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A ROBUST NUMERICAL METHOD FOR SINGULARLY PERTURBED SEMILINEAR CONVECTION-DIFFUSION PROBLEMS

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ABSTRACT. A singularly perturbed semilinear convection-diffusion problem is considered. The leading term is multiplied by a small positive parameter ε. The solution to this problem exhibits boundary layer at the left end of the domain. To solve this problem numerically, we develop a B-spline collocation method on a piecewise-uniform Shishkin mesh. The error analysis is given and the method is proved to be almost second-order convergent in the maximum norm uniformly in ε. Numerical results are presented in support of the theory.

Key Words. Singular perturbation; convection-diffusion problems; Boundary layer; B-spline collocation method; Shishkin mesh

AMS Subject Classification. 65L10, 65L12

1. INTRODUCTION

We consider the following singularly perturbed semilinear convection-diffusion equation

$$Lu(x) := -\varepsilon u''(x) - a(x)u'(x) + f(x, u(x)) = 0, \quad x \in \Omega = [0, 1] \tag{1.1}$$

subject to boundary conditions

$$u(0) = 0, \qquad u(1) = 0, \tag{1.2}$$

where $0 < \varepsilon \ll 1$ is a small parameter. We assume that a and f are sufficiently smooth functions with

$$a(x) \geq \gamma > 0 \quad \text{for } x \in \Omega, \qquad f_u(x, u) \geq f_* > 0 \quad \text{for } (x, u) \in \Omega \times \mathbb{R}. \tag{1.3}$$

With these assumptions, there exists a unique solution u to the problem (1.1)–(1.2); see [1]. The solution u generally has an exponential boundary layer at $x = 0$. The solution u and its derivatives can be bounded as follows [2]:

$$|u^{(m)}(x)| \leq C(1 + \varepsilon^{-m} \exp(-\gamma x/\varepsilon)) \quad \text{for } m = 0, \ldots, 4 \text{ and } x \in \Omega. \tag{1.4}$$

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Singular perturbation problems arise frequently in many areas of science and engineering such as the heat transfer problem with large Peclet numbers, Navier-Strokes flows with large Reynolds numbers, chemical reactor theory, aerodynamics, quantum mechanics, optimal control, etc. It is well known that classical numerical methods on uniform meshes are not adequate for solving such problems. Therefore special measures are needed to efficiently obtain good numerical approximations. Properly layer adapted meshes have been proven to overcome these difficulties ([3, 4, 5, 6]).

A wide class of spline approximation method for the numerical solution of singularly perturbed problems have been studied by various researchers. Some spline approximation methods for the numerical solution of nonlinear singularly perturbed boundary problem are given in [7, 8, 9, 10, 11, 12], and the references therein. Among the various classes of splines, the polynomial spline has received a greater attention primarily because it admits a B-splines basis which can be computed efficiently. Kadalbajoo and Gupta [13] gave a B-spline collocation method on a piecewise-uniform Shishkin mesh for singularly perturbed linear convection-diffusion problem and proved the almost second order uniform convergence of the method. Also they extended this method for singularly perturbed one-dimensional time dependent linear convection-diffusion problem ([14]).

We reformulate the problem (1.1)–(1.2) to an equivalent problem of the form

$$L_\varepsilon u(x) := -\varepsilon u''(x) - a(x)u'(x) + b(x)u(x) = g(x, u(x)), \quad x \in \ \Omega = [0, 1] \tag{1.5}$$

$$u(0) = 0, \qquad u(1) = 0, \tag{1.6}$$

where $b(x) \geq \beta > 0$ for $x \in \Omega$, and $g(x, u) = b(x)u(x) - f(x, u)$, that satisfies

$$|\, g(x, u_1) - g(x, u_2)| \leq M|\, u_1 - u_2|, \quad \forall\, x \in \Omega \quad \text{(Lipschitz condition)} \tag{1.7}$$

such that, for $K = (\min_{\forall x} 2b(x))^{-1}$, $1 - 6KM > 0$.

