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BDF-FEM FOR PARABOLIC SINGULARLY PERTURBED PROBLEMS WITH EXPONENTIAL LAYERS ON LAYER-ADAPTED MESHES IN SPACE

VIT DOLEJŠÍ AND HANS-GÖRG ROOS

Faculty of Mathematics and Physics, Charles University Prague
dolejsi@karlin.mff.cuni.cz
Department of Mathematics, TU Dresden
Hans-Goerg.Roos@tu-dresden.de

ABSTRACT. A parabolic initial-boundary value problem with solutions displaying exponential layers is solved using layer-adapted meshes. The paper combines finite elements in space, for instance, a pure Galerkin technique or stabilizations with certain properties on a Shishkin-type mesh, with two-step backward differencing in time. We prove optimal robust error estimates and present numerical results.

Key words: convection-diffusion, transient, finite element, Shishkin mesh, time discretization

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1. INTRODUCTION

We consider unsteady convection-diffusion problems of the type

$$u_t + Lu = f \quad \text{in } Q = \Omega \times (0,T], \tag{1.1a}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega = (0,1)^2, \tag{1.1b}$$

$$u|_{\partial\Omega} = 0 \quad \text{for } t \in (0,T]. \tag{1.1c}$$

Here the differential operator L is given by,

$$Lu := -\varepsilon \triangle u + b \cdot \bigtriangledown u + cu, \tag{1.2}$$

$0 < \varepsilon \ll 1$ is a small parameter and $b = b(x), c = c(x)$ and $f = f(x,t)$ are sufficiently smooth with

$$-b = (-b_1, -b_2) > (\beta_1, \beta_2) > 0 \qquad \text{on } \bar{\Omega}. \tag{1.3}$$

By changing the dependent variable we may also assume that

$$c - \frac{1}{2}\nabla \cdot b \geq c_0 > 0. \tag{1.4}$$

The exact solution of (1.1) has, in general, exponential boundary layers at $x = 0$ and $y = 0$, moreover, a corner layer at $(0,0)$ for all t. Additionally, a discontinuity in

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the initial-boundary data at the point $x = (1,1)$, $t = 0$ would lead to an interior layer along the subcharacteristics through that point. We assume sufficient compatibility of the data to exclude the existence of an interior layer, see [12].

In recent years many numerical methods have been developed to solve the corresponding stationary problem on layer-adapted meshes, resulting in error estimates that are uniform with respect to the parameter ε, see [12, 9]. For unsteady problems, however, the situation is different.

Most existing papers deal with low order finite difference schemes, beginning with [14], analysing backward differencing in time and upwind differencing in space on a Shishkin mesh. This result was extended in [8], and [6]; in the last paper defect correction in both space and time is applied to enhance the accuracy of the computed solution. In [2] Clavero et. al. combine a ADI dimension-splitting method with a HODIE technique on a one-dimensional Shishkin mesh.

Concerning finite elements in space on a Shishkin mesh, we only know the pointwise error estimates of [5] for problems one-dimensional in space using space-time finite elements that are linear and continuous in space but discontinuous in time. Additionally streamline diffusion stabilization in space is applied. Recently in [7] the authors studied a Galerkin finite element technique on Shishkin meshes in space combined with the θ-scheme or discontinuous Galerkin (dG) in time. While for the θ-scheme and linear or bilinear elements estimates of the type

$$\|u - U\| \leq C\left((N^{-1}\ln N)^q + \tau^p\right) \tag{1.5}$$

in the L_2 norm with $p = 1$ or $p = 2$ were proved, the estimates obtained for dG in time were less nice. For instance, for piecewise constant approximations in time the following estimate was derived:

$$\|u - U\| \leq C\left(N^{-1}\ln N + \tau^{1/2} + \frac{(N^{-1}\ln N)^2}{\tau^{1/2}}\right).$$

Similar terms where N, τ were coupled appeared as well for higher order dG methods with respect to time. Remark that in the estimates above and in the following C denotes a generic constant independent of ε, N, τ.

It is the aim of the present paper to prove estimates of the type (1.5) where the errors in space and time are separated for the time discretization with BDF methods.

2. THE CONTINUOUS PROBLEM

It is well known that for $f \in L_2(Q)$ and $u_0 \in L_2(\Omega)$ problem (1.1) has a unique solution $u \in L_2(0,T; H_0^1(\Omega))$. Moreover, the time derivative u' satisfies $u' \in L_2(0,T; H^{-1}(\Omega))$.

If we introduce the ε-weighted $H^1(\Omega)$ norm defined by

$$\|v\|_\varepsilon^2 := \varepsilon|v|_1^2 + \|v\|_0^2, \tag{2.1}$$

standard arguments lead us to the stability estimate (see [10], Theorem 11.1.1)

$$\sup_{t\epsilon(0,\tau)} \|u(t)\|_0 + \left(\int_0^T \|u(t)\|_\varepsilon^2 dt\right)^{1/2} \leq C\left(\|f\|_{0,Q} + \|u_0\|_0\right) \tag{2.2}$$

for the solution of (1.1). Therefore it is natural that we shall later prove error estimates in "$L^\infty(L^2)$"- and "$\sqrt{\varepsilon}L^2(H^1)$"-norms or their discrete analogues.

