Neural, Parallel, and Scientific Computations 18 (2010) 167-194

NUMERICAL APPROXIMATION OF MODIFIED BURGERS' EQUATION VIA HYBRID FINITE DIFFERENCE SCHEME ON LAYER-ADAPTIVE MESH

VIKAS GUPTA AND MOHAN K. KADALBAJOO

Department of Mathematics and Statistics, Indian Institute of Technology,
Kanpur-208016, India

ABSTRACT. In this paper, a numerical method is constructed for solving one-dimensional time dependent modified Burgers' equation for various values of Reynolds number. At high Reynolds number, an inviscid boundary layer is produced in the neighborhood of right part of the lateral surface of the domain and the problem can be considered as a non-linear singularly perturbed problem involving a small parameter ε. Using singular perturbation analysis, asymptotic bounds for the derivatives of the solution are established by decomposing the solution into smooth and singular components. We construct a numerical scheme that comprises of Implicit-Euler method to discretize in temporal direction on uniform mesh and a monotone hybrid finite difference operator to discretize the spatial variable with piecewise uniform Shishkin mesh. Quasi-linearization process is used to tackle the non-linearity and shown that quasi-linearization process converges quadratically. The method has been shown to be first order uniformly accurate in the temporal variable and first order parameter uniform convergent on the non-boundary layer domain and almost second order parameter uniform convergent on the boundary layer domain in the spatial variable. Uniform convergence of the method is demonstrated by numerical examples and an estimate of the error is given.

Key Words: Singular perturbation; Modified Burgers' equation; Implicit Euler method; Quasi-linearization; Shishkin mesh; Hybrid finite difference; Stability and convergence analysis.

1. INTRODUCTION

The present study deals with the following one-dimensional cubic non-linear modified Burgers' turbulence model on the domain $D = (0,1) \times (0,T]$, with the smooth boundary $\partial D = \bar{D} \backslash D$ and the Dirichlet boundary conditions:

$$L_\varepsilon u(x,t) = -\varepsilon \frac{\partial^2 u}{\partial x^2} + u^2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0, \tag{1.1a}$$

$$(x,t) \in D \equiv \Omega_x \times \Omega_t \equiv (0,1) \times (0,T], \tag{1.1b}$$

$$u(x,0) = u_0(x), \qquad x \in \bar{\Omega}_x, \tag{1.1c}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \qquad t \in \bar{\Omega}_t, \tag{1.1d}$$

where, the first term in Eq. (1.1) represents the viscous dissipation, second term models non-linear convection and third is an unsteady term. In the fluid dynamics

Received March 15, 2010

model, ε is interpreted as kinematic viscosity or dissipation coefficient and reciprocal of ε is the effective Reynolds number R of the problem. In the case, where ε is small or R is large, it can be considered as a one-dimensional non-linear singularly perturbed parabolic problem with a small singular perturbation parameter ε. This equation describes the interplay between non-linear convection and diffusion. For $\partial D = \Gamma_l \cup \Gamma_i \cup \Gamma_r$, we distinguish the left lateral boundary $\Gamma_l = \{(x,t) : x = 0, t \in \bar{\Omega}_t\}$, right lateral boundary $\Gamma_r = \{(x,t) : x = 1, t \in \bar{\Omega}_t\}$ and the initial boundary $\Gamma_i = \{(x,t) : t = 0, x \in \bar{\Omega}_x\}$.

Modified Burgers' equation has varied applications in the field of physics and more particularly of continuum mechanics, in which dissipation is a significant aspect of wave propagation. Certain representations of viscoelastic solid behaviour modelled by modified Burgers' equation [19, 23]. Nariboli and Lin [19] also show the applicability of modified Burgers' equation to the problem of magnetohydrodynamic 'switch-on' shock waves. This model equation also arise in the case of torsional waves in a thin viscoelastic rod [22] and for the case of transverse electromagnetic waves in a non-linear dielectric [2]. There are further situation in many practical problems in which modified Burgers' equation is the appropriate model, such as non-linear waves in a medium with low-frequency pumping or absorption, ion reflection at quasi-perpendicular shocks, physics of ionized gases, explosions and sonic boom theory, turbulence transport, wave processes in thermoelastic medium, acoustic waves generated by laser radiations, dispersion of pollutants in rivers and sediment transport and unsteady infiltration of water into homogeneous soil [4, 12, 16, 14].

The case of quadratic non-linearity corresponds to the customary Burgers' equation. For a small value of ε, Burgers' equation behaves merely as hyperbolic partial differential equation. This equation admits a transformation which transform it into the linear diffusion equation (Hopf [11] and Cole [6]). Lighthill [18] deduced a N-waves analytical solution for weak plane shock waves by using the above transformation. Moreover it has been proved that there does not exist Backlund transformation for the modified Burgers' equation with cubic non-linearity, and in particular, therefore, modified Burgers' equation does not admit exact linearisation since no Hopf-Cole like transformation seems to exist for the purpose. The only known solutions to the modified Burgers' equation correspond to the steady shock wave (analogous to the well-known Taylor shock wave in a thermoviscous fluid) or to a similarity form. Attempts to understand the physics underlying the modified Burgers' equation must then, for the moment, rest on asymptotic studies and direct numerical computation. Using singular perturbation techniques Lee-Bapty and Crighton [2] and Harris [13] studied the asymptotic solutions for sufficiently small values of the dissipation coefficient ε, whereas Sachdev and Rao [21] presented an analytical exact solution for the Eq. (1.1) with N-wave initial condition. In the present paper, our principal aim is to provide a ε-uniform numerical method and analyse the proposed method by means of singular perturbation theory.

In general, the solutions of this class of problems may have a multiscale character. For small values of the viscosity coefficient ε, the solutions of these problems presenting rapid variations in some narrow region called boundary layer, in the neighborhood of the right lateral surface Γ_r, which correspond to the steeping effect of the non-linear convection term. An innovative and robust numerical method have not been sufficiently developed yet for seeking accurate and efficient numerical solutions of modified Burgers equation with small values of ε, and remains as a challenging task. Sachdev and Seebass [20] study the finite difference solution of the nonplaner Burgers' equation with $\varepsilon = 10^{-2}$. By choosing the initial conditions which have $\max(|u_x|, |u_{xx}|)$ not much greater than $O(1)$, they show that a uniform mesh size of $O(10^{-2})$ in both the spatial and temporal direction is adequate for the predictor-corrector finite difference scheme of Douglas and Jones [8]. However the use of a uniform mesh would be impractical due to the occurrence of much smaller values of ε. Chong [5] used the variable mesh finite difference scheme and gave more accurate result in comparison to [20] in the boundary layer region specially when ε is very small. But still, his method is not uniformly convergent with respect to ε. Recently Kadalbajoo and Awasthi [15] proposed a method comprises a standard implicit Euler method to discretize the temporal variable and a standard upwind finite difference with piecewise uniform Shishkin mesh to discretize in spatial direction and proved that the method is parameter uniform.

If rapid variations of the solution are confined to thin isolated regions, than the total number of mesh points may be made manageable by using a fine mesh only inside these boundary layers. Therefore to construct a parameter uniform numerical method, it is crucial to have the information about the behavior of the solution, which may be required in the *a priori* mesh refinement strategy as well as in the error analysis.

In the present work, we investigate a monotone finite difference operator for the problem class (1.1). In the present paper we use piecewise uniform Shishkin mesh for its simple structure to resolve the boundary layer. We focus on decomposing the global error in two components which are analyzed separately. At the first stage, we discretize the temporal variable by means of implicit Euler method with the constant time step and freezing the coefficients of the resulting non-linear ordinary differential equations. Then we use the quasi-linearization method given in Bellman and Kalaba [3] to linearize the stationary differential equations and show that the sequence of solutions of linearized problems converges quadratically to the solution of the original non-linear problem at each time step. We prove uniform convergence with respect to both the parameters Δt and ε at temporal semi-discretization stage. At the second stage to discretize the spatial variable, we use a hybrid finite difference operator consisting of upwind and central difference operator on the linearized ordinary differential equations at each time step resulting from the temporal semi-discretization with piecewise uniform Shishkin mesh. These two difference operators are monotone in the various subdomain of the parameter space $P = (\varepsilon, N)$, where N is the number of mesh points in the spatial direction. The upwind finite difference operator is always monotone and has first order truncation error for all values of ε, whereas the central difference operator is monotone if ε is relatively large, *i.e*, $\varepsilon > C_1 N^{-1}$ and it has second order truncation error away from the transition point of the piecewise uniform mesh. Therefore we use upwind difference operator in coarse mesh region and central difference operator in fine mesh region. To analyse the proposed scheme in space, we split the solution into smooth and singular component and use analytical finite difference techniques consisting of truncation error bounds, discrete comparison principle and appropriate choices of discrete barrier functions. Finally, we prove that the proposed hybrid finite difference scheme is first order ε-uniform convergent away from the boundary layer region and second order ε-uniform convergent inside the boundary layer region at the spatial discretization stage. Combining the results obtained in both the stages, we conclude that our scheme is ε-uniformly convergent and independent of mesh parameters N and Δt. Throughout the paper we use C (sometimes subscripted) as a generic positive constant independent of ε and mesh parameters.

2. A PRIORI ESTIMATES AND TEMPORAL SEMI-DISCRETIZATION

In this section, we give some *a priori* estimates for the solution of the continuous problem and then discretize the temporal variable by means of Implicit Euler method.

2.1. A priori Estimates.

Here, we discuss the continuous maximum principle and derive the bounds for the analytical solution of the modified Burgers' equation. We assume enough smoothness and compatibility conditions at the corner points. Let S be a bounded and convex domain then for any given function $g(x,t) \in C^0(S)$, the maximum norm over the domain S is define by

$$\|g\|_S = \max_{(x,t)\in S} |g(x,t)|.$$

In the following, we prove that the operator L_ε as defined in Eq. (1.1), satisfies a continuous maximum principle.