In this paper, we develop a B-spline collocation method on a piecewise-uniform Shishkin mesh for the numerical solution of the modified problem (1.5)–(1.6). It is interesting to note, that in all the numerical experiments, there is no difficulty in solving (1.1)–(1.2) directly by present B-spline collocation method, and they rendered exactly the same result as the modified form (1.5)–(1.6). The reformulation of the problem (1.1)–(1.2) to (1.5)–(1.6) is solely for the theoretical purpose. Two test problems are considered to demonstrate the efficiency of the proposed B-spline collocation method.

This paper is arranged as follows. In section 2, the B-spline collocation method on a piecewise-uniform Shishkin mesh is developed for the numerical solution of the problem (1.5)–(1.6). In section 3, the error analysis is given and the method is proved to be almost second order convergent in the maximum norm uniformly in ε. In section 4, numerical experiments are conducted to validate the theoretical results. Finally, discussion is included in section 5.

NOTATIONS: Throughout the paper we use C, to denote a generic positive constant independent of ε and the discretization parameter N. For a real valued function $y \in C(\Omega)$, define $\|y\|_\Omega = \max_{x\in\Omega} |y(x)|$. For a mesh function $y_N = (g_0, \ldots, g_N)$, define $\|y_N\| = \max_{0\le i\le N} |g_i|$, and denote the corresponding subordinate matrix norm in the same way.

2. THE DISCRETIZATION

In this section, we develop a B-spline collocation method on a piecewise-uniform Shishkin mesh for the numerical solution of singularly perturbed problem (1.5)–(1.6). First we construct a piecewise-uniform Shishkin mesh $\Omega^N = \{x_i\}_{i=0}^N$, in such a way that more mesh points are generated in the boundary layer region than outside of it. Let $N = 2^k$, $k \ge 2$ be a positive integer. Set

$$\sigma = \min\left\{\frac{1}{2}, \frac{2\varepsilon}{\gamma}\ln N\right\}.$$

We divide each of the subintervals $[0,\sigma]$, $[\sigma, 1]$ into $N/2$ equidistant subintervals. Set $i_0 = \frac{N}{2}$, then $x_{i_0} = \sigma$ is the transition point. Let $x_i = x_{i-1} + h_i, \forall\ i = 1, \ldots, N$. Then the resulting piecewise-uniform Shishkin mesh is represented as

$$\tilde{h} := \begin{cases} h_i = \dfrac{2\sigma}{N} & \text{for } i = 1, \ldots, i_0; \\[2ex] h_i = \dfrac{2(1-\sigma)}{N} & \text{for } i = i_0 + 1, \ldots, N; \end{cases} \tag{2.1}$$

Note that if $\sigma = 1/2$, then the mesh is uniform, N^{-1} is very small with respect to ε and therefore a classical analysis could be used to prove the uniform convergence of the method. So, in the convergence analysis of method we only consider the case $\sigma = 2\varepsilon \ln N/\gamma$.

Now we describe the B-spline collocation method for the problem (1.5)–(1.6) on a piecewise-uniform mesh Ω^N. We extend the partition Ω^N by introducing $x_{-3} < x_{-2} < x_{-1}$ mesh points on the left side and $x_{N+1} < x_{N+2} < x_{N+3}$ mesh points on the right side. Then, for $i = -1, 0, \ldots, N+1$, the cubic B-splines are defined by ([13, 14])

$$B_i(x) = \frac{1}{\tilde{h}^3}\begin{cases} (x - x_{i-2})^3 & \text{for } x \in [x_{i-2}, x_{i-1}]; \\ \tilde{h}^3 + 3\tilde{h}^2(x - x_{i-1}) + 3\tilde{h}(x - x_{i-1})^2 - 3(x - x_{i-1})^3 & \text{for } x \in [x_{i-1}, x_i]; \\ \tilde{h}^3 + 3\tilde{h}^2(x_{i+1} - x) + 3\tilde{h}(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3 & \text{for } x \in [x_i, x_{i+1}]; \\ (x_{i+2} - x)^3 & \text{for } x \in [x_{i+1}, x_{i+2}]; \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Suppose $V = \text{span}\{B_{-1}(x), B_0(x), \ldots, B_N(x), B_{N+1}(x)\}$. It is well known that cubic B-splines $\{B_i(x)\}_{i=-1}^{N+1}$ are linearly independent and dim $V = N + 3$. Suppose the approximate solution to the problem (1.5)–(1.6) is given by