Remark 2.1. In [10], Proposition 11.1.1, we additionally can find an estimate with respect to the norm $\max_{t\epsilon(0,T)}\|u\|_1$. But, in the singularly perturbed case, it seems not possible to follow the proof of Proposition 11.1.1 in such a way that the constants arising are independent of ε (if moreover, $\|u\|_1$ is replaced by $\|u\|_\varepsilon$). □

Under certain compatibility conditions there exists a classical solution of problem (1.1). Further compatibility of the data excludes interior layers and guarantees the existence of an S-decomposition of the solution [12], [14]:

We decompose $u = S + V_1 + V_2 + V_{12}$. It can be shown that the components of u satisfy the estimates

$$\left|\frac{\partial^{i+j+k} S(x_1,x_2,t)}{\partial x_1^i \partial x_2^j \partial t^k}\right| \leq C, \tag{2.3a}$$

$$\left|\frac{\partial^{i+j+k} V_1(x_1,x_2,t)}{\partial x_1^i \partial x_2^j \partial t^k}\right| \leq C\varepsilon^{-i}\exp^{-\beta_1 x_1/\varepsilon}, \tag{2.3b}$$

$$\left|\frac{\partial^{i+j+k} V_2(x_1,x_2,t)}{\partial x_1^i \partial x_2^j \partial t^k}\right| \leq C\varepsilon^{-j}\exp^{-\beta_2 x_2/\varepsilon}, \tag{2.3c}$$

$$\left|\frac{\partial^{i+j+k} V_{12}(x_1,x_2,t)}{\partial x_1^i \partial x_2^j \partial t^k}\right| \leq C\varepsilon^{-i-j}\min\left\{\exp^{-\beta_1 x_1/\varepsilon}, \exp^{-\beta_2 x_2/\varepsilon}\right\}, \tag{2.3d}$$

for $i + j + 2k \leq l$ with some given l and $(x, y, t) \in \bar{Q}$. While S represents the smooth part of the solution, V_1 and V_2 are the boundary layer terms and V_{12} corresponds to the corner layer. Remark, that the bounds (2.3) exclude the presence of interior layers.

3. FINITE ELEMENTS ON LAYER ADAPTED MESHES REVISITED AND THE RITZ PROJECTION

We shall discretize (1.1) by linear finite elements in space on a Shishkin-type mesh and with BDF with respect to time. In the error analysis of the method the Ritz projection $R^N u$ of u plays an important role. Let us introduce the bilinear form

$$a(w, v) := \varepsilon(\nabla w, \nabla v) + (b \cdot \nabla w + cw, v)$$

and denote by $V^N \subset V = H_0^1(\Omega)$ our finite element space of linear elements. Then, the Ritz projection is defined by:

For a given $w \in H^1(\Omega)$ find $R^N w \in V^N$ such that this projection satisfies

$$a(R^N w, v) = a(w, v) \quad \forall v \in V^N. \tag{3.1}$$

Additionally we denote by $u^I \in V^N$ the standard nodal interpolant of u. Because the difference $v^N := u^I - R^N u$ satisfies

$$\alpha \|v^N\|_\varepsilon^2 \le a(v^N, v^N) = a(u^I - u, v^N)$$

with $\alpha = \min(c_0, 1)$, one can estimate $\|u^I - R^N u\|_\varepsilon$ following the standard arguments in the error estimate for finite elements on layer adapted meshes. Additionally, estimates for $\|u - R^N u\|_\varepsilon$ then follow from the (known) estimates of $\|u - u^I\|_\varepsilon$.

Next we describe the layer-adapted meshes which we want to use. Let N, our discretization parameter in space, be an even positive integer. We introduce the mesh points

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1, \quad 0 = y_0 < y_1 < \cdots < y_{N-1} < y_N = 1$$

and consider a tensor-product mesh with mesh points (x_i, y_j). For linear elements we additionally bisect every mesh rectangle into two triangles. Because both meshes have the same structure we only describe the meshes in the x_1-direction (for the mesh in the x_2-direction take $\beta_1 := \beta_2$).