Lemma 2.1 (Maximum Principle)**.** *Let $y \in C^{2,1}(\bar{D})$. If $y(x,t) \geq 0 \ \forall \ (x,t) \in \partial D$ and $L_\varepsilon y(x,t) \geq 0 \ \forall \ (x,t) \in D$, then $y(x,t) \geq 0 \ \forall \ (x,t) \in \bar{D}$.*

Proof. The proof easily follows from contradiction. Assume that there exist a point $(\hat{x},\hat{t}) \in \bar{D}$, such that $y(\hat{x},\hat{t}) < 0$. It follows from the hypotheses that the point $(\hat{x},\hat{t}) \notin \partial D$. Define the auxiliary function $v(x,t) = \exp(-t)y(x,t)$ and note that $v(\hat{x},\hat{t}) < 0$. Furthermore, choose a point $(\tilde{x},\tilde{t}) \in D$ such that

$$v(\tilde{x},\tilde{t}) = \min_{(x,t)\in D} v(x,t) < 0.$$

Therefore from the definition of $(\tilde{x},\tilde{t})$ we have

$$\frac{\partial^2 v}{\partial x^2}|_{(\tilde{x},\tilde{t})} \geq 0, \quad \frac{\partial v}{\partial x}|_{(\tilde{x},\tilde{t})} = 0, \quad \frac{\partial v}{\partial t}|_{(\tilde{x},\tilde{t})} = 0.$$

Using the above estimates, we have

$$L_\varepsilon y(\tilde{x},\tilde{t}) = \left(-\varepsilon\frac{\partial^2 v}{\partial x^2} + \exp(2t)v^2\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} + v\right)(\tilde{x},\tilde{t})\exp(t) < 0,$$

which is a contradiction. Therefore we can conclude that $y(x,t)$ is non-negative. □

An immediate consequence of maximum principle for the solution of problem (1.1), is the following uniform stability estimate.

Lemma 2.2. *Let $u(x,t)$ be the solution of* (1.1)*, then* $\forall\varepsilon > 0$*, we have*

$$\|u\|_{\bar{D}} \leq T\|u_0\|_{\Gamma_i} + \|u\|_{\partial D}.$$

Proof. Let us define the two comparison functions

$$\Psi^{\pm}(x,t) = t\|u_0\|_{\Gamma_i} + \|u\|_{\partial D} \pm u(x,t) \qquad \forall (x,t) \in \bar{D}.$$

It is easily seen that $\Psi^{\pm}(x,t) \geq 0 \; \forall \; (x,t) \in \partial D$. Furthermore, we have

$$L_\varepsilon\Psi^{\pm}(x,t) = \|u_0\|_{\Gamma_i} \pm 0 \geq 0, \qquad \forall (x,t) \in D.$$

Therefore, by applying the maximum principle (Lemma 2.1), we get $\Psi^{\pm}(x,t) \geq 0, \forall (x,t) \in \bar{D}$, which gives the desired estimate. □

2.2. **Temporal Semi-discretization.** At the first stage, we discretize temporal variable by means of the implicit Euler method, with constant step size Δt. Such a semi-discretization yields the following system of non-linear elliptic differential equations:

(2.1a)
$$u^0 = u(x,0) = u_0(x), \qquad x \in \bar{\Omega}_x,$$

(2.1b)
$$(I + \Delta t L_{x,\varepsilon})u^{n+1} \equiv -\varepsilon\Delta t\frac{\partial^2 u^{n+1}}{\partial x^2} + \Delta t(u^{n+1})^2\frac{\partial u^{n+1}}{\partial x} + u^{n+1} = u^n, \quad x \in \Omega_x, \; n \geq 0,$$

(2.1c)
$$u^{n+1}(0) = 0, \qquad u^{n+1}(1) = 0, \qquad n \geq 0,$$

where, u^{n+1} is the solution of the Eq. (2.1), at the $(n+1)$-th time level. Here $u^n \equiv u(x, t^n)$, and Δt is the uniform time step. We define the discrete mesh $\bar{\Omega}_t^n$ that discretizes $\bar{\Omega}_t$ with uniform mesh elements as

$$\bar{\Omega}_t^n = \{t^n \mid t^n = n\Delta t, \quad n \leq T/\Delta t\}.$$

Clearly, the operator $(I + \Delta t L_{x,\varepsilon})$ satisfies the maximum principle, which ensures the stability of the temporal semi-discretization.

The local truncation error of the time semi-discretization is given by $\mu_{n+1} \equiv u^{n+1} - \hat{u}^{n+1}$, where $\hat{u}^{n+1}$ is the computed solution of the following boundary value problem

$$(I + \Delta t L_{x,\varepsilon})\hat{u}^{n+1} = u^n, \quad x \in \Omega_x,\ n \geq 0, \tag{2.2a}$$

$$\hat{u}^{n+1}(0) = 0, \qquad \hat{u}^{n+1}(1) = 0, \qquad n \geq 0. \tag{2.2b}$$

Local error estimates of each time step contributes to the global error of the temporal semi-discretization which is defined, at the instant t^n, as $E_n \equiv u(x, t^n) - u^n(x)$. Then, the following consistency result holds.

Lemma 2.3 (Local Error Estimate)**.** *If*

$$\left|\frac{\partial^j}{\partial t^j} u(x,t)\right| \leq C, \qquad \forall (x,t) \in \bar{D}, \qquad 0 \leq j \leq 2, \tag{2.3}$$

then the local error estimates in the temporal direction is given by

$$\|\mu_{n+1}\|_\infty \leq C(\Delta t)^2. \tag{2.4}$$

Proof. Since the solution of Eq. (1.1) is smooth enough, therefore we have

$$\begin{aligned} u^n &= u^{n+1} - \frac{1}{1!}\Delta t \frac{\partial u^{n+1}}{\partial t} + \int_{t^n}^{t^{n+1}} (t^n - s)\frac{\partial^2 u(s)}{\partial t^2} ds \\ &= u^{n+1} - \Delta t \left\{ \varepsilon \frac{\partial^2 u^{n+1}}{\partial x^2} - (u^{n+1})^2 \frac{\partial u^{n+1}}{\partial x} \right\} + O(\Delta t^2). \end{aligned}$$

Therefore, μ_{n+1} is the solution of a boundary value problem

$$(I + \Delta t L_{x,\varepsilon})\mu^{n+1} \equiv -\varepsilon \Delta t \frac{\partial^2 \mu_{n+1}}{\partial x^2} + \Delta t (\mu_{n+1})^2 \frac{\partial \mu_{n+1}}{\partial x} + \mu_{n+1} = O(\Delta t^2), \tag{2.5a}$$

$$\mu_{n+1}(0) = \mu_{n+1}(1) = 0, \tag{2.5b}$$

and therefore, desired estimates follows by using the stability estimate for the operator $(I + \Delta t L_{x,\varepsilon})$. □

Now combining the stability and consistency of the temporal semi-discretization process, we lead to the following global error estimate.

Lemma 2.4 (Global Error Estimate)**.** *Under the hypotheses of Lemma* 2.3*, we have*

$$\|E_n\|_\infty \le C\Delta t, \qquad \forall n \le T/\Delta t.$$

Therefore, the temporal semi-discretization process is of uniformly convergent of first order.

3. QUASI-LINEARIZATION

In this section, we use the quasi-linearization process to linearize the above non-linear ordinary differential equations. The non-linear ordinary differential equation linearized around a nominal solution, which satisfies the specified boundary conditions. Then, we solve a sequence of two-point boundary-value problems in which the solution of the kth linear two-point boundary-value problem satisfies the specified boundary conditions and is taken as the nominal solution for the $(k+1)$th linear two-point boundary-value problem. Assume that the $u_{(k)}(x)$ as the kth nominal solution of the problem (2.1). An application of the quasi-linearization process [3] to the non-linear problem (2.1) introduce a sequence $\langle u_{(k)}\rangle_{k=0}^{\infty}$ of linear equations determined by the following recurrence relation

$$u^0_{(k+1)} = u_0(x), \qquad x \in \bar{\Omega}_x, \tag{3.1a}$$

$$-\varepsilon\Delta t\frac{\partial^2 u^{n+1}_{(k+1)}}{\partial x^2} + \Delta t(u^{n+1}_{(k)})^2\frac{\partial u^{n+1}_{(k+1)}}{\partial x} + \left(1 + 2\Delta t u^{n+1}_{(k)}\frac{\partial u^{n+1}_{(k)}}{\partial x}\right)u^{n+1}_{(k+1)}$$

$$= u^n_{(k+1)} + 2\Delta t(u^{n+1}_{(k)})^2\frac{\partial u^{n+1}_{(k)}}{\partial x} \qquad x \in \Omega_x,\ n \ge 0, \tag{3.1b}$$

$$u^{n+1}_{(k+1)}(0) = 0, \qquad u^{n+1}_{(k+1)}(1) = 0, \qquad n \ge 0, \tag{3.1c}$$

where $k = 0, 1, 2, \ldots$ is the iteration index. This is an application of the Newton-Rapson-Kantorovich approximation method in function space. We choose a reasonable initial guess $u_{(0)}(x)$ satisfying the initial condition $u_0(x)$. For the sake of convenience, we let $u_{(k+1)} = \bar{u}$. Therefore the above equation leads to the following initial-boundary value problem

$$\bar{u}^0 = u_0(x), \qquad x \in \bar{\Omega}_x, \tag{3.2a}$$

$$(I + \Delta t\tilde{L}_{x,\varepsilon})\bar{u}^{n+1} \equiv -\varepsilon\Delta t\frac{\partial^2 \bar{u}^{n+1}}{\partial x^2} + a(x)\Delta t\frac{\partial \bar{u}^{n+1}}{\partial x} + (1 + \Delta t b(x))\bar{u}^{n+1} = \bar{u}^n + \Delta t f(x), \quad x \in \Omega_x,\ n \ge 0, \tag{3.2b}$$

$$\bar{u}^{n+1}(0) = 0, \qquad \bar{u}^{n+1}(1) = 0, \qquad n \ge 0, \tag{3.2c}$$

where,

$$a(x) = a_{(k)}(x, t^{n+1}) = (u^{n+1}_{(k)})^2, \quad b(x) = b_{(k)}(x, t^{n+1}) = 2u^{n+1}_{(k)}\frac{\partial u^{n+1}_{(k)}}{\partial x},$$

$$f(x) = f_{(k)}(x, t^{n+1}) = \left(2(u^{n+1}_{(k)})^2 \frac{\partial u^{n+1}_{(k)}}{\partial x} \right).$$

Further, we assume that the functions $a(x), b(x)$ and $f(x)$ are sufficiently smooth functions in the spatial direction with

$$a(x) \geq \alpha > 0, \qquad x \in \bar{\Omega}_x, \tag{3.3a}$$

$$b(x) \geq \beta > 0, \qquad x \in \bar{\Omega}_x. \tag{3.3b}$$

These conditions ensure that the boundary layer is located at $x = 1$ and also ensure the uniqueness of the solution [9]. Thus, by using quasi-linearization process, we get the linear boundary value problem (3.2) for the function $\bar{u}^{n+1} = u^{n+1}_{(k+1)}$ and in lieu of solving the original non-linear problem (2.1), we will solve the sequence of second order singularly perturbed linear elliptic problems (3.2), for $k = 0, 1, 2, \ldots$ and $n \geq 0$ by using monotone finite difference operator, which is introduced in the next section. Analytically, for the solution $u^{n+1}(x)$ of original non-linear problem (2.1) we require that

$$\lim_{k \to \infty} u^{n+1}_{(k)}(x) = u^{n+1}(x), \qquad x \in \bar{\Omega}_x,$$

whereas numerically, we require that

$$\left| \bar{u}^{n+1}(x) - u^{n+1}_{(k)}(x) \right| < \nu, \qquad x \in \bar{\Omega}_x,$$

where ν is the small prescribed value to terminate the computation. This is the requisite criterion for terminating the iteration and the solution $\bar{u}^{n+1}(x)$ is used as the numerical solution of the non-linear boundary value problem (2.1).