$$u_N(x) = \sum_{i=-1}^{N+1} \alpha_i B_i(x),$$

where α_i are the unknown parameters and B_i's are cubic B-spline functions. We force the function u_N to satisfy the differential equation at the mesh points of the partition Ω^N and also the boundary conditions. Thus we have

$$L_\varepsilon u_N(x_i) = g(x_i, u_N(x_i)), \quad i = 0, \ldots, N, \tag{2.3}$$

with

$$u_N(x_0) = 0, \quad u_N(x_N) = 0, \tag{2.4}$$

where

$$L_\varepsilon u_N(x_i) := -\varepsilon u_N''(x_i) - a(x_i)u_N'(x_i) + b(x_i)u_N(x_i).$$

Explicitly, the collocation equation (2.3) can be rewritten as

$$\begin{aligned}
\alpha_{i-1}(&-\varepsilon B_{i-1}''(x_i) - a(x_i)B_{i-1}'(x_i) + b(x_i)B_{i-1}(x_i)) \\
&+ \alpha_i(-\varepsilon B_i''(x_i) - a(x_i)B_i'(x_i) + b(x_i)B_i(x_i)) \\
&+ \alpha_{i+1}(-\varepsilon B_{i+1}''(x_i) - a(x_i)B_{i+1}'(x_i) + b(x_i)B_{i+1}(x_i)) \\
&\quad = g(x_i, u_N(x_i)), \quad i = 0, \ldots, N.
\end{aligned}$$

Each basis function B_i is twice continuously differentiable. The values of the basis functions B_i at the mesh points in Ω^N can be determined from (2.2). Putting the values of basis functions B_i and their derivative at the mesh points in Ω^N, we obtain

$$\begin{aligned}
&\alpha_{i-1}\left(\frac{-6\varepsilon}{\tilde{h}^2} + \frac{3a(x_i)}{\tilde{h}} + b(x_i)\right) + \alpha_i\left(\frac{12\varepsilon}{\tilde{h}^2} + 4b(x_i)\right) \\
&\quad + \alpha_{i+1}\left(\frac{-6\varepsilon}{\tilde{h}^2} - \frac{3a(x_i)}{\tilde{h}} + b(x_i)\right) = g(x_i, u_N(x_i)), \quad i = 0, \ldots, N
\end{aligned} \tag{2.5}$$

with

$$\alpha_{-1} + 4\alpha_0 + \alpha_1 = 0, \quad \alpha_{N-1} + 4\alpha_N + \alpha_{N+1} = 0. \tag{2.6}$$

On eliminating α_{-1} and α_{N+1}, the resulting system (2.5)–(2.6) can be represented as

$$\boldsymbol{A}\boldsymbol{\alpha} = \boldsymbol{g}(\boldsymbol{\alpha}), \tag{2.7}$$

where $\boldsymbol{A}$ is an $(N+1) \times (N+1)$ tridiagonal matrix, $\boldsymbol{\alpha}$ is an $(N+1)$-dimensional column vector with components α_i and $\boldsymbol{g}(\boldsymbol{\alpha})$ is the right hand side vector of dimension $N + 1$.

The elements of the tridiagonal matrix $\boldsymbol{A} = (a_{ij})$ are

$$\begin{cases} a_{0,0} = \dfrac{36\varepsilon}{\tilde{h}^2} - \dfrac{12a(x_0)}{\tilde{h}}, \\ a_{0,1} = -\dfrac{6a(x_0)}{\tilde{h}}, \\ a_{i,i-1} = \dfrac{-6\varepsilon}{\tilde{h}^2} + \dfrac{3a(x_i)}{\tilde{h}} + b(x_i), & i = 1, \dots, N-1, \\ a_{i,i} = \dfrac{12\varepsilon}{\tilde{h}^2} + 4b(x_i), & i = 1, \dots, N-1, \\ a_{i,i+1} = \dfrac{-6\varepsilon}{\tilde{h}^2} - \dfrac{3a(x_i)}{\tilde{h}} + b(x_i), & i = 1, \dots, N-1, \\ a_{N,N-1} = \dfrac{6a(x_N)}{\tilde{h}}, \\ a_{N,N} = \dfrac{36\varepsilon}{\tilde{h}^2} + \dfrac{12a(x_N)}{\tilde{h}}, \\ a_{i,j} = 0, & \forall\ |i-j| > 1. \end{cases}$$

Following the arguments in [13, 14], we prove that the collocation matrix $\boldsymbol{A}$ is strictly diagonally dominant and hence nonsingular. Moreover

$$\|\boldsymbol{A}^{-1}\| \leq \frac{1}{2\beta} \equiv K(say). \tag{2.8}$$

This bound on $\boldsymbol{A}^{-1}$ will be useful in the convergence analysis of the present B-spline collocation method.