The mesh is graded in $[0, x_{N/2}]$ but equidistant in $[x_{N/2}, 1]$. The graded part of the mesh is based on a mesh generating function ϕ with $\phi(0) = 0$, $\phi(1/2) = \ln N$, moreover we assume ϕ to be continuous, monotonically increasing and differentiable. Set

$$x_i = \begin{cases} \dfrac{\sigma\varepsilon}{\beta_1}\phi(t_i) \quad \text{with } t_i = i/N & \text{for } i = 0, 1, \dots, N/2 \\ 1 - (1 - x_{N/2})2(N-i)/N & \text{for } i = N/2 + 1, \dots, N. \end{cases} \tag{3.2}$$

Here σ is some positive constant which characterizes the order of the smallness of the layer term in $x_{N/2}$. We call this class of meshes S-type meshes. Typical examples are the original Shishkin meshes with $\phi(t) := 2(\ln N)t$ or the Bakhvalov-Shishkin mesh with

$$\phi(t) := -\ln[1 - 2(1 - N^{-1})t]. \tag{3.3}$$

Following [11] the mesh characterizing function ψ is defined by

$$\psi := \exp(-\phi). \tag{3.4}$$

Let now $w \in V$, $w^N \in V^N$ solve for some given f^*

$$a(w, v) = (f^*, v) \quad \forall v \in V, \quad a(w^N, v) = (f^*, v) \quad \forall v \in V^N.$$

Then in [11] the following result is proved (see also [9]):
If w allows a decomposition of the type (2.3)*, $\sigma \geq 2$ and ψ satisfies*

$$\frac{\max |\psi'|}{\psi} \leq C\,N, \tag{3.5}$$

then the finite element error can be estimated by

$$\|w - w^N\|_\varepsilon \leq CN^{-1} \max |\psi'|. \tag{3.6}$$

As we remarked above the same estimate is valid for $\|u - R^N u\|_\varepsilon$. While on a Shishkin mesh $\max |\psi'|$ generates the factor $\ln N$, a Bakhvalov-Shishkin mesh is optimal because $\max |\psi'| \leq C$.

Consequently we have the following result for the Ritz projection:

Lemma 3.1. *Assume that u allows a decomposition* (2.3) *with $l = 2$. Then its Ritz projection on S-type meshes with the property proposed in* (3.5) *satisfies for $\sigma \geq 2$ and linear elements*

$$\|u - R^N u\|_\varepsilon \leq CN^{-1} \max |\psi'|. \tag{3.7}$$

Remark that for bilinear elements assuming more smoothness ($l = 3$) and $\sigma \geq 2.5$ one can prove a supercloseness property [9]:

$$\|w^I - w^N\|_\varepsilon \leq CN^{-2}(\max |\psi'| + \ln^{1/4} N)^2. \tag{3.8}$$

Together with the results for the interpolation error

$$\|w - w^I\|_\varepsilon \leq CN^{-1} \max |\psi'|, \quad \|w - w^I\|_0 \leq C\,(N^{-1} \max |\psi'|)^2 \tag{3.9}$$

one gets for the L_2 error and bilinear elements

$$\|w^I - w^N\|_0 \leq CN^{-2}(\max |\psi'| + \ln^{1/4} N)^2. \tag{3.10}$$

An analogous result holds true for $\|u^I - R^N u\|_0$ if bilinear elements are used. For linear elements, however, we are not able to improve the first order result for the L_2 error which follows from (3.7).

4. ANALYSIS OF BDF FOR THE DISCRETIZATION IN TIME

For the discretization in space of (1.1) we use linear elements on a S-type mesh. Then the semi-discrete problem is given by

$$\left(\frac{du^N}{dt}, v\right) + a(u^N, v) = (f, v) \quad \forall v \in V^N, \quad u^N(0) = u_0^N \tag{4.1}$$

Let us introduce a mesh in time that is equidistant for simplicity, with mesh width τ and $\tau \cdot M = T$. We denote by $U^l \in V^N$ some approximation of $u(t_l)$. Then the

k-step BDF scheme defines these approximations by

$$\frac{1}{\tau}\Big(\sum_{\nu=0}^{k}\alpha_\nu U^{s+\nu}, v\Big) + a(U^{s+k}, v) = (f(t_{s+k}), v) \quad \forall v \in V^N \tag{4.2}$$

for $s = 0, 1, \ldots, M-k$ and $U^0 = u_0^N$. The α_ν are well known parameters for the BDF scheme, for instance, for $k = 2$ we have $\alpha_2 = 3/2$, $\alpha_1 = -2$, $\alpha_0 = 1/2$. We assume, additionally, that the starting values of our multi-step scheme $U^1, \ldots, U^{k-1}$ are given (computed by some other method).