The following theorem shows that not only the convergence of this sequence is quadratic, but also its proportionality constant is independent of k.

Theorem 3.1 (Convergence of quasi-linearization process). *Let $\langle u^{n+1}_{(k)} \rangle_{k=0}^{\infty}$ be the sequence produced by quasi-linearization technique at $(n+1)$th time level. Then there exist a constant $C > 0$, independent of k, such that*

$$\left|\left| u^{n+1}_{(k+1)} - u^{n+1}_{(k)} \right|\right|_{\bar{\Omega}_x} \leq C \left|\left| u^{n+1}_{(k)} - u^{n+1}_{(k-1)} \right|\right|^2_{\bar{\Omega}_x},$$

i.e., the quasi-linearization process converge quadratically.

Proof. To prove the convergence of the quasi-linearization process, we consider the following equation for the sake of convenience

$$\varepsilon \frac{\partial^2 u^{n+1}}{\partial x^2} = H(u^{n+1}), \quad x \in \Omega_x, n \geq 0, \tag{3.4a}$$

$$u^{n+1}(0) = 0, \qquad u^{n+1}(1) = 0, \quad n \geq 0. \tag{3.4b}$$

We assume, $u_{(0)}^{n+1}$ be the initial guess. By using quasi-linearization process, we obtain a sequence $\langle u_{(k)}^{n+1}\rangle_{k=0}^{\infty}$ of linear equations determined by the following recurrence relation

$$\varepsilon\frac{\partial^2 u_{(k+1)}^{n+1}}{\partial x^2} \approx H(u_{(k)}^{n+1}) + (u_{(k+1)}^{n+1} - u_{(k)}^{n+1})\frac{\partial H}{\partial u_{(k)}^{n+1}}(u_{(k)}^{n+1}), \quad x \in \Omega_x, n \geq 0, \tag{3.5a}$$

$$u_{(k+1)}^{n+1}(0) = 0, \qquad u_{(k+1)}^{n+1}(1) = 0, \quad n \geq 0. \tag{3.5b}$$

Thus we have

$$\begin{aligned}\varepsilon\left(\frac{\partial^2 u_{(k+2)}^{n+1}}{\partial x^2} - \frac{\partial^2 u_{(k+1)}^{n+1}}{\partial x^2}\right) &= H(u_{(k+1)}^{n+1}) - H(u_{(k)}^{n+1}) - (u_{(k+1)}^{n+1} - u_{(k)}^{n+1})\frac{\partial H}{\partial u_{(k)}^{n+1}}(u_{(k)}^{n+1}) \\ &+ (u_{(k+2)}^{n+1} - u_{(k+1)}^{n+1})\frac{\partial H}{\partial u_{(k+1)}^{n+1}}(u_{(k+1)}^{n+1}), \quad x \in \Omega_x, n \geq 0.\end{aligned} \tag{3.6}$$

The above equation is a second order differential equation for $(u_{(k+2)}^{n+1} - u_{(k+1)}^{n+1})$. Converting it into an integral function by using Green's function, we have

$$\begin{aligned}\varepsilon(u_{(k+2)}^{n+1} - u_{(k+1)}^{n+1}) &= \int_0^1 G(x,s)[H(u_{(k+1)}^{n+1}) - H(u_{(k)}^{n+1}) - (u_{(k+1)}^{n+1} - u_{(k)}^{n+1})\frac{\partial H}{\partial u_{(k)}^{n+1}}(u_{(k)}^{n+1}) \\ &+ (u_{(k+2)}^{n+1} - u_{(k+1)}^{n+1})\frac{\partial H}{\partial u_{(k+1)}^{n+1}}(u_{(k+1)}^{n+1})]ds, \quad x \in \Omega_x, n \geq 0,\end{aligned} \tag{3.7}$$

where the Green's function $G(x,s)$ is defined by

$$G(x,s) = \begin{cases} x(s-1), & 0 \leq x \leq s \leq 1, \\ (x-1)s, & 0 \leq s \leq x \leq 1, \end{cases} \tag{3.8}$$

and

$$\max_{x,s} G(x,s) = \frac{1}{4}. \tag{3.9}$$

The mean-value theorem gives us

$$H(u_{(k+1)}^{n+1}) - H(u_{(k)}^{n+1}) = (u_{(k+1)}^{n+1} - u_{(k)}^{n+1})\frac{\partial H}{\partial u_{(k)}^{n+1}}(u_{(k)}^{n+1}) + \frac{(u_{(k+1)}^{n+1} - u_{(k)}^{n+1})^2}{2}\frac{\partial^2 H}{\partial (u^{n+1})^2}(\theta), \tag{3.10}$$

where $u_{(k)}^{n+1} \leq \theta \leq u_{(k+1)}^{n+1}$. Now putting the value of $H(u_{(k+1)}^{n+1}) - H(u_{(k)}^{n+1})$, in Eq. (3.7), we get the following estimate

$$\begin{aligned}\varepsilon(u_{(k+2)}^{n+1} - u_{(k+1)}^{n+1}) &= \int_0^1 G(x,s)[\frac{(u_{(k+1)}^{n+1} - u_{(k)}^{n+1})^2}{2}\frac{\partial^2 H}{\partial (u^{n+1})^2}(\theta) \\ &+ (u_{(k+2)}^{n+1} - u_{(k+1)}^{n+1})\frac{\partial H}{\partial u_{(k+1)}^{n+1}}(u_{(k+1)}^{n+1})]ds, \quad x \in \Omega_x, n \geq 0.\end{aligned} \tag{3.11}$$

Let

$$\max_{|u^{n+1}|\leq 1}\left|\frac{\partial^2 H}{\partial (u^{n+1})^2}(u^{n+1})\right| = p, \text{ and } \max_{|u^{n+1}|\leq 1}\left|\frac{\partial H}{\partial u^{n+1}}(u^{n+1})\right| = q. \tag{3.12}$$

Therefore, using Eqs. (3.9),(3.12), in Eq. (3.11) we obtain

$$\left|u^{n+1}_{(k+2)} - u^{n+1}_{(k+1)}\right| \leq \frac{1}{4\varepsilon}\int_0^1 \left\{\frac{p}{2}(u^{n+1}_{(k+1)} - u^{n+1}_{(k)})^2 + q\left|u^{n+1}_{(k+2)} - u^{n+1}_{(k+1)}\right|\right\} ds. \tag{3.13}$$

Taking the maximum norm over the spatial domain and after some simplification, we have

$$\begin{aligned}\left\|u^{n+1}_{(k+2)} - u^{n+1}_{(k+1)}\right\|_{\bar{\Omega}_x} &\leq \frac{p}{(8\varepsilon - 2q)}\left\|u^{n+1}_{(k+1)} - u^{n+1}_{(k)}\right\|^2_{\bar{\Omega}_x}, \\ &\leq C\left\|u^{n+1}_{(k+1)} - u^{n+1}_{(k)}\right\|^2_{\bar{\Omega}_x}.\end{aligned} \tag{3.14}$$

Thus with the judicious choice of initial approximation $u^{n+1}_{(0)}$, the sequence of quasi-linearization process converges quadratically. □

4. A PRIORI ESTIMATES FOR SPATIAL DISCRETIZATION

In this section, bounds for the solution of the semi-discretized problem (3.2) and its derivatives are derived. Further we analyse the asymptotic behavior of the solution and obtain bounds for the smooth and singular components of the solution separately which are used in the convergence analysis of the totally discrete scheme.

Lemma 4.1. *If $\bar{u}^{n+1}(x)$ is the solution of the problem (3.2), then $\forall\varepsilon > 0$, there exists a constant C such that*

$$\left\|\frac{\partial^i \bar{u}^{n+1}}{\partial x^i}\right\|_{\bar{\Omega}_x} \leq C\left(1 + \varepsilon^{-i}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \qquad 0 \leq i \leq 4.$$

Proof. The proof follows inductively by differentiating the problem (3.2) with respect to x up to $i = 4$. To derive these bounds, we rewrite (3.2) in the form

$$\tilde{L}_{x,\varepsilon}\bar{u}^{n+1} = -\frac{\bar{u}^{n+1} - \bar{u}^n}{\Delta t} + f(x) \equiv h_1(x) + f(x), \qquad x \in \Omega_x,\ n \geq 0, \tag{4.1a}$$

$$\bar{u}^{n+1}(0) = 0, \qquad \bar{u}^{n+1}(1) = 0, \quad n \geq 0, \tag{4.1b}$$

One can easily see that the operator $(I + \Delta t\tilde{L}_{x,\varepsilon})$ satisfy the maximum principle, which directly gives $\|\bar{u}^{n+1}\|_{\bar{\Omega}_x} \leq C$. Also, under the sufficient smoothness of the function $f(x)$, the function $h_1(x) + f(x)$ is continuous and ε-uniformly bounded in the spatial domain. Fix the temporal variable $t \in \bar{\Omega}_t$ at $(n+1)$th time level. Now using the technique of Kellogg & Tsan [17], it is easy to deduce that

$$\left|\frac{\partial \bar{u}^{n+1}}{\partial x}\right| \leq C\left(1 + \varepsilon^{-1}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \qquad \forall x \in \bar{\Omega}_x. \tag{4.2}$$

This approach can be extended to higher-order derivatives, by differentiating the Eq. (4.1) with respect to x. For example, the function $y(x) \equiv \partial\bar{u}^{n+1}/\partial x$ is the solution of the following boundary value problem:

$$\tilde{L}_{x,\varepsilon}y = \frac{\partial h_1}{\partial x} + \frac{\partial f}{\partial x} - \frac{\partial a}{\partial x}y - \frac{\partial b}{\partial x}\bar{u}^{n+1} \equiv h_2(x), \qquad x \in \Omega_x, \tag{4.3a}$$

(4.3b) $$y(0) = C_1, \quad y(1) = C_2\varepsilon^{-1},$$

Now, by using the stability of the operator $(I + \Delta t\tilde{L}_{x,\varepsilon})$ and Eq. (4.2) with the assuming smoothness conditions, it is easily seen that the function $h_2(x)$ is continuous and uniformly bounded *i.e.*, we have

(4.4) $$|h_2(x)| \leq C\left(1 + \varepsilon^{-1}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \qquad \forall x \in \bar{\Omega}_x.$$

Therefore, the bound on $h_2(x)$, gives the required bound on second derivative of $\bar{u}^{n+1}(x)$, by applying the same technique of Kellogg and Tsan [17] for Eq. (4.3). In the same manner, bounds are established for the third and fourth order derivative. □

In order to obtain more precise error estimates, we need to derive stronger bounds on the derivatives of the solution of the semi-discretized problem (3.2). These sharper bounds are obtained by decomposition of the solution $\bar{u}^{n+1}(x)$ into smooth and singular components at the $(n+1)th$ time step.