3. CONVERGENCE ANALYSIS

In this section, we estimate the error $\|u - u_N\|_\Omega$. Let u_N be the unique cubic spline collocation approximate solution of (1.5)–(1.6) given by

$$u_N(x) = \sum_{i=-1}^{N+1} \alpha_i B_i(x), \tag{3.1}$$

and let $\overline{u}_N$ be the unique cubic spline interpolate from the space V to the exact solution u of (1.5)–(1.6) given by

$$\overline{u}_N(x) = \sum_{i=-1}^{N+1} \overline{\alpha}_i B_i(x). \tag{3.2}$$

An application of the triangle inequality gives

$$\|u - u_N\|_\Omega \leq \|u - \overline{u}_N\|_\Omega + \|\overline{u}_N - u_N\|_\Omega. \tag{3.3}$$

We now estimate both terms on the right hand side of (3.3). First consider the interpolation error $\|u - \overline{u}_N\|_\Omega$. Let $\overline{y}$ be the unique cubic spline interpolant of $y \in$

$C^4(\Omega)$. Then by the standard cubic spline interpolation error estimates ([15, 16]) for $x \in \Omega_i := [x_{i-1}, x_i] \subset \Omega$,

$$|(\overline{y} - y)^{(k)}(x)| \leq \begin{cases} Ch_i^{4-k} \|y^{(4)}\|_{\Omega_i}; \\ C\|y^{(k)}\|_{\Omega_i}; \qquad k = 0, 1, 2. \end{cases} \tag{3.4}$$

First consider the case when $\Omega_i \subset [0, \sigma]$. In this case $h_i \leq C\varepsilon N^{-1} \ln N$. Furthermore $\|u^{(4)}\|_\Omega \leq C\varepsilon^{-4}$ by (1.4). Using first interpolation error estimate of (3.4) with $k = 0$, we get

$$|(u - \overline{u}_N)(x)| \leq CN^{-4} \ln^4 N, \quad x \in \Omega_i \subset [0, \sigma].$$

Next consider the case when $\Omega_i \subset [\sigma, 1]$. Here we need a special decomposition of the exact solution u into regular part v and layer part w. Set $x^* = 4\varepsilon\gamma^{-1} \ln(1/\varepsilon)$ and define, for $x \in \overline{\Omega}$

$$v(x) = \begin{cases} \displaystyle\sum_{\ell=0}^{4} \frac{(x - x^*)^\ell}{\ell!} u^{(\ell)}(x^*) & \text{for } 0 \leq x \leq x^*; \\ u(x) & \text{for } x^* \leq x \leq 1; \end{cases}$$

and $w(x) = u(x) - v(x)$. Then equation (1.4) and the choice of x^* yields

$$|v^{(m)}(x)| \leq C \quad \text{and} \tag{3.5}$$

$$|w^{(m)}(x)| \leq C\varepsilon^{-m} e^{-x\gamma/\varepsilon} \quad \text{for } m = 0, \ldots, 4. \tag{3.6}$$

Decomposing the interpolation error according to the the decomposition of u, we get

$$|(u - \overline{u}_N)(x)| \leq |(v - \overline{v}_N)(x)| + |(w - \overline{w}_N)(x)|. \tag{3.7}$$

For the first term on the right hand side of (3.7), we use the first interpolation error estimate of (3.4) with $k = 0$, $h_i \leq CN^{-1}$ and $\|v^{(4)}\|_\Omega \leq C$, and for the second term on the right hand side of (3.7), we use the second interpolation error estimate of (3.4) with $k = 0$ and (3.6). Thus, we get

$$\begin{aligned} |(u - \overline{u}_N)(x)| &\leq CN^{-4} + C\|w\|_{\Omega_i} \\ &\leq CN^{-4} + C \max_{x \in [\sigma, 1]} \exp(-x\gamma/\varepsilon) \\ &\leq CN^{-4} + C \exp(-\sigma\gamma/\varepsilon) \\ &\leq CN^{-4} + CN^{-2} \leq CN^{-2}. \end{aligned}$$