To analyze our discretization we use the Ritz projection and define

$$R^N u^s := R^N(u(t_s)), \quad \eta^s := R^N u^s - u(t_s), \quad \xi^s := U^s - R^N u^s \in V^N.$$

ξ satisfies the error equation

$$\left(\sum_{\nu=0}^{k}\alpha_\nu \xi^{s+\nu}, v\right) + \tau a(\xi^{s+k}, v) = (W^{s+k}, v) \quad \forall v \in V^N \tag{4.3}$$

with

$$W^{s+k} := -\sum_{\nu=0}^{k}\alpha_\nu \eta^{s+\nu} + \left\{\tau\, u'(t_{s+k}) - \sum_{\nu=0}^{k}\alpha_\nu u(t_{s+\nu})\right\}. \tag{4.4}$$

The second term in W^{s+k} represents the approximation of u' by our backward differentiation formulas. Because the solution of our given problem (1.1) is smooth with respect to t, one has

$$\|\tau u'(t_{s+k}) - \sum_{\nu=0}^{k}\alpha_\nu u(t_{s+\nu})\|_0 \le C\tau^{k+1}. \tag{4.5}$$

When estimating the first term we follow [4], see Lemma 10,11. By Lemma 1 the Ritz projection satisfies

$$\|u - R^N u\|_0 \le E_{RP}^N := C\, N^{-1} \max|\psi'|$$

and we get

$$\|\sum_{\nu=0}^{k}\alpha_\nu \eta^{s+\nu}\| \le C\,\tau\, E_{RP}^N. \tag{4.6}$$

Matching (4.5) and (4.6) together we obtain

$$|(W^{s+k}, v)| \le \tau\|v\|_0^2 + B \quad \text{with} \quad B = C\,\tau(\tau^{2k} + (E_{RP}^N)^2). \tag{4.7}$$

Next, we need a stability result for the error equation (4.3). We derive this estimate following [4] for $k = 2$. Note, that the stability result in [15] assumes symmetry of the underlying elliptic operator and therefore does not apply here.

Let $k = 2$. We choose $v := \xi^{s+2}$ and obtain

$$\frac{3}{2}\|\xi^{s+2}\|_0^2 - 2\|\xi^{s+1}\|_0^2 + \frac{1}{2}\|\xi^s\|_0^2 + \|\xi^{s+2} - \xi^{s+1}\|_0^2 - \|\xi^{s+1} - \xi^s\|_0^2 \le \tau\,\|\xi^{s+2}\|_0^2 + B. \tag{4.8}$$

Now we introduce the auxiliary constants γ_i by

$$\gamma_i := 1 - \left(\frac{1}{3}\right)^{i+1}.$$

We multiply (4.8) for $s = 0$ with γ_{m-2}, for $s = 1$ with γ_{m-1} and so on, finally for $s = m-2$ with γ_0 and sum up all equations. The γ_i have been chosen in such a way that many terms cancel. We get

$$\|\xi^m\|_0^2 \le \tau \left(\gamma_0 \|\xi^m\|_0^2 + \sum_{s=0}^{m-3} \gamma_{m-2-s} \|\xi^{s+2}\|_0^2 \right) + C(\|\xi^1\|_0^2 + \|\xi^0\|_0^2) + B \sum_{s=0}^{m-2} \gamma_{m-2-s}.$$

Note that all γ_i are smaller than 1, but $\sum_{s=0}^{m-2} \gamma_{m-2-s} = O(1/\tau)$. Therefore we conclude

$$\|\xi^m\|_0^2 \le c\tau \sum_{s=0}^{m-1} \|\xi^s\|_0^2 + C(\|\xi^1\|_0^2 + \|\xi^0\|_0^2) + \frac{C}{\tau} B = c\tau \sum_{s=0}^{m-1} \|\xi^s\|_0^2 + Y.$$

It follows inductively

$$\|\xi^s\|_0^2 \le Y(1 + c\tau)^s \le \exp(ct_s) Y. \tag{4.9}$$

Using

$$U^s - u(t_s) = U^s - R^N u^s + R^N u^s - u(t_s)$$

we get

Theorem 4.1. *The error of our BDF-FE scheme for $k = 2$ satisfies*

$$\|U^M - u(t_M)\|_0 \le C \exp(cT) \left(\|U^0 - R^N u^0\|_0 + \|U^1 - R^N u^1\|_0 + \tau^2 + E_{RP}^N, \right) \tag{4.10}$$

bounds for the error E_{RP}^N of the Ritz projection are given in Lemma 1.

Remark 4.2. For $k = 3$, a stability estimate like (4.9) was proved with a similar technique [4]. The starting point is an inequality related to (4.8), namely

$$\frac{11}{6}\|\xi^{s+3}\|_0^2 - \frac{18}{6}\|\xi^{s+2}\|_0^2 + \frac{9}{6}\|\xi^{s+1}\|_0^2 - \frac{2}{6}\|\xi^s\|_0^2 \le \tau \|\xi^{s+3}\|_0^2 + 3\|\xi^{s+2} - \xi^{s+1}\|_0^2 + B.$$

The γ_i for the γ-trick are more complicated and satisfy

$$\begin{aligned} \frac{11}{6}\gamma_1 - \frac{18}{6}\gamma_0 &= 0, \\ \frac{11}{6}\gamma_2 - \frac{18}{6}\gamma_1 + \frac{9}{6}\gamma_0 &= 0, \\ \frac{11}{6}\gamma_{j+3} - \frac{18}{6}\gamma_{j+2} + \frac{9}{6}\gamma_{j+1} - \frac{2}{6}\gamma_j &= 0. \end{aligned}$$