Theorem 4.2. *Assume that the solution $\bar{u}^{n+1}(x)$ of the semi-discretized boundary value problem* (3.2) *is decomposed into regular and singular components as*

$$\bar{u}^{n+1}(x) = v^{n+1}(x) + w^{n+1}(x), \qquad \forall x \in \bar{\Omega}_x,$$

then for all non-negative integer i such that $0 \leq i \leq 4$, the regular component $v^{n+1}(x)$ satisfies

$$\left\|\frac{\partial^i v^{n+1}}{\partial x^i}\right\|_{\bar{\Omega}_x} \leq C\left(1 + \varepsilon^{(3-i)}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right),$$

and the singular component $w^{n+1}(x)$ satisfies

$$\left\|\frac{\partial^i w^{n+1}}{\partial x^i}\right\|_{\bar{\Omega}_x} \leq C\left(\varepsilon^{-i}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right).$$

Proof. The regular (smooth) component $v^{n+1}(x)$ satisfies the following non-homogeneous problem at the $(n+1)$th time level

(4.5) $$(I + \Delta t\tilde{L}_{x,\varepsilon})v^{n+1}(x) = \bar{u}^n + \Delta t f(x), \qquad \forall x \in \Omega_x,$$

(4.6) $$v^{n+1}(0) = \bar{u}^{n+1}(0),$$

and the singular component $w^{n+1}(x)$ satisfies the homogeneous problem

(4.7a) $$(I + \Delta t\tilde{L}_{x,\varepsilon})w^{n+1}(x) = 0, \qquad \forall x \in \Omega_x,$$

(4.7b) $$w^{n+1}(0) = 0, \quad w^{n+1}(1) = \bar{u}^{n+1}(1) - v^{n+1}(1).$$

Taking the four term asymptotic expansion for the regular component as

(4.8) $$v^{n+1}(x) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \varepsilon^3 v_3(x), \qquad \forall x \in \bar{\Omega}_x,$$

where, $v_0(x)$ is the solution of the following reduced problem

$$\Delta t a(x)\frac{\partial v_0}{\partial x} + (1+\Delta t b(x))v_0 = \bar{u}^n + \Delta t f(x), \qquad \forall x \in \Omega_x, \tag{4.9}$$

$$v_0(0) = \bar{u}^{n+1}(0). \tag{4.10}$$

Also, $v_1(x), v_2(x)$, and $v_3(x)$ satisfy the following relation

$$\Delta t a(x)\frac{\partial v_1}{\partial x} + (1+\Delta t b(x))v_1 = \Delta t\frac{\partial^2 v_0}{\partial x^2}, \qquad \forall x \in \Omega_x, \tag{4.11a}$$

$$v_1(0) = 0, \tag{4.11b}$$

$$\Delta t a(x)\frac{\partial v_2}{\partial x} + (1+\Delta t b(x))v_2 = \Delta t\frac{\partial^2 v_1}{\partial x^2}, \qquad \forall x \in \Omega_x, \tag{4.11c}$$

$$v_2(0) = 0, \tag{4.11d}$$

$$(I + \Delta t\tilde{L}_{x,\varepsilon})v_3(x) = \Delta t\frac{\partial^2 v_2}{\partial x^2}, \qquad \forall x \in \Omega_x, \tag{4.11e}$$

$$v_3(0) = 0, \qquad v_3(1) = 0. \tag{4.11f}$$

Thus the regular component $v^{n+1}(x)$ is the solution of

$$(I + \Delta t\tilde{L}_{x,\varepsilon})v^{n+1}(x) = \bar{u}^n + \Delta t f(x), \qquad \forall x \in \Omega_x, \tag{4.12a}$$

$$v^{n+1}(0) = \bar{u}^{n+1}(0), \tag{4.12b}$$

$$v^{n+1}(1) = v_0(1) + \varepsilon v_1(1) + \varepsilon^2 v_2(1). \tag{4.12c}$$

Since $v_0(x)$ is the solution of the reduced problem, with bounded coefficient and independent of ε, therefore for all non-negative integers i, such that $0 \le i \le 4$, we have

$$\left\|\frac{\partial^i v_0}{\partial x^i}\right\|_{\bar{\Omega}_x} \le C, \tag{4.13}$$

In the same manner assuming the sufficient smoothness of the data, $v_1(x), v_2(x)$ are independent of the ε, therefore for all non-negative integers i, such that $0 \le i \le 4$, we have

$$\left\|\frac{\partial^i v_k}{\partial x^i}\right\|_{\bar{\Omega}_x} \le C, \qquad k = 1,\, 2. \tag{4.14}$$

Since, $v_3(x)$ is the solution of the boundary value problem similar to the problem (3.2), therefore by using Lemma 4.1, for all non-negative integers i, such that $0 \le i \le 4$, we have

$$\left\|\frac{\partial^i v_3}{\partial x^i}\right\|_{\bar{\Omega}_x} \le C\left(1 + \varepsilon^{-i}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right). \tag{4.15}$$

Now using above estimates from Eqs. (4.13)-(4.15) into Eq. (4.8), we obtain desired estimates for regular component $v^{n+1}(x)$ and its derivatives. To obtain the required bounds on the singular component $w^{n+1}(x)$, construct two barrier functions defined as

$$\Psi^{\pm}(x) = C\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right) \pm w^{n+1}(x), \qquad \forall x \in \bar{\Omega}_x.$$

Now, for the sufficiently large value of C, we have

$$\Psi^{\pm}(0) = C\exp(-\alpha/\varepsilon) \geq 0,$$
$$\Psi^{\pm}(1) = C \pm (\bar{u}^{n+1}(1) - (v_0(1) + \varepsilon v_1(1) + \varepsilon^2 v_2(1))) \geq 0.$$

and

$$(I + \Delta t\tilde{L}_{x,\varepsilon})\Psi^{\pm}(x) = C\left\{\Delta t\frac{\alpha}{\varepsilon}(-\alpha + a(x)) + (1 + \Delta t b(x))\right\}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right),$$

since, $a(x) \geq \alpha > 0$ and $b(x) \geq \beta > 0$, therefore we have

$$(I + \Delta t\tilde{L}_{x,\varepsilon})\Psi^{\pm}(x) \geq 0, \qquad \forall x \in \Omega_x.$$

Furthermore, the operator $(I + \Delta t\tilde{L}_{x,\varepsilon})$ satisfies the maximum principle, therefore we have $\Psi^{\pm}(x) \geq 0, \forall x \in \bar{\Omega}_x$, which yields

$$\left\|w^{n+1}(x)\right\|_{\bar{\Omega}_x} \leq C\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right). \tag{4.16}$$

Furthermore, to derive the bound for $\partial w^{n+1}/\partial x$, we use the fact that

$$w^{n+1}(x) = \int_0^x H_w(s)ds + \kappa\int_0^x \exp(-A(s))ds, \qquad \forall x \in \bar{\Omega}_x, \tag{4.17}$$

where,

$$H_w(x) = -\frac{1}{\varepsilon}\int_x^1 a(s)w^{n+1}(s)\exp(A(s) - A(x))ds, \;\; A(x) = \frac{1}{\varepsilon}\int_x^1 a(s)ds, \;\; \forall x \in \bar{\Omega}_x.$$

The lower bound on the coefficient $a(x)$ and the bound for the singular term $w^{n+1}(x)$ lead to the following estimate for $H_w(x)$

$$|H_w(x)| \leq \frac{C}{\varepsilon}\int_x^1 \exp\left(-\frac{\alpha(1-s)}{\varepsilon}\right).\exp\left(-\frac{\alpha(s-x)}{\varepsilon}\right)ds$$
$$\leq \frac{C}{\varepsilon}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right), \qquad \forall x \in \bar{\Omega}_x.$$

The coefficient κ is determined by using the boundary condition for $w(x)$ at the boundary $x = 1$, such that

$$\kappa = \frac{1}{\gamma}\left(\bar{u}^{n+1}(1) - v(1) - \int_0^1 H_w(s)ds\right),$$

where,

$$\gamma = \int_0^1 \exp(-A(s))\,ds \geq \int_0^1 \exp\left(-\frac{\|a\|_{\bar{\Omega}_x}(1-s)}{\varepsilon}\right)ds \geq \frac{C\varepsilon}{\|a\|_{\bar{\Omega}_x}}.$$

Thus

$$|\kappa| \leq C\varepsilon^{-1}.$$

Now, from the Eq. (4.17), we have

$$\frac{\partial w^{n+1}}{\partial x} = H_w(x) + \kappa \exp(-A(x)), \qquad \forall x \in \bar{\Omega}_x, \tag{4.18}$$

therefore, using the ε-uniform bounds for $H_w(x)$ and κ, we have

$$\left\| \frac{\partial w^{n+1}}{\partial x} \right\|_{\bar{\Omega}_x} \le C\varepsilon^{-1} \exp\left(-\frac{\alpha(1-x)}{\varepsilon} \right). \tag{4.19}$$

Similarly, for $i = 2, 3, 4$, one can easily obtain desired bounds by repeated differentiations of $(I + \Delta t \tilde{L}_{x,\varepsilon}) w^{n+1}(x) = 0$. Following these arguments for all non-negative integers i such that $0 \le i \le 4$, we obtain desired bounds on the singular component. □

Remark 4.3. In general, under the sufficient smoothness requirements on data, above theorem holds for any arbitrary value of i. In particular, by taking further decompositions of the solution, this theorem shows that the smooth component $v^{n+1}(x)$ and its derivatives are bounded by a constant value C, which is independent of ε, while the singular component $w^{n+1}(x)$ satisfies the same estimates as in the first decomposition.