Collecting all the interpolation error estimates, we have

$$\|u - \overline{u}_N\|_\Omega \leq CN^{-2}. \tag{3.8}$$

Now we estimate $\|\overline{u}_N - u_N\|_\Omega$. For this consider the quantities $L_\varepsilon \overline{u}_N(x_i)$, $i = 0, \ldots, N$. Using interpolation error estimates (3.4) and the arguments that we have used to estimate $\|u - \overline{u}_N\|_\Omega$, we obtain $\|L_\varepsilon \overline{u}_N - L_\varepsilon u\|_\Omega \le CN^{-2}\ln^2 N$. At the mesh points, in particular, we write

$$L_\varepsilon \overline{u}_N(x_i) = g(x_i, \overline{u}_N(x_i)) + r(x_i), \quad i = 0, \dots, N,$$

where $r(x)$ is the error function with the order of magnitude $O(N^{-2}\ln^2 N)$. With the boundary conditions $\overline{u}_N(x_0) = 0$, $\overline{u}_N(x_N) = 0$, this lead to the nonlinear system

$$\boldsymbol{A}\overline{\boldsymbol{\alpha}} = \boldsymbol{g}(\overline{\boldsymbol{\alpha}}) + \boldsymbol{r}, \tag{3.9}$$

where $\boldsymbol{A}$ is the same matrix as in (2.7), $\overline{\boldsymbol{\alpha}}$, $\boldsymbol{g}(\overline{\boldsymbol{\alpha}})$ and $\boldsymbol{r}$ are vectors of dimension $(N+1)$ with components $\overline{\alpha}_i$, $g(x_i, \overline{u}_N(x_i))$ and $r(x_i)$ respectively. Let $\boldsymbol{e} = (e_0, \dots, e_N)^T$, where $e_i = \overline{\alpha_i} - \alpha_i$. Subtraction of (2.7) from (3.9) results in

$$\boldsymbol{A}\boldsymbol{e} = \boldsymbol{r} + \boldsymbol{g}(\overline{\boldsymbol{\alpha}}) - \boldsymbol{g}(\boldsymbol{\alpha}). \tag{3.10}$$

The Lipschitz condition (1.7) implies

$$\begin{aligned} g(x_i, \overline{u}_N(x_i)) - g(x_i, u_N(x_i)) &= M_i(\overline{u}_N(x_i) - u_N(x_i)) \\ &= \begin{cases} M_i[e_{i-1} + 4e_i + e_{i+1}] & \text{for } 1 \le i \le N-1 \\ 0 & \text{for } i = 0, N, \end{cases} \end{aligned} \tag{3.11}$$

for some constants M_i, where $|M_i| \le M$, for $i = 0, \dots, N$. Thus (3.10) can be written as

$$\boldsymbol{A}\boldsymbol{e} = \boldsymbol{r} + \widetilde{\boldsymbol{M}}\,\boldsymbol{T}\boldsymbol{e}, \tag{3.12}$$

where $\widetilde{\boldsymbol{M}} = \mathrm{diag}(M_0, M_1, \dots, M_N)$,

$$\boldsymbol{T} = \begin{pmatrix} 0 & 0 & 0 & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & 0 & 0 & 0 \end{pmatrix}$$

and satisfy

$$\|\widetilde{\boldsymbol{M}}\| \le M, \quad \|\boldsymbol{T}\| = 6. \tag{3.13}$$

Thus, from

$$\boldsymbol{e} = \boldsymbol{A}^{-1}\boldsymbol{r} + \boldsymbol{A}^{-1}\widetilde{\boldsymbol{M}}\,\boldsymbol{T}\boldsymbol{e},$$

and the bound on $\|\boldsymbol{A}^{-1}\|$, it follows that

$$\|\boldsymbol{e}\| \le K\|\boldsymbol{r}\| + 6KM\|\boldsymbol{e}\|.$$

As $(1 - 6KM) > 0$, and $\|\boldsymbol{r}\| \le CN^{-2}\ln^2 N$, we have

$$\|\boldsymbol{e}\| \le CN^{-2}\ln^2 N. \tag{3.14}$$

We have

$$(\overline{\alpha}_{-1} - \alpha_{-1}) + 4(\overline{\alpha}_0 - \alpha_0) + (\overline{\alpha}_1 - \alpha_1) = 0,$$

$$(\overline{\alpha}_{N-1} - \alpha_{N-1}) + 4(\overline{\alpha}_N - \alpha_N) + (\overline{\alpha}_{N+1} - \alpha_{N+1}) = 0.$$