However, it turns out that the term $\|\xi^{s+2} - \xi^{s+1}\|_0^2$ on the right hand side causes new difficulties. Therefore, in [4] the symmetry of the bilinear form $a(\cdot,\cdot)$ was used to control that term. That means: We can prove error estimates for reaction-diffusion problems with $b = 0$ in (1.2) but so far not for the more interesting convection-diffusion case. □

Next we derive from our error estimate in the $L_\infty(L_2)$ norm an estimate in a discrete $\sqrt{\varepsilon}L_2(H^1)$ norm. For $k=2$ we start from (4.3). Set $v=\xi^{s+2}$ and sum all the inequalities obtained after the transformation to a form analogous to (4.8). This gives

$$\frac{3}{2}\|\xi^M\|_0^2-\frac{1}{2}\|\xi^{M-1}\|_0^2-\frac{3}{2}\|\xi^1\|_0^2+\frac{1}{2}\|\xi^0\|_0^2+\|\xi^M-\xi^{M-1}\|_0^2-\|\xi^1-\xi^0\|_0^2$$
$$+\alpha\tau\sum_{r=2}^{M}\|\xi^r\|_\varepsilon^2\leq\tau\sum_{r=2}^{M}\|\xi\|_0^2+\sum_{r=2}^{M}B.$$

We cancel some nonnegative terms on the left hand side and take into account that we can already bound the error in L_2:

$$\left(\tau\sum_{r=2}^{M}\|\xi^r\|_\varepsilon^2\right)^{1/2}\leq C\left(\max_s\|\xi^s\|_0+\tau^2+E_{RP}^N.\right)$$

Thus we obtain our second main result:

Theorem 4.3. *The error ξ between the numerical solution and the Ritz projection of the exact solution of our BDF-FE scheme for $k=2$ can be estimated by*

$$\left(\tau\sum_{r=1}^{M}\|\xi^r\|_\varepsilon^2\right)^{1/2}\leq C\left(\max_s\|\xi^s\|_0+\tau\|\xi^1\|_\varepsilon+\tau^2+E_{RP}^N\right). \tag{4.11}$$

Using the triangle inequality and the known results for the interpolation error we get an identical estimate for the error $u-U$.

Remark that for bilinear elements, however, the estimate in the $\sqrt{\varepsilon}L_2(H^1)$ norm is a supercloseness result because we can replace E_{RP}^N by a second order term. For the error $U-u$ itself we get for bilinear elements the same estimate as for linear elements, i.e.,

$$\left(\tau\sum_{r=1}^{M}\|U^r-u(t_r)\|_\varepsilon^2\right)^{1/2}\leq C\left(\max\|\xi^s\|_0+\tau\|\xi^1\|_\varepsilon+\tau^2+N^{-1}\max|\psi'|\right). \tag{4.12}$$

Remark 4.4. Let us assume that instead of (4.2) we start from

$$\frac{1}{\tau}\left(\sum_{\nu=0}^{k}\alpha_\nu U^{s+\nu},v\right)+a_{st}(U^{s+k},v)=(f_{st}(t_{s+k}),v)\quad\forall v\in V^N$$

i.e., we replace the Galerkin scheme by some stabilized FEM based on a modified bilinear form $a_{st}(\cdot,\cdot)$. Define the modified Ritz projection by

$$a_{st}(R^Nu,v)=a_{st}(u,v)\quad\forall v\in V^N.$$

Let, moreover, the stabilized FEM be consistent, i.e., the exact solution u satisfies

$$\left(\frac{du}{dt},v\right)+a_{st}(u,v)=(f_{st},v).$$

Then we can analogously as above derive the error equation

$$\left(\sum_{\nu=0}^{k} \alpha_\nu \xi^{s+\nu}, v\right) + \tau\, a_{st}(\xi^{s+k}, v) = (W^{s+k}, v) \quad \forall v \in V^N.$$

Therefore we conclude: if for the modified Ritz projection error estimates of the type

$$\|u - R^N u\|_\varepsilon \le CN^{-1} \max|\psi'|. \tag{4.13}$$

are available, then we can without problems extend our analysis to a stabilized method. While the continuous interior penalty technique satisfies all these assumptions (see [13]), local projection is not consistent but it seems possible additionally also to estimate the consistency error. But for streamline diffusion stabilization the situation is more complicated. □

5. NUMERICAL EXPERIMENTS

In this section we present several numerical experiments which verify the error estimates (4.10) and (4.11) presented in the previous section. We try to investigate the dependence of the computational error on h and τ independently. We expect the error dependence in $\|\cdot\|_0$-norm and $\|\cdot\|_\varepsilon$-norm according to the formula

$$e_{N,\tau} \approx c_{\mathrm{s}} N^{-p} + c_{\mathrm{t}} \tau^k, \tag{5.1}$$

where N and τ are the space and time discretization parameters, p and k are the order of the space and time convergences, c_h and c_τ are constants and $e_{N,\tau}$ is the corresponding computational error. Our aim is to determine the orders of convergence p and k and to show that constants c_{s} and c_{t} are independent of ε.