5. Spatial Discretization

In this section, we construct the totally discrete scheme by using a monotone difference operator in the spatial direction. Since the problem (1.1) has an inviscid boundary layer in the neighborhood of the outflow boundary Γ_r, therefore to resolve this boundary layer we use a special piecewise uniform mesh, called Shishkin mesh, which will condense large number of mesh points in the boundary layer region as $\varepsilon \to 0$. Shishkin mesh is much simpler than the Bakhvalov [1] and Gartland [10] meshes. Shishkin mesh is define as follows:

5.1. **Shishkin Mesh.** Shishkin meshes are piecewise equidistant meshes, constructed *a priori* as a function, that partly resolve the boundary layers. To construct them correctly, it is crucial to have a precise knowledge of the asymptotic behavior of the exact solution. For $N \ge 2^r$, where $r \ge 2$ is an integer, the piecewise uniform Shishkin mesh $\bar{\Omega}_x^N$ is designed by partitioning the spatial domain $\bar{\Omega}_x$ into two subintervals $\Omega_1 = [0, 1-\tau]$ and $\Omega_2 = (1-\tau, 1]$ such that $\bar{\Omega}_x = \Omega_1 \cup \Omega_2$. Here, transition parameter τ is defined by the following function of ε, α and N as

$$\tau = \min\left\{ \frac{1}{2}, \frac{2\varepsilon}{\alpha} \log N \right\},$$

Moreover, mesh spacing $\tilde{h}$ in spatial direction is given by

$$\tilde{h} = \begin{cases} \tilde{h}_1 = h_i = (2(1-\tau))/N, & \text{if } i = 1, 2, \ldots, N/2, \\ \tilde{h}_2 = h_i = 2\tau/N, & \text{if } i = N/2+1, \ldots, N. \end{cases} \tag{5.1}$$

Therefore, set of mesh points $\bar{\Omega}_x^N = \{x_i\}_{i=0}^N$ is given by

$$x_i = \begin{cases} (2(1-\tau)/N)i, & \text{if } i = 0, 1, 2, \dots, N/2, \\ (1-\tau) + (2\tau/N)(i - N/2) & \text{if } i = N/2+1, \dots, N. \end{cases} \tag{5.2}$$

Thus, when $\tau = 1/2$, the mesh is uniform, otherwise mesh condenses near the right part Γ_r of the lateral surface. Now, we define the resulting piecewise uniform fitted mesh to be the tensor product $D^N = \Omega_x^N \times \Omega_t^n$ and its boundary points Γ^N are $\Gamma^N = \bar{D}^N \cap \Gamma$.

5.2. **Hybrid Finite Difference Scheme.** The monotone hybrid difference scheme is a composition of upwinding and central differencing on a special piecewise equidistant mesh in the spatial domain $\bar{\Omega}_x$. We employ the upwind finite difference operator on the coarse mesh region Ω_1 and central difference operator on the fine mesh region Ω_2, whenever the local mesh size allows us to do this without losing stability. The totally discrete approximation is considered as

$$\bar{u}_i^0 = \bar{u}^0(x_i), \qquad i = 0, 1, \dots, N, \tag{5.3a}$$

$$\left(I + \Delta t \tilde{L}_{x,\varepsilon}^N\right) \bar{u}_i^{n+1} = g_i^n, \quad i = 1, 2, \dots, N-1, \tag{5.3b}$$

$$\bar{u}_0^{n+1} = 0, \qquad \bar{u}_N^{n+1} = 0 \qquad n \geq 0, \tag{5.3c}$$

where, discrete linear operator $\tilde{L}_{x,\varepsilon}^N$ is defined as

(5.4)

$$\tilde{L}_{x,\varepsilon}^N \bar{u}_i^{n+1} = \begin{cases} \tilde{L}_{x,\varepsilon,\mathrm{up}}^N \bar{u}_i^{n+1} = \left(-\varepsilon\delta_x^2 + a_i D_x^{-1} + b_i I\right)\bar{u}_i^{n+1}, & i = 1, 2, \dots N/2, \\ \tilde{L}_{x,\varepsilon,\mathrm{c}}^N \bar{u}_i^{n+1} = \left(-\varepsilon\delta_x^2 + a_i D_x^{0} + b_i I\right)\bar{u}_i^{n+1}, & i = N/2+1, \dots N-1, \end{cases}$$

and

$$g_i^n = \bar{u}_i^n + \Delta t f_i, \qquad i = 1, 2, \dots, N-1. \tag{5.5}$$

Here,

$$a_i = a(x_i),\ b_i = b(x_i),\ f_i = f(x_i),\ g_i^n = g(x_i, t^n),$$

First order derivatives of $u(x,t)$ with respect to the spatial variable at the point (x_i, t^n) corresponding to forward, backward and central difference operators, are given by

$$D_x^+ u_i^n = \frac{u_{i+1}^n - u_i^n}{h_{i+1}},\ D_x^- u_i^n = \frac{u_i^n - u_{i-1}^n}{h_i},\ D_x^0 u_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{h_i + h_{i+1}},$$

respectively. We shall approximate second-order derivative at (x_i, t^n) by

$$\delta_x^2 u_i^n = \frac{1}{\bar{h}_i}\left(D_x^+ u_i^n - D_x^- u_i^n\right) \text{ where } \bar{h}_i = \frac{h_i + h_{i+1}}{2}.$$

Finally, after simplification, the totally discrete approximation (5.3) takes the following form

$$\bar{u}_i^0 = \bar{u}^0(x_i), \qquad i = 0, 1, \dots, N, \tag{5.6a}$$

(5.6b)
$$\begin{cases} p_i^-\bar{u}_{i-1}^{n+1} + p_i^c\bar{u}_i^{n+1} + p_i^+\bar{u}_{i+1}^{n+1} = g_i^n, & i = 1, 2, \ldots, N/2, \\ q_i^-\bar{u}_{i-1}^{n+1} + q_i^c\bar{u}_i^{n+1} + q_i^+\bar{u}_{i+1}^{n+1} = g_i^n, & i = N/2+1, \ldots, N-1, \end{cases}$$

(5.6c)
$$\bar{u}_0^{n+1} = 0, \qquad \bar{u}_N^{n+1} = 0, \qquad n \geq 0,$$

where, elements in the system matrix $(I + \Delta t\tilde{L}_{x,\varepsilon}^N)$ are as follows

$$\begin{aligned} p_i^- &= -\left(\frac{\Delta t\varepsilon}{h_i\bar{h}_i} + \frac{\Delta t a_i}{h_i}\right), & p_i^c &= \left(1 + \Delta t b_i - p_i^- - p_i^+\right), \\ p_i^+ &= -\left(\frac{\Delta t\varepsilon}{h_{i+1}\bar{h}_i}\right), & & i = 1, 2, \ldots, N/2, \\ q_i^- &= -\left(\frac{\Delta t\varepsilon}{h_i\bar{h}_i} + \frac{\Delta t a_i}{h_i + h_{i+1}}\right), & q_i^c &= \left(1 + \Delta t b_i - q_i^- - q_i^+\right) \\ q_i^+ &= -\left(\frac{\Delta t\varepsilon}{h_{i+1}\bar{h}_i} - \frac{\Delta t a_i}{h_i + h_{i+1}}\right), & & i = N/2+1, \ldots, N-1. \end{aligned}$$

6. STABILITY AND CONVERGENCE ANALYSIS

In this section, we establish the stability and ε-uniform error estimate for the totally discrete scheme by decomposing the approximate solution $\bar{u}_i^n$ in an analogous manner as that of the continuous solution $\bar{u}^n(x)$ at nth time step. For the sake of simplicity, we denote the discrete solution u_i^n by $u^N(x_i, t^n)$ during convergence analysis. In order to attain a monotone discrete operator $(I + \Delta t\tilde{L}_{x,\varepsilon}^N)$, we impose the following mild assumption on the minimum number of mesh points

(6.1)
$$\frac{\tilde{h}_2\|a\|_{\bar{\Omega}_x}}{2\varepsilon} < 1, \ \textit{i.e.}, \ \frac{N}{\log N} > 2\frac{\|a\|_{\bar{\Omega}_x}}{\alpha}.$$

The analysis is based on the discrete maximum principle and barrier function technique introduced by Kellogg and Tsan [17]. We start by stating the following discrete maximum principle.

Lemma 6.1 (Discrete Maximum Principle)**.** *Under the assumption* (6.1)*, the totally discrete scheme* (5.6) *satisfies a discrete maximum principle for any mesh function* ψ^N *defined on* $\bar{D}^N = \bar{\Omega}_x^N \times \bar{\Omega}_t^n$ *such that if* $\psi^N(x_i, t^n) \geq 0\ \forall (x_i, t^n) \in \Gamma^N$ *and* $(I + \Delta t\tilde{L}_{x,\varepsilon}^N)\psi^N(x^i, t^n) \geq 0\ \forall (x_i, t^n) \in D^N$*, then* $\psi^N(x_i, t^n) \geq 0\ \forall (x_i, t^n) \in \bar{D}^N$*.*

Proof. To establish the discrete maximum principle, we will simply check the following inequalities to show that the associated system matrix is an M-matrix:

$$p_i^- < 0, \quad p_i^+ < 0, \quad p_i^- + p_i^c + p^+ > 0, \quad i = 1, 2, \ldots, N/2,$$

and under the assumption (6.1), we have

$$q_i^- < 0, \quad q_i^+ < 0, \quad q_i^- + q_i^c + q^+ > 0, \quad i = N/2+1, \ldots, N-1.$$

From these sign patterns, it is easily seen that the system matrix $(I+\Delta t\tilde{L}^N_{x,\varepsilon})$ is an $(N-1)\times(N-1)$ irreducible M-matrix and so has a positive inverse. Moreover, discrete system (5.3) satisfies the desired discrete maximum principle. Discrete maximum principle ensures the stability of the spatial discretization process. □

Lemma 6.2. *Let Z^N be any mesh function defined on $\bar{D}^N$, such that*

$$Z^N(x_i,t^n)=0, \qquad \forall (x_i,t^n)\in\Gamma^N,$$

then

$$|Z^N(x_i,t^n)|\le \frac{1}{\alpha\Delta t}\max_{D^N}|(I+\Delta t\tilde{L}^N_{x,\varepsilon})Z^N(x_i,t^n)|, \quad \forall (x_i,t^n)\in\bar{D}^N.$$

Proof. Construct the mesh functions

$$\psi^{\pm}(x_i,t^n)=\frac{1}{\alpha\Delta t}\max_{D^N}|(I+\Delta t\tilde{L}^N_{x,\varepsilon})Z^N(x_i,t^n)|x_i\pm Z^N(x_i,t^n).$$

Now applying the discrete maximum principle (Lemma 6.1), we get the desired estimate. □

To prove the uniform convergence of the proposed difference scheme, we construct the following barrier functions for all $n\Delta t\le T$.

$$\psi_i^n(\alpha)=\begin{cases}\prod_{j=1}^{i}\left(1+\frac{\alpha h_j}{\varepsilon}\right), & i=1,2,\dots,N,\\ 1, & i=0.\end{cases} \tag{6.2}$$

Lemma 6.3. *The barrier functions $\psi_i^n(\alpha)$ satisfy the following inequalities*

$$\left(I+\Delta t\tilde{L}^N_{x,\varepsilon}\right)\psi_i^n(\alpha)\ge\frac{C(\alpha)\Delta t}{(\varepsilon+\alpha h_i)}\psi_i^n(\alpha), \quad i=1,2,\dots,N-1,\ n\Delta t\le T,$$

for some positive constant $C(\alpha)$.