Using (3.14), we get $|\,\overline{\alpha}_{-1} - \alpha_{-1}| \leq CN^{-2}\ln^2 N$, $|\,\overline{\alpha}_{N+1} - \alpha_{N+1}| \leq CN^{-2}\ln^2 N$. Hence

$$\max_{-1\leq i\leq N+1} |\overline{\alpha}_i - \alpha_i| \leq CN^{-2}\ln^2 N. \tag{3.15}$$

From (3.1)–(3.2), we get

$$|\overline{u}_N(x) - u_N(x)| \leq \max_{-1\leq i\leq N+1} |\overline{\alpha}_i - \alpha_i| \sum_{i=-1}^{N+1} |B_i(x)|. \tag{3.16}$$

By [13, 14], we have the following inequality

$$\sum_{i=-1}^{N+1} |B_i(x)| \leq 10, \quad 0 \leq x \leq 1. \tag{3.17}$$

Thus from (3.15)–(3.17), we get

$$\|\overline{u}_N - u_N\|_\Omega \leq CN^{-2}\ln^2 N. \tag{3.18}$$

Finally we combine (3.3),(3.8) and (3.18), to get our main convergence result.

Theorem 3.1. *Let u be the exact solution of the problem* (1.5)–(1.6) *and let u_N be the cubic B-spline collocation approximate solution on a piecewise-uniform Shishkin mesh. Then*

$$\|u - u_N\|_\Omega \leq CN^{-2}\ln^2 N.$$

4. NUMERICAL RESULTS

The present B-spline collocation method on a piecewise-uniform Shishkin mesh is implemented on two test problems.

Example 4.1. Consider the singularly perturbed problem

$$-\varepsilon u'' - (1 + x)u' + u + (1 + x)^2 = 0, \quad x \in \Omega$$

$$u(0) = 0, \quad u(1) = 0.$$

Example 4.2. Consider the singularly perturbed problem

$$-\varepsilon u'' - (2 - x)u' + \exp(u) = 0, \quad x \in \Omega$$

$$u(0) = 0, \quad u(1) = 0.$$

Table 4.1: Maximum Errors E_ε^N, E^N and parameter-uniform numerical rate p^N for the Example 4.1.

$\varepsilon = 10^{-K}$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
K=0	6.46E-06	1.62E-06	4.04E-07	1.01E-07	2.53E-08	6.31E-09
1	1.69E-03	4.18E-04	1.04E-04	2.61E-05	6.52E-06	1.63E-06
2	3.48E-03	1.31E-03	4.17E-04	1.38E-04	4.41E-05	1.37E-05
3	3.32E-03	1.24E-03	4.32E-04	1.44E-04	4.62E-05	1.37E-05
4	3.30E-03	1.23E-03	4.27E-04	1.41E-04	4.51E-05	1.40E-05
5	3.30E-03	1.23E-03	4.26E-04	1.41E-04	4.50E-05	1.40E-05
10	3.30E-03	1.23E-03	4.26E-04	1.41E-04	4.50E-05	1.40E-05
15	3.30E-03	1.23E-03	4.26E-04	1.41E-04	4.50E-05	1.40E-05
E^N	3.48E-03	1.31E-03	4.32E-04	1.44E-04	4.62E-05	1.40E-05
p^N	1.41	1.60	1.59	1.64	1.72	

To solve the corresponding nonlinear systems, the Newton's method is used with the initial guess $u_N^{(0)} = (0, u_0(x_1), \ldots, u_0(x_{N-1}), u_0(x_N))^T$, where $u_0(x)$ is the solution of the reduced problem. The stopping criterion is $\|u_N^{(k)} - u_N^{(k-1)}\| < 10^{-9}$. Here $u_N^{(k)}$, for $k = 1, 2, \ldots$, represent the successive approximates to u_N computed iteratively. For each N and ε in the tables, it takes only about 5 iterations to satisfy this criterion.

