method on a uniform mesh throughout the region. The latter is referred to as a standard finite difference method (SFDM).

Example 5.1. Consider

$$\frac{\partial u(t,x)}{\partial t} - \varepsilon \frac{\partial^2 u(t,x)}{\partial x^2} = \frac{1}{2}\left(\left(2x\sqrt{\varepsilon}-\varepsilon\right)e^{-\left(t+x/\sqrt{\varepsilon}\right)} - \left(2x\sqrt{\varepsilon}+\varepsilon\right)e^{-\left(t+(1-x)/\sqrt{\varepsilon}\right)}\right) - 2e^{-1}u(t-1,x), \quad (t,x)\in[0,2]\times[0,1],$$

with the initial data

$$u(t,x) = (2+x^2)(e^{-(t+x/\sqrt{\varepsilon})} + e^{-(t+(1-x)/\sqrt{\varepsilon})}), \quad (t,x)\in[-\tau,0]\times[0,1],$$

and boundary conditions

$$u(t,0) = e^{-t} + e^{-t-1/\sqrt{\varepsilon}}, \quad t\in[0,2]$$

and

$$u(t,1) = \frac{3}{2}(e^{-t} + e^{-t-1/\sqrt{\varepsilon}}), \quad t\in[0,2].$$

The exact solution of the above problem is given by

$$u(t,x) = \left(2+x^2\right)\left(e^{-\left(t+x/\sqrt{\varepsilon}\right)} + e^{-\left(t+(1-x)/\sqrt{\varepsilon}\right)}\right).$$

By taking $N_t = N_x = N$, the maximum errors (denoted by $E_{N,\varepsilon}$) at all grid points are evaluated using the formula

$$E_{N,\varepsilon} := \max_{0\le m,\, n\le N} |u(t_n, x_m) - U_m^n|.$$

We also tabulate the errors

$$E_N = \max_{0<\varepsilon\le 1} E_{N,\varepsilon}.$$

These errors are presented in tables 1 and 2. The acronym SFDM in the caption of Table 1 stands for the standard finite difference method which is defined by (3.1)–(3.4) by setting $\sigma = 0.25$.

The numerical rates of convergence are computed using the formula [4]:

$$r_i \equiv r_{i,\varepsilon} := \log_2\left(E_{N_i,\varepsilon}/E_{2N_i,\varepsilon}\right), \quad i = 1,2,\cdots$$

whereas those of uniform convergence are computed using

$$R_N := \log_2\left(E_N/E_{2N}\right).$$

These rates are presented in Table 3.

6. CONCLUSIONS

In this paper, we constructed a fitted mesh finite difference method (FMFDM) based on the Crank-Nicolson method for solving a singularly perturbed delay parabolic partial differential equation. The method is analyzed for convergence. A test example is solved to confirm the theoretical estimates.

The proposed FMFDM is unconditionally stable and is converging with the order $\mathcal{O}(N_t^{-2} + N_x^{-2}\ln^2 N_x)$ which is an improvement over the estimate presented in Ansari

TABLE 1. Maximum Errors obtained by SFDM for Example 5.1 using $N_x = N_t = N$

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
1	6.64E-06	1.66E-06	4.15E-07	1.04E-07	2.59E-08	6.40E-09
10^{-2}	4.64E-04	1.16E-04	2.91E-05	7.26E-06	1.82E-06	4.54E-07
10^{-4}	3.09E-02	9.10E-03	2.48E-03	6.25E-04	1.57E-04	3.93E-05
10^{-6}	4.28E-03	1.63E-02	4.05E-02	3.83E-02	1.42E-02	3.76E-03
10^{-8}	4.31E-05	1.72E-04	6.89E-04	2.76E-03	1.08E-02	3.30E-02
10^{-10}	4.31E-07	1.72E-06	6.90E-06	2.76E-05	1.10E-04	4.41E-04
10^{-12}	4.31E-09	1.72E-08	6.90E-08	2.76E-07	1.10E-06	4.42E-06
10^{-14}	4.31E-11	1.72E-10	6.90E-10	2.76E-09	1.10E-08	4.42E-08
10^{-16}	4.31E-12	1.72E-11	6.90E-11	2.76E-10	1.10E-09	4.42E-10

TABLE 2. Maximum Errors obtained by FMFDM for Example 5.1 using $N_x = N_t = N$

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
1	6.64e-06	1.66e-06	4.15e-07	1.04e-07	2.59e-08	6.44e-09
10^{-1}	7.00e-05	1.75e-05	4.38e-06	1.09e-06	2.74e-07	6.84e-08
10^{-3}	4.08e-03	1.04e-03	2.61e-04	6.53e-05	1.63e-05	4.08e-06
10^{-4}	4.34e-03	1.49e-03	4.92e-04	1.56e-04	4.82e-05	1.46e-05
10^{-5}	4.28e-03	1.47e-03	4.85e-04	1.54e-04	4.76e-05	1.44e-05
10^{-6}	4.26e-03	1.47e-03	4.83e-04	1.53e-04	4.74e-05	1.43e-05
10^{-7}	4.25e-03	1.47e-03	4.82e-04	1.53e-04	4.73e-05	1.43e-05
10^{-8}	4.25e-03	1.46e-03	4.82e-04	1.53e-04	4.73e-05	1.43e-05
10^{-12}	4.25e-03	1.46e-03	4.82e-04	1.53e-04	4.73e-05	1.43e-05
10^{-13}	4.25e-03	1.46e-03	4.82e-04	1.53e-04	4.73e-05	1.43e-05
10^{-16}	4.25e-03	1.46e-03	4.82e-04	1.53e-04	4.73e-05	1.43e-05
E_N	4.25e-03	1.46e-03	4.82e-04	1.53e-04	4.73e-05	1.43e-05

TABLE 3. Rates of Convergence obtained by FMFDM for Example 5.1 using $N_x = N_t = N = 2^i$, $i = 6(1)10$

ε	r_1	r_2	r_3	r_4	r_5
1	2.00	2.00	2.00	2.00	2.01
10^{-1}	2.00	2.00	2.00	2.00	2.00
10^{-3}	1.98	1.99	2.00	2.00	2.00
10^{-4}	1.54	1.60	1.66	1.69	1.72
10^{-5}	1.54	1.60	1.66	1.69	1.72
10^{-6}	1.54	1.60	1.66	1.69	1.73
10^{-7}	1.54	1.60	1.66	1.69	1.73
10^{-8}	1.54	1.60	1.66	1.69	1.73
10^{-12}	1.54	1.60	1.66	1.69	1.73
10^{-13}	1.54	1.60	1.66	1.69	1.73
10^{-16}	1.54	1.60	1.66	1.69	1.73
R_N	1.54	1.60	1.66	1.69	1.73

et al. in [1] for the problem under consideration. These improved results can be seen from the results presented in Tables 2–3. For the sake of comparison, the results obtained by the corresponding standard finite difference method (the Crank-Nicolson method on uniform mesh) are presented in Table 1. One can see that the latter does not converges to a specific order.

A further improvement to the results can be made if we use the proposed method on a mesh of Bakhvalov type rather than a mesh of Shishkin type. Due the absence of the locking term in the error in the Bakhvalov mesh, one would expect the accuracy of order $\mathcal{O}(N_t^{-2} + N_x^{-2})$ if a Crank-Nicolson method is used on this mesh. We are currently investigating these issues.

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