Proof. From the definition of barrier functions, we have

$$\left(\frac{\psi_i^n(\alpha)-\psi_{i-1}^n(\alpha)}{h_i}\right)=\frac{\alpha\psi_{i-1}^n(\alpha)}{\varepsilon},\ \left(\frac{\psi_{i+1}^n(\alpha)-\psi_{i-1}^n(\alpha)}{h_i+h_{i+1}}\right)=\frac{\alpha}{\varepsilon}\left(\frac{h_{i+1}\psi_i^n(\alpha)+h_i\psi_{i-1}^n(\alpha)}{h_i+h_{i+1}}\right),$$

therefore, applying the operator $(I+\Delta t\tilde{L}^N_{x,\varepsilon})$ to the discrete function ψ_i^n, and doing some simplifications, we arrive at the estimate,

$$(I+\Delta t\tilde{L}^N_{x,\varepsilon})\psi_i^n(\alpha)=\begin{cases}\left[1+\Delta t\left(\frac{\alpha}{\varepsilon+\alpha h_i}\left(-\frac{2\alpha h_i}{h_i+h_{i+1}}+a_i+b_i\frac{\varepsilon+\alpha h_i}{\alpha}\right)\right)\right]\psi_i^n(\alpha), & \\ & i=1,2,\dots,N/2,\\ \left[1+\Delta t\left(\frac{\alpha}{\varepsilon+\alpha h_i}\left(-\frac{2\alpha h_i}{h_i+h_{i+1}}+a_i+b_i\frac{\varepsilon+\alpha h_i}{\alpha}\right)\right)\right]\psi_i^n(\alpha)+ & \\ b_i\left(\frac{\alpha^2h_ih_{i+1}}{\varepsilon^2(h_i+h_{i+1})}\right)\psi_{i-1}^n(\alpha), & i=N/2+1,\dots,N-1,\end{cases}$$

from which we deduce the desired result. □

Furthermore, we give the following truncation error bounds for upwind and central difference operators employed in $\tilde{L}^N_{x,\varepsilon}$.

Lemma 6.4. *At the time level t^n, for $u^n(x) \in C^4(\bar{\Omega}_x)$, the local truncation error at spatial discretization stage for the operator $(I + \Delta t\tilde{L}^N_{x,\varepsilon})$ is given by*

$$\begin{aligned}
|\rho_i^{up}| &= \left|(I + \Delta t\tilde{L}^N_{x,\varepsilon})(u_i^n) - \left((I + \Delta t\tilde{L}_{x,\varepsilon})u^n\right)(x_i)\right| \\
&\le C\Delta t\left[\varepsilon\int_{x_{i-1}}^{x_{i+1}}\left|\frac{\partial^3 u^n}{\partial s^3}\right| ds + \int_{x_{i-1}}^{x_i}\left|\frac{\partial^2 u^n}{\partial s^2}\right| ds\right], \qquad i = 1, 2, \ldots, N/2, \\
|\rho_i^{c}| &= \left|(I + \Delta t\tilde{L}^N_{x,\varepsilon})(u_i^n) - \left((I + \Delta t\tilde{L}_{x,\varepsilon})u^n\right)(x_i)\right| \\
&\le C\Delta t h_i\left[\varepsilon\int_{x_{i-1}}^{x_{i+1}}\left|\frac{\partial^4 u^n}{\partial s^4}\right| ds + \int_{x_{i-1}}^{x_{i+1}}\left|\frac{\partial^3 u^n}{\partial s^3}\right| ds\right], \quad i = N/2 + 1 \ldots, N-1,
\end{aligned}$$

where, C is a positive constant depends on $\|a\|$ and $\|a'\|$.

Proof. By using the valid Taylor series expansion with the integral form of the remainder, or by Peano's theorem [7], we obtain the desired truncation error estimates. □

In order to prove the uniform convergence of the proposed hybrid finite difference scheme, we use the following estimate.

Lemma 6.5. *For each i and $\alpha > 0$, we have*

$$\prod_{j=i+1}^{N}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1} \ge \exp\left(-\alpha\frac{(1 - x_i)}{\varepsilon}\right),$$

Proof. The proof follows by an easy computation. □

Lemma 6.6. *For the Shishkin mesh defined above, there exists a constant C, such that*

$$\prod_{j=i+1}^{N}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1} \le \begin{cases} CN^{-2}, & \forall i = 0, 1, \ldots, N/2, \\ CN^{-4(1-i/N)}, & \forall i = N/2, \ldots, N-1. \end{cases}$$

Proof. In particular, for $i = 0, 1, \ldots, N/2$, we have

$$\begin{aligned}
\prod_{j=i+1}^{N}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1} &\le \prod_{j=N/2+1}^{N}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1} \\
&\le \exp\left(-\alpha(1 - x_{N/2})/(\varepsilon + \alpha\tilde{h}_2)\right), \text{ (by Lemma 4.1(b) in [17])} \\
&= \exp\left(-\alpha\tau/(\varepsilon + 2\alpha\tau N^{-1})\right) \\
&= \exp\left(-2\log N/(1 + 4N^{-1}\log N)\right) \\
&= N^{-2/(1+4N^{-1}\log N)} \\
&\le CN^{-2}.
\end{aligned}$$

The required bound for $i = N/2, \ldots, N-1$ also follows using the same argument given in Lemma 4.1(b) in [17]. □

To estimate the error in the regular (smooth) component and singular component separately in the spatial direction at the n-th time step, we decompose the numerical solution into a regular and singular part as

$$\bar{u}^N(x_i,t^n) = v^N(x_i,t^n) + w^N(x_i,t^n), \qquad \forall x_i \in \bar{\Omega}_x^N,\ n\Delta t \le T,$$

where, the regular component $v^N(x_i,t^n)$ satisfies the non-homogeneous equation

$$\left(I + \Delta t \tilde{L}_{x,\varepsilon}^N\right)(v^N(x_i,t^n)) = g(x_i,t^n), \qquad \forall x_i \in \Omega_x^N,\ n\Delta t \le T, \tag{6.3a}$$

$$v^N(x_i,t^n) = v(x_i,t^n), \qquad \forall (x_i,t^n) \in \Gamma^N, \tag{6.3b}$$

and singular part $w^N(x_i,t^n)$ is the solution of the problem

$$\left(I + \Delta t \tilde{L}_{x,\varepsilon}^N\right)(w^N(x_i,t^n)) = 0, \qquad \forall x_i \in \Omega_x^N,\ n\Delta t \le T, \tag{6.4a}$$

$$w^N(x_i,t^n) = w(x_i,t^n), \qquad \forall (x_i,t^n) \in \Gamma^N. \tag{6.4b}$$

Therefore, we have

$$(\bar{u}^N - \bar{u})(x_i,t^n) = (v^N - v)(x_i,t^n) + (w^N - w)(x_i,t^n), \qquad \forall x_i \in \Omega_x^N,\ n\Delta t \le T,$$

and we estimate the error in the regular and singular component separately.

Theorem 6.7 (Error in the Regular Component)**.** *Assume* (6.1)*. The error in the regular component* $v^N(x_i,t^n)$ *satisfies the following error estimate at the nth time level*

$$|(v^N - v)(x_i,t^n)| \le \begin{cases} CN^{-1}, & i = 0,1,\dots,N/2,\ n\Delta t \le T, \\ CN^{-2}, & i = N/2+1\dots,N,\ n\Delta t \le T. \end{cases}$$

Proof. From Lemma 6.4, the truncation error in the smooth component is given by

$$|(I + \Delta t \tilde{L}_{x,\varepsilon}^N)(v^N - v)(x_i,t^n)| \le \begin{cases} C\Delta t \left[\varepsilon(h_{i+1}+h_i)\left\|\frac{\partial^3 v}{\partial x^3}\right\|_{\bar{\Omega}_x} + h_i\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{\bar{\Omega}_x}\right], \\ \qquad\qquad i = 0,1,\dots,N/2,\ n\Delta t \le T, \\ C\Delta t\left[h_i(h_{i+1}+h_i)\left(\varepsilon\left\|\frac{\partial^4 v}{\partial x^4}\right\|_{\bar{\Omega}_x} + \left\|\frac{\partial^3 v}{\partial x^3}\right\|_{\bar{\Omega}_x}\right)\right], \\ \qquad\qquad i = N/2+1\dots,N,\ n\Delta t \le T. \end{cases}$$

Note that $h_{i+1} + h_i \le 2N^{-1}$ is always true for both the cases of uniform mesh and piecewise uniform Shishkin mesh. Furthermore, using the bounds on the derivatives of v given in Theorem 4.2, we get

$$|(I + \Delta t \tilde{L}_{x,\varepsilon}^N)(v^N - v)(x_i,t^n)| \le \begin{cases} C\Delta t N^{-1}(\varepsilon+1), & i = 0,1,\dots,N/2,\ n\Delta t \le T, \\ C\Delta t N^{-2}, & i = N/2+1\dots,N,\ n\Delta t \le T. \end{cases}$$

An application of Lemma 6.2 yield the desired bounds. ∎

Theorem 6.8 (Error in the Singular Component)**.** *Under the assumption* (6.1)*, the error in the singular component satisfies the following estimate*

$$|(w^N - w)(x_i, t^n)| \le \begin{cases} CN^{-2}, & i = 0, 1, \ldots, N/2, \; n\Delta t \le T, \\ CN^{-2}(\log N)^2, & i = N/2 + 1 \ldots, N, \; n\Delta t \le T. \end{cases}$$

Proof. In the sub-interval with no boundary layer Ω_1, both w^N and w are small. After applying triangle inequality, it is sufficient to find the bounds on w and w^N separately. Here, we note that

$$(I + \Delta t \tilde{L}^N_{x,\varepsilon}) w^N(x_i, t^n) = 0, \qquad \forall x_i \in \Omega^N_x, \; n\Delta t \le T,$$

$$|w^N(x_0, t^n)| = |w(0, t^n)| \le C \exp(-\alpha/\varepsilon) \le C \prod_{j=1}^{N} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}, \; \forall n\Delta t \le T,$$

and

$$|w^N(x_N, t^n)| = |w(1, t^n)| \le C, \qquad \forall n\Delta t \le T.$$

Furthermore, to obtain the bound on w^N, we consider the following mesh function $\phi^n_i(\alpha)$ for sufficiently large C and $n\Delta t \le T$,

$$\phi^n_i(\alpha) = C \left[\prod_{j=1}^{N} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}\right] \psi^n_i(\alpha).$$