As the exact solutions of the test problems are not known, so we used a double mesh method to estimate the errors. Let u_N be the solution of the present B-spline collocation method on the original mesh with N discretization parameters and $\tilde{u}_N$ that on the mesh obtained by uniformly bisecting the original mesh. We then estimate the maximum errors and the parameter-uniform errors by

$$E_\varepsilon^N = \max_{0 \le i \le N} |(u_N)_i - (\tilde{u}_{2N})_{2i}|, \quad E^N = \max_{\forall \varepsilon} E_\varepsilon^N.$$

The ε-uniform numerical rate of convergence p^N is calculated by

$$p^N = \log_2(E^N / E^{2N}).$$

For the different values of ε and N, Tables 4.1 and 4.2 represent the maximum errors E_ε^N, E^N and the ε-uniform numerical rate of convergence p^N of the present B-spline collocation method on a piecewise-uniform Shishkin mesh for the Examples 4.1 and 4.2 respectively.

5. DISCUSSION

The cubic B-spline collocation method on a piecewise-uniform Shishkin mesh for the numerical solution of singularly perturbed semilinear convection-diffusion two point boundary value problem is presented. The original problem (1.1)–(1.2) is reformulated to an equivalent problem (1.5)–(1.6). A B-spline collocation method for

Table 4.2: Maximum Errors E_ε^N, E^N and parameter-uniform numerical rate p^N for the Example 4.2.

$\varepsilon = 10^{-K}$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
K=0	4.85E-06	1.21E-06	3.03E-07	7.58E-08	1.89E-08	4.74E-09
1	1.27E-03	3.14E-04	7.82E-05	1.96E-05	4.89E-06	1.22E-06
2	8.35E-03	3.00E-03	9.73E-04	3.19E-04	1.01E-04	3.14E-05
3	8.35E-03	2.99E-03	1.01E-03	3.31E-04	1.05E-04	3.15E-05
4	8.35E-03	2.99E-03	1.01E-03	3.30E-04	1.04E-04	3.21E-05
5	8.35E-03	2.99E-03	1.01E-03	3.30E-04	1.04E-04	3.21E-05
10	8.35E-03	2.99E-03	1.01E-03	3.30E-04	1.04E-04	3.21E-05
15	8.35E-03	2.99E-03	1.01E-03	3.30E-04	1.04E-04	3.21E-05
E^N	8.35E-03	3.00E-03	1.01E-03	3.31E-04	1.05E-04	3.21E-05
p^N	1.48	1.57	1.61	1.66	1.71	

the modified problem (1.5)–(1.6) on a piecewise-uniform Shishkin mesh is proposed. It is interesting to note, that in all the numerical experiments, there is no difficulty in solving (1.1)–(1.2) directly by the present B-spline collocation method, and they rendered exactly the same result as the modified form (1.5)–(1.6). The reformulation of the problem (1.1)–(1.2) to (1.5)–(1.6) is solely for the theoretical purpose. The essential idea in this method is to use the cubic B-spline basis on a piecewise-uniform Shishkin mesh to approximate the solution of the modified problem (1.5)–(1.6) via. collocation approach. The cubic B-spline basis function are defined in section 2 and has a finite support on the four consecutive intervals $[x_{i+jh}, x_{i+(j+1)h}]_{j=-2}^{1}$, and results in a tridiagonal system which can be solved using the standard algorithm.

From the Tables 4.1 and 4.2, it can be observed that, for fixed value of ε, the maximum errors E_ε^N decreases as the mesh points increases. The last row in each of the Tables (4.1 and 4.2) represent the ε-uniform numerical rate of convergence p^N of the present method. Clearly these results are in good agreement with the Theorem 3.1.

The most significant virtue of the spline collocation procedure is its ease in application; e.g. matrix elements of the defining equation are evaluated directly, rather than by numerical integration as in the Galerkin method. Therefore the collocation system is set up rather easily. Also this method ensure that the solution is, at least, continuous in the domain Ω, whereas the finite difference methods give the solution only at the chosen mesh points.

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