Although we proved only estimates for k-step BDF with $k = 2$ we present results for $k = 1, 2, 3$. For $k = 1$ the related estimates are known, for $k = 3$ the proof is open.

5.1. **Data settings.** We consider the problem (similarly as in [2]) with layers at $x_1 = 0$ and $x_2 = 0$

$$u_t - \varepsilon \triangle u - u_{x_1} - u_{x_2} = f \quad \text{in } (0,1)^2 \times (0,T], \tag{5.2}$$

where f and initial and boundary conditions are chosen that the exact solution is

$$u(x,t) = P(t)\,(c_1 + c_2(1-x_1) + \exp(-x_1/\varepsilon))\,(c_1 + c_2(1-x_2) + \exp(-x_2/\varepsilon)) \tag{5.3}$$

with $c_1 = -\exp(-1/\varepsilon)$, $c_2 = -1 - c_1$ and $P(t) : [0,T] \to \mathbb{R}$. We deal with two sets of numerical experiments.

(i) In order to investigate the *order of convergence with respect to* N we put $T = 2$ in (5.2) and $P(t) = 1 - \exp(t)$ in (5.3). In order to eliminate discretization error caused by the time discretization, we employ 3-steps BDF scheme with a small time step $\tau = 0.01$. For such data setting we expect that $c_{\mathrm{s}} h^p \gg c_{\mathrm{t}} \tau^k$. The computations were carried out on the Bakhvalov-Shishkin meshes defined by

(3.2)–(3.3) with $\beta = 1$, $\sigma = 3$ and $N = 16$, 32, 64, 128 and 256. We investigate the dependence of the $\|u(T) - U(t)\|_0$ and $\|u(T) - U(t)\|_\varepsilon$ on N and evaluate the corresponding *experimental order of convergence* (EOC).

(ii) In order to investigate the *order of convergence with respect to* τ we put $T = 1$ in (5.2) and $P(t) = (\exp(10t) - 1)/(\exp(10) - 1)$ in (5.3). In order to eliminate discretization error caused by the space discretization, we employ the Bakhvalov-Shishkin meshes defined by (3.2)–(3.3) with $\beta = 1$, $\sigma = 3$ and $N = 256$. For such data setting we expect that $c_s h^p \ll c_t \tau^k$. The computations were carried out for the time steps $\tau = 0.2$, 0.1, 0.05, 0.025 and 0.0125 using k-step BDF for $k = 1, 2, 3$. We investigate the dependence of the $\|u(T) - U(t)\|_0$ and $\|u(T) - U(t)\|_\varepsilon$ on τ and evaluate the corresponding EOC.

In order to demonstrate the uniformity of the errors with respect to the parameter ε, the previous experiments in case (i) as well as case (ii) were carried out for $\varepsilon = 10^{-2}$, 10^{-3}, 10^{-4}, 10^{-5} and 10^{-6}. The corresponding linear algebra systems were solved by the direct solver UMFPACK (see [3] and the references therein).

5.2. **Computational results.** Table 1 shows the computational errors $e_{N,\tau} := u(T) - U(t)$ in the L^2-norm $\|\cdot\|_0$ and the ε-weighted H^1-norm $\|\cdot\|_\varepsilon$ and the corresponding experimental orders of convergence for the case (i). We simply observe the order of convergence $O(N^{-2})$ in the L^2-norm and the order $O(N^{-1})$ in the ε-weighted H^1-norm which is in agreement with (4.10) and (4.11). Moreover, we observe that the magnitude of the error in the ε-weighted H^1-norm is uniform with respect to ε.

Tables 2–4 show the computational errors in the L^2-norm $\|\cdot\|_0$ and the ε-weighted H^1-norm $\|\cdot\|_\varepsilon$ and the corresponding experimental orders of convergence for the case (ii). We simply observe the order of convergence close to the theoretical values, i.e., $O(\tau^k)$ in the L^2-norm as well as in the ε-weighted H^1-norm. The decrease of the order of convergence for the 3-steps BDF for the smallest time step is caused by the influence of the space discretization error, i.e., our expectation $c_s h^p \ll c_t \tau^k$ is not valid. Again, we observe that the magnitude of the error is uniform with respect to ε.