Moreover, using Lemma (6.3) we have,

$$(I + \Delta t \tilde{L}^N_{x,\varepsilon}) \phi^n_i(\alpha) \ge \frac{C\Delta t}{\varepsilon + \alpha h_i} \prod_{j=i+1}^{N} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1} \ge 0, \quad i = 1, 2, \ldots, N-1, \; n\Delta t \le T,$$

and

$$\phi^n_0(\alpha) = C \prod_{j=1}^{N} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}, \quad \phi^n_N(\alpha) = C, \qquad \forall n\Delta t \le T.$$

Therefore, $\phi^n_i(\alpha)$ is a barrier function for $\{w^N(x_i, t^n)\}$. Again, by the discrete maximum principle, we have,

$$|w^N(x_i, t^n)| \le \phi^n_i(\alpha) = C \prod_{j=i+1}^{N} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}, \quad i = 0, 1, \ldots, N, \; n\Delta t \le T. \tag{6.5}$$

Using triangle inequality, we have

$$\begin{aligned} |(w^N - w)(x_i, t^n)| &\le |w(x_i, t^n)| + |w^N(x_i, t^n)| \\ &\le C \exp\left(-\frac{\alpha(1 - x_i)}{\varepsilon}\right) + \prod_{j=i+1}^{N} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1} \\ &\le C \prod_{j=i+1}^{N} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1} \quad \text{(using Lemma 6.5).} \end{aligned}$$

In particular, using Lemma 6.6, we get

$$|(w^N - w)(x_i, t^n)| \le CN^{-2}, \qquad i = 0, 1, \ldots, N/2,\ n\Delta t \le T, \tag{6.6}$$

On the other hand, we use consistency estimate given in Lemma 6.4 and barrier function technique to estimate $|(w-w^N)(x_i,t^n)|$ in the fine mesh region Ω_2. Moreover, truncation error estimate in Lemma 6.4 for $i = N/2+1, \ldots, N-1$ leads to the following estimate

$$\begin{aligned}
&|(I + \Delta t\tilde{L}^N_{x,\varepsilon})(w^N - w)(x_i, t^n)| \\
&\le C\Delta t h_i \left[\varepsilon \int_{x_{i-1}}^{x_{i+1}} \left\|\frac{\partial^4 w}{\partial x^4}\right\|_{\bar{\Omega}_x} dx + \int_{x_{i-1}}^{x_{i+1}} \left\|\frac{\partial^3 w}{\partial x^3}\right\|_{\bar{\Omega}_x} dx\right] \\
&\le C\Delta t h_i \left[\frac{1}{\varepsilon^3}\int_{x_{i-1}}^{x_{i+1}} \exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right) dx\right], \\
&\qquad \text{(using the bounds given in Theorem 4.2)} \\
&= \frac{C\Delta t h_i}{\varepsilon^2\alpha}\left[\exp\left(-\frac{\alpha(1-x_{i+1})}{\varepsilon}\right) - \exp\left(-\frac{\alpha(1-x_{i-1})}{\varepsilon}\right)\right] \\
&= \frac{C\Delta t h_i}{\varepsilon^2\alpha}\exp\left(-\frac{\alpha(1-x_i)}{\varepsilon}\right)\sinh\left(\frac{\alpha\tilde{h}_2}{\varepsilon}\right) \\
&\le \frac{C\Delta t}{\varepsilon}N^{-2}(\log N)^2\exp\left(-\frac{\alpha(1-x_i)}{\varepsilon}\right), \quad \text{(since } \sinh t \le Ct \text{ for } 0 \le t \le 1\text{)},
\end{aligned}$$

$$\le \frac{C\Delta t}{\varepsilon}N^{-2}(\log N)^2\prod_{j=i+1}^{N}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}, \quad \text{(by Lemma 6.5)}. \tag{6.7}$$

It is easy to see that for $i = N/2$ in Eq. (6.6), we have

$$|(w^N - w)(x_{N/2}, t^n)| \le CN^{-2}.$$

Furthermore,

$$|(w^N - w)(x_N, t^n)| = 0.$$

Using Eq. (6.7), construct the mesh function

$$\phi_i^n(\alpha) = CN^{-2}\left(1 + (\log N)^2\left[\prod_{j=1}^{N}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}\right]\psi_i^n(\alpha)\right), \quad i = N/2, \ldots, N,$$

for sufficiently large value of C. With the help of Lemma 6.3, it is easy to see that

$$|(I + \Delta t\tilde{L}_{x,\varepsilon})\phi_i^n(\alpha)| \ge |(I + \Delta t\tilde{L}_{x,\varepsilon})(w^N - w)(x_i, t^n)|,$$
$$i = N/2 + 1, \ldots, N - 1,\ n\Delta t \le T,$$

$$|\phi_{N/2}^n(\alpha)| = CN^{-2}\left[1 + (\log N)^2\prod_{j=N/2+1}^{N}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}\right] \ge |(w - w^N)(x_{N/2}, t^n)|,$$

and

$$|\phi_N^n(\alpha)| = CN^{-2}\left(1 + (\log N)^2\right) \ge |(w - w^N)(x_N, t^n)|.$$

Therefore, $\phi_i^n(\alpha)$ is a barrier function for $(w^N - w)(x_i, t^n)$ and consequently by the discrete maximum principle (Lemma 6.1), we have

$$|(w^N - w)(x_i, t^n)| \le \phi_i^n(\alpha), \quad i = N/2, \ldots, N, \; n\Delta t \le T,$$

Now Lemma 6.6 gives the estimate

$$|(w^N - w)(x_i, t^n)| \le C \max\{N^{-2}, N^{-6+4i/N}(\log N)^2\}, \quad i = N/2, \ldots, N, \; n\Delta t \le T, \tag{6.8}$$

Hence combining the results from the Eq. (6.6) and Eq. (6.8) proves the theorem. □

Theorem 6.9 (Error in the Spatial Direction)**.** *Let* $\bar{u}^N(x_i, t^n)$ *be the hybrid finite difference approximation in the spatial direction to the solution* $\bar{u}^n(x) \in C^4(\bar{\Omega}_x)$ *of the problem* (3.2) *at n-th time level. Then under the assumption* (6.1)*, following error estimates hold for the proposed hybrid finite difference scheme in the spatial discretization process at the n-th time level*

$$\|(\bar{u}^N - \bar{u})(x_i, t^n)\|_{\bar{\Omega}_x^N} \le \begin{cases} CN^{-1}, & i = 0, 1, \ldots, N/2, \; n\Delta t \le T, \\ CN^{-2}(\log N)^2, & i = N/2+1, \ldots, N, \; n\Delta t \le T. \end{cases}$$

Proof. Follows from Theorem 6.7 and Theorem 6.8. □

Now, we deduce the main result of the paper.

Theorem 6.10 (Error in the Totally Discrete Scheme)**.** *Let* $u(x,t)$ *be the continuous solution of the modified Burger's equation* (1.1)*,* $\bar{u}^n(x)$ *be the solution of the semi-discrete problem* (3.2) *after the temporal discretization and quasi-linearization process and* $\bar{u}^N(x_i, t^n)$ *be the solution of the totally discrete problem* (5.3)*, then under the assumption* (6.1) *following error estimates satisfies for the totally discrete scheme*

$$\|(\bar{u}^N - u)(x_i, t^n)\|_{\bar{D}^N} \le \begin{cases} C\left(\Delta t + N^{-1}\right), & i = 0, 1, \ldots, N/2, \; n\Delta t \le T, \\ C\left(\Delta t + N^{-2}(\log N)^2\right), & i = N/2+1, \ldots, N, \; n\Delta t \le T. \end{cases}$$

where C *is a positive constant independent of* ε *and mesh parameters.*

Proof. The proof easily follows by combining the estimates given in Lemma 2.4 and Theorem 6.9. □

7. NUMERICAL EXPERIMENTS AND RESULTS

In this section, to demonstrate applicability, accuracy and the convergence order of the method presented in this paper, we report some numerical results. Range $(0, 1]$ of the parameter ε shows our interest in the singularly perturbed case.

Example 1. This example corresponds to the following singularly-perturbed non-linear parabolic initial-boundary value problem:

$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}, \qquad (x,t) \in D, \tag{7.1a}$$

with sinusoidal initial condition

$$u(x,0) = \sin(\pi x), \qquad x \in \bar{\Omega}_x, \tag{7.1b}$$

and boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \qquad t \in \bar{\Omega}_t. \tag{7.1c}$$

For the numerical computation, we begin with $N = 16, T = 1$ and $\Delta t = 0.1$ and we multiply N by 2 and divide Δt by 2. For small values of the parameter ε, the exact solution of the modified Burgers' equation is not available, therefore to illustrate the performance of the proposed scheme at low viscosity coefficient ε, we use the double mesh principle to estimate the pointwise error as follows

$$e_\varepsilon^{N,\Delta t}(x_i,t^n) = |u^N(x_i,t^n) - u^{2N}(x_i,t^n)|. \tag{7.2}$$

where the superscript N denotes the number of mesh points in the spatial direction, $t^n = n\Delta t$ and Δt is the time step. For each ε, the maximum nodal error is given by

$$E_\varepsilon^{N,\Delta t} = \max_{i,n} e_\varepsilon^{N,\Delta t}, \tag{7.3}$$

and, for each N and Δt, the ε-uniform maximum pointwise error is define by

$$E^{N,\Delta t} = \max_{\varepsilon} E_\varepsilon^{N,\Delta t}. \tag{7.4}$$

We also tabulate the numerical rate of convergence in the following way

$$p_\varepsilon^{N,\Delta t} = \frac{\log(E_\varepsilon^{N,\Delta t}/E_\varepsilon^{2N,\Delta t/2})}{\log 2}. \tag{7.5}$$

The numerical ε-uniform order of convergence is given by

$$p^{N,\Delta t} = \frac{\log(E^{N,\Delta t}/E^{2N,\Delta t/2})}{\log 2}. \tag{7.6}$$

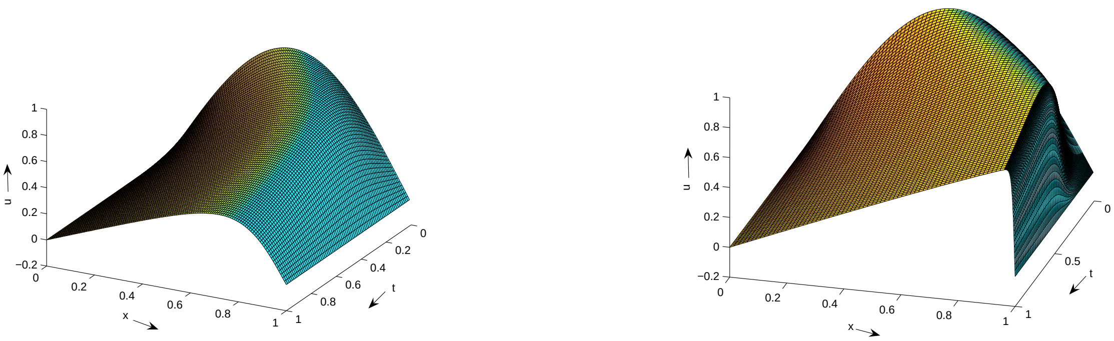
Numerical results are tabulated in Table 1 with piecewise uniform Shishkin mesh for various values of ε.

Table 1. Maximum pointwise errors $E_\varepsilon^{n,\Delta t}$ and numerical order of convergence $p_\varepsilon^{N,\Delta t}$ for Example 1 with Shishkin mesh

$\varepsilon \downarrow$	N=16	N=32	N=64	N=128	N=256
2^{0}	8.4807E-6	5.4448E-7	1.4570E-7	3.4354E-8	1.95E-8
	3.9612	1.9019	2.0844	0.8170	
2^{-2}	1.1621E-3	6.3448E-4	3.2706E-4	1.6858E-4	8.5102E-5
	0.8731	0.9560	0.9562	0.9862	
2^{-4}	5.0003E-3	1.4547E-3	8.1233E-4	4.3922E-4	1.7593E-4
	1.7813	0.8406	0.8871	1.3199	
2^{-6}	1.7164E-2	5.9542E-3	2.8138E-3	8.7279E-4	3.4687E-4
	1.5274	1.0814	1.6888	1.3312	
2^{-8}	3.9275E-2	1.2973E-2	4.5016E-3	2.0154E-3	8.2690E-4
	1.5981	1.5270	1.1593	1.2853	
2^{-10}	3.6458E-2	1.7454E-2	5.3819E-3	2.4288E-3	1.0921E-3
	1.0627	1.6973	1.1479	1.1531	
2^{-12}	3.7310E-2	1.7481E-2	5.7008E-3	2.5924E-3	1.1665E-3
	1.0937	1.6166	1.1369	1.1521	
2^{-14}	2.9973E-2	1.4676E-2	5.7897E-3	2.6373E-3	1.1868E-3
	1.0302	1.3419	1.1344	1.1520	
2^{-16}	2.9658E-2	1.3941E-2	5.8152E-3	2.6220E-3	1.2999E-3
	1.0891	1.2615	1.1492	1.0122	
2^{-18}	2.9574E-2	1.3756E-2	5.8260E-3	2.5338E-3	1.1967E-3
	1.1042	1.2395	1.2012	1.0822	
2^{-20}	2.9553E-2	1.3710E-2	5.8397E-3	2.5118E-3	1.1705E-3
	1.1080	1.2313	1.2172	1.1016	
2^{-22}	2.9547E-2	1.3699E-2	5.8759E-3	2.5063E-3	1.1640E-3
	1.1090	1.2212	1.2293	1.1065	
2^{-24}	2.9546E-2	1.3696E-2	5.9676E-3	2.5047E-3	1.1623E-3
	1.1092	1.1985	1.2525	1.1076	
$\mathbf{E^{N,\Delta t}}$	**3.9275E-2**	**1.7481E-2**	**5.9676E-3**	**2.6373E-3**	**1.2999E-3**
$\mathbf{p^{N,\Delta t}}$	**1.1678**	**1.5506**	**1.1781**	**1.0207**	

8. Discussions and Conclusions

A numerical scheme has been developed to solve modified Burgers' equation subjected to various values of Reynolds numbers. The qualitative aspects of the modified Burgers' equation have been studied by means of singular perturbation theory. Modified Burgers' equation is a non-linear problem and at high Reynolds number, it produces a sharp gradient in the boundary layer region with a smooth initial data, when the Dirichlet boundary condition is employed. Solutions of such problem at



(a) (b)

FIGURE 1. Numerical solution profiles of Example-1 with $N = 128$ and $\Delta t = 1/80$ and different values of ε (a) $\varepsilon = 2^{-4}$, and (b) $\varepsilon = 2^{-8}$.

high values of R, have a rather complicated behavior in a neighborhood of boundary layer. It is well known that an accurate resolution of such boundary layer is a challenging numerical task. Taking more mesh points in the boundary layer can lead to an outstanding result for a much larger value of R, therefore we have used a special piecewise uniform Shishkin mesh for its simple structure. To tackle the non-linearity, quasi-linearization process is used and shown that the quasi-linearization process converges quadratically to the solution of the original non-linear problem. A brief analysis has been carried out to prove the uniform convergence of the proposed scheme and show the parameter free linear convergence in the temporal direction and first order uniform convergence in the region Ω_1 outside from the boundary layer region and almost quadratic uniform convergence in the boundary layer region Ω_2 for the spatial variable. The proposed method comprises of Implicit Euler scheme to discretize the temporal variable and a hybrid finite difference scheme on a special piecewise equidistant mesh to discretize the spatial variable. Our hybrid finite difference scheme in spatial direction is based on simple upwinding but employ central differencing in the boundary layer region Ω_2, where Shishkin mesh allows us to do this without losing stability. The proposed method is fully implicit and has no time step size restriction for stability considerations as opposed to the explicit finite difference scheme. ε-uniform error estimate for simple upwind scheme is bounded by $N^{-1}(\log N)^2$(see [9]) whereas for the proposed hybrid monotone difference operator, error estimate is bounded by N^{-1} in spatial domain $\bar{\Omega}_x$ with Shishkin mesh. Thus hybrid finite difference method in spatial direction has superior convergence properties than simple upwinding, but is of same computational cost.

The numerical accuracy of the present scheme is tested at low viscosity coefficient ε and the results are presented in Table 1. Numerical results show that for a fixed value of ε, pointwise errors and maximal nodal errors decrease as the number of mesh points increases. We observe that the computational order of local convergence are in good agreement with the theoretical estimates. It has been seen that in some cases proposed scheme has a order of convergence greater than two. From the numerical solution profiles given in Figure 1, we observe that the propagation front is steeper in the neighborhood of Γ_r, the right part of the lateral surface for the small values of the viscosity coefficient *i.e.*, at the high Reynolds number, which validate the physical behavior of the solution.

Thus the present method works nicely for low as well as high Reynolds number and the numerical results support the theoretical predictions and exhibit good physical behavior.

REFERENCES

[1] A. S. Bakhvalov, K optimizacii metodov resenia kraevyh zadac prinalicii pograanicnogo sloja, *Zh. Vychisl. Mat. Mat. Fiz.*, 9:841–859, 1969.

[2] I.P. Lee-bapty and D.G. Crighton, non-linear wave motion governed by the modified Burgers' equation, *Phil. Trans. R. Soc. Lond. A*, 323:173–209, 1987.

[3] R.E. Bellman and R.E. Kalaba, *Quasilineaization and non-linear Boundary-value problems*, American Elsevier Publishing Company, Inc., New York, 1965.

[4] T. Chong and L. Sirovich, non-linear effects in supersonic dissipative gas dynamics, *J. Fluid Mech.*, 58:55-63, 1973.

[5] T. Chong, A variable mesh finite difference method for solving a class of parabolic differential equations in one variable, *SIAM J. Numer. Anal.*, 15:835–857, 1978.

[6] J.D. Cole, On a quasi-linear parabolic equations occurring in aerodynamics, *Quart. Appl. Math.*, 9:225–236, 1951.

[7] P.J. Davis, *Interpolation and approximations*, Blaisdell, Waltham, Mass., 1963.

[8] J. Douglas and B. Jones, On predictor corrector methods for non-linear parabolic differential equations, *J. Fluid Mech.*, 11:195–204, 1963.

[9] P.A. Farrell, A.F. Hegarty, J.J.H. Miller E.O'Riordan and G.I. Shishkin, *Robust Computational Techniques for Boundary Layers*, Chapman & Hall London 2000.

[10] E.C. Gartland, Graded-mesh difference schemes for singularly perturbed two-point boundary value problems, *Math. Comput.*, 51:631–657, 1988.

[11] E. Hopf, The partial differential equation $u_t+uu_x = u_{xx}$, *Comm. Pure Appl. Math.*, 3:201–230, 1950.

[12] L. Halabisky and L. Sirovich, Evolution of finite disturbances in dissipative gas dynamics, *Phy. Fluid*, 16:360–368, 1973.

[13] S.L. Harris, Sonic shocks governed by the modified Burgers' equation, *EJAM*, 6:75–107, 1996.

[14] D. Hilhorst, A non-linear evolution problem arising in the physics of ionized gases, *SIAM J. Math. Anal.*, 13:16–39, 1982.

[15] M.K. Kadalbajoo and A. Awasthi, Uniformly convergent numerical method for solving modified Burgers' equations on a non-uniform mesh, *J. Numer. Math.*, 16:217–235, 2008.

[16] A.A. Karabutov and O.V. Rudenko, Excitation of non-linear acoustic waves by surface absorption of laser radiation, *Sov. Phys. Tech. Phys.*, 20:920–922, 1976.

[17] R.B. Kellogg and A. Tsan, Analysis of some difference approximations for a singular perturbation problem without turning points, *Math. Comp.*, 32:1025–1039, 1978.

[18] M.J. Lighthill, *Viscosity effects in sound waves of finite amplitude*, in: G.K. Batchelor and R.M. Davies (Eds.), *Surveys in Mechanics*, 250–351, Cambridge University press, 1956.

[19] G.A. Nariboli and W.C. Lin, A new type of Burgers' equation, *Z. Angew. Math. Mech.*, 53:505–510, 1973.

[20] P.L. Sachdev and R. Seebass, A propogation of spherical and cylindrical N-waves, *J. Fluid Mech.*, 58:192–205, 1973.

[21] P.L. Sachdev and CH. Srinivasa Rao, N-wave solution of the modified Burgers' equation, *Appl. Math. Lett.*, 13:1–6, 2000.

[22] N. Sugumoto, Y. Yamane and T. Kakutani, *S*hock wave propagation in a viscoelastic rod, In non-linear Deformation Waves, IUTAM Symposium, Tallinn 1982 (ed. U. Nigul and J. Engelbrecht), pp. 203-208, Berlin: Springer-Verlag.

[23] M. Teymur and E. Suhubi, Wave propagation in dissipative or dispersive non-linear media, *J. Inst. Math. Applics.*, 21:25–40, 1978.