ε	N	τ	$\|e_{N,\tau}\|_0$	EOC	$\|e_{N,\tau}\|_\varepsilon$	EOC
1E-02	16	1.000E-02	1.196E-03	–	1.619E-02	–
1E-02	32	1.000E-02	2.928E-04	2.030	8.195E-03	0.983
1E-02	64	1.000E-02	7.189E-05	2.026	4.081E-03	1.006
1E-02	128	1.000E-02	2.662E-05	1.433	2.040E-03	1.000
1E-02	256	1.000E-02	3.266E-05	-.295	1.175E-03	0.796
1E-03	16	1.000E-02	1.342E-03	–	1.270E-02	–
1E-03	32	1.000E-02	3.305E-04	2.021	6.400E-03	0.989
1E-03	64	1.000E-02	7.987E-05	2.049	3.237E-03	0.983
1E-03	128	1.000E-02	1.986E-05	2.008	1.633E-03	0.987
1E-03	256	1.000E-02	4.954E-06	2.003	8.206E-04	0.993
1E-04	16	1.000E-02	1.411E-03	–	1.217E-02	–
1E-04	32	1.000E-02	3.850E-04	1.873	6.090E-03	0.999
1E-04	64	1.000E-02	9.234E-05	2.060	3.066E-03	0.990
1E-04	128	1.000E-02	2.059E-05	2.165	1.542E-03	0.992
1E-04	256	1.000E-02	4.982E-06	2.047	7.739E-04	0.994
1E-05	16	1.000E-02	1.420E-03	–	1.212E-02	–
1E-05	32	1.000E-02	3.988E-04	1.833	6.057E-03	1.000
1E-05	64	1.000E-02	1.044E-04	1.934	3.047E-03	0.991
1E-05	128	1.000E-02	2.558E-05	2.029	1.531E-03	0.993
1E-05	256	1.000E-02	5.633E-06	2.183	7.681E-04	0.995
1E-06	16	1.000E-02	1.421E-03	–	1.211E-02	–
1E-06	32	1.000E-02	4.003E-04	1.828	6.054E-03	1.000
1E-06	64	1.000E-02	1.062E-04	1.915	3.045E-03	0.991
1E-06	128	1.000E-02	2.742E-05	1.953	1.530E-03	0.993
1E-06	256	1.000E-02	6.783E-06	2.015	7.675E-04	0.995

TABLE 1. Case (i), 3-step BDF, computational errors and the corresponding EOC in $\|\cdot\|_0$-norm and $\|\cdot\|_\varepsilon$-norm with respect to N

ε	N	τ	$\|e_{N,\tau}\|_0$	EOC	$\|e_{N,\tau}\|_\varepsilon$	EOC
1E-02	256	2.000E-01	2.276E-01	–	4.852E-01	–
1E-02	256	1.000E-01	1.193E-01	0.932	2.513E-01	0.949
1E-02	256	5.000E-02	6.002E-02	0.991	1.256E-01	1.000
1E-02	256	2.500E-02	2.995E-02	1.003	6.250E-02	1.007
1E-02	256	1.250E-02	1.494E-02	1.004	3.115E-02	1.004
1E-03	256	2.000E-01	2.384E-01	–	5.055E-01	–
1E-03	256	1.000E-01	1.248E-01	0.933	2.615E-01	0.951
1E-03	256	5.000E-02	6.278E-02	0.992	1.307E-01	1.001
1E-03	256	2.500E-02	3.132E-02	1.003	6.498E-02	1.008
1E-03	256	1.250E-02	1.562E-02	1.004	3.237E-02	1.005
1E-04	256	2.000E-01	2.395E-01	–	5.076E-01	–
1E-04	256	1.000E-01	1.254E-01	0.933	2.626E-01	0.951
1E-04	256	5.000E-02	6.305E-02	0.992	1.312E-01	1.001
1E-04	256	2.500E-02	3.145E-02	1.003	6.523E-02	1.008
1E-04	256	1.250E-02	1.569E-02	1.004	3.249E-02	1.005
1E-05	256	2.000E-01	2.396E-01	–	5.078E-01	–
1E-05	256	1.000E-01	1.254E-01	0.933	2.627E-01	0.951
1E-05	256	5.000E-02	6.308E-02	0.992	1.312E-01	1.001
1E-05	256	2.500E-02	3.147E-02	1.003	6.525E-02	1.008
1E-05	256	1.250E-02	1.569E-02	1.004	3.251E-02	1.005
1E-06	256	2.000E-01	2.396E-01	–	5.078E-01	–
1E-06	256	1.000E-01	1.254E-01	0.933	2.627E-01	0.951
1E-06	256	5.000E-02	6.308E-02	0.992	1.312E-01	1.001
1E-06	256	2.500E-02	3.147E-02	1.003	6.526E-02	1.008
1E-06	256	1.250E-02	1.569E-02	1.004	3.251E-02	1.005

TABLE 2. Case (ii), 1-step BDF, computational errors and the corresponding EOC in $\|\cdot\|_0$-norm and $\|\cdot\|_\varepsilon$-norm with respect to τ

ε	N	τ	$\|e_{N,\tau}\|_0$	EOC	$\|e_{N,\tau}\|_\varepsilon$	EOC
1E-02	256	2.000E-01	1.251E-01	–	2.636E-01	–
1E-02	256	1.000E-01	4.560E-02	1.456	9.531E-02	1.468
1E-02	256	5.000E-02	1.449E-02	1.654	3.021E-02	1.657
1E-02	256	2.500E-02	4.184E-03	1.792	8.824E-03	1.776
1E-02	256	1.250E-02	1.135E-03	1.882	2.735E-03	1.690
1E-03	256	2.000E-01	1.309E-01	–	2.744E-01	–
1E-03	256	1.000E-01	4.769E-02	1.456	9.912E-02	1.469
1E-03	256	5.000E-02	1.515E-02	1.655	3.139E-02	1.659
1E-03	256	2.500E-02	4.374E-03	1.792	9.112E-03	1.785
1E-03	256	1.250E-02	1.186E-03	1.883	2.644E-03	1.785
1E-04	256	2.000E-01	1.315E-01	–	2.755E-01	–
1E-04	256	1.000E-01	4.790E-02	1.456	9.950E-02	1.469
1E-04	256	5.000E-02	1.521E-02	1.655	3.151E-02	1.659
1E-04	256	2.500E-02	4.393E-03	1.792	9.141E-03	1.785
1E-04	256	1.250E-02	1.191E-03	1.883	2.634E-03	1.795
1E-05	256	2.000E-01	1.315E-01	–	2.756E-01	–
1E-05	256	1.000E-01	4.792E-02	1.457	9.954E-02	1.469
1E-05	256	5.000E-02	1.522E-02	1.655	3.152E-02	1.659
1E-05	256	2.500E-02	4.395E-03	1.792	9.144E-03	1.786
1E-05	256	1.250E-02	1.192E-03	1.883	2.632E-03	1.797
1E-06	256	2.000E-01	1.315E-01	–	2.756E-01	–
1E-06	256	1.000E-01	4.792E-02	1.457	9.954E-02	1.469
1E-06	256	5.000E-02	1.522E-02	1.655	3.153E-02	1.659
1E-06	256	2.500E-02	4.396E-03	1.792	9.144E-03	1.786
1E-06	256	1.250E-02	1.192E-03	1.883	2.632E-03	1.797

TABLE 3. Case (ii), 2-step BDF, computational errors and the corresponding EOC in $\|\cdot\|_0$-norm and $\|\cdot\|_\varepsilon$-norm with respect to τ

ε	N	τ	$\|e_{N,\tau}\|_0$	EOC	$\|e_{N,\tau}\|_\varepsilon$	EOC
1E-02	256	2.000E-01	8.154E-02	–	1.712E-01	–
1E-02	256	1.000E-01	2.128E-02	1.938	4.439E-02	1.948
1E-02	256	5.000E-02	4.254E-03	2.323	8.969E-03	2.307
1E-02	256	2.500E-02	6.962E-04	2.611	1.994E-03	2.169
1E-02	256	1.250E-02	1.062E-04	2.713	1.374E-03	0.537
1E-03	256	2.000E-01	8.531E-02	–	1.782E-01	–
1E-03	256	1.000E-01	2.225E-02	1.939	4.614E-02	1.949
1E-03	256	5.000E-02	4.448E-03	2.323	9.263E-03	2.316
1E-03	256	2.500E-02	7.271E-04	2.613	1.791E-03	2.371
1E-03	256	1.250E-02	1.049E-04	2.794	9.768E-04	0.874
1E-04	256	2.000E-01	8.568E-02	–	1.789E-01	–
1E-04	256	1.000E-01	2.235E-02	1.939	4.632E-02	1.949
1E-04	256	5.000E-02	4.467E-03	2.323	9.293E-03	2.317
1E-04	256	2.500E-02	7.303E-04	2.613	1.767E-03	2.395
1E-04	256	1.250E-02	1.053E-04	2.794	9.246E-04	0.935
1E-05	256	2.000E-01	8.572E-02	–	1.789E-01	–
1E-05	256	1.000E-01	2.236E-02	1.939	4.633E-02	1.949
1E-05	256	5.000E-02	4.469E-03	2.323	9.296E-03	2.317
1E-05	256	2.500E-02	7.306E-04	2.613	1.764E-03	2.397
1E-05	256	1.250E-02	1.053E-04	2.794	9.182E-04	0.942
1E-06	256	2.000E-01	8.573E-02	–	1.789E-01	–
1E-06	256	1.000E-01	2.236E-02	1.939	4.634E-02	1.949
1E-06	256	5.000E-02	4.469E-03	2.323	9.296E-03	2.317
1E-06	256	2.500E-02	7.306E-04	2.613	1.764E-03	2.398
1E-06	256	1.250E-02	1.053E-04	2.794	9.175E-04	0.943

TABLE 4. Case (ii), 3-step BDF, computational errors and the corresponding EOC in $\|\cdot\|_0$-norm and $\|\cdot\|_\varepsilon$-norm with respect to τ

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