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DEVELOPMENT OF NONLINEAR STOCHASTIC MODELS BY USING STOCK PRICE DATA AND BASIC STATISTICS

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ABSTRACT. We employ the classical model building process to develop nonlinear stochastic models for stock market. In this work, three different nonlinear stochastic models are presented. Furthermore, under different data partitioning with equal and unequal intervals, a few modified nonlinear models are developed. Empirical comparisons between the constructed models and GBM (linear) models are also outlined.

AMS (MOS) Subject Classification. 60H10.

1. INTRODUCTION

In financial engineering, it is common to model a continuous time price process described by the Itô-Doob type stochastic differential equation [1, 2, 3, 4, 13, 14]. A general stochastic differential equation takes the form:

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t, S_{t_0} = S_0. \tag{1.1}$$

Here, $t \geq t_0, W_t$ is a Brownian motion, and $S_t > 0$,which is the price process [2, 5]. In our previous study [16], we initiate the usage of a classical modeling approach [14] to develop modified Geometric Brownian motion models for the price movement of individual stocks. GBM model is linear stochastic model, since the drift and volatility terms in equation (1.1) are linear in terms of S.

In this paper, the classical modeling approach [14] is extended to develop nonlinear stochastic models. In Section 2, three different nonlinear models are presented. We explore the utilization of three nonlinear stochastic models under equal and unequal interval data partitioning with jumps in Section 3 and 4 respectively. In Section 5, by following empirical comparison techniques [6, 7], we compare the presented models with each other, and also compare with modified linear models [16].

In this work, three data sets selected from Fortune 500 companies and S&P 500 Index are used. The daily adjusted closing prices can be free download from the web site http://finance.yahoo.com/.

2. NONLINEAR STOCHASTIC MODELS

There are several nonlinear stochastic differential equations that have been used to describe asset price in the area of finance [2]. Our previous study [16] exhibits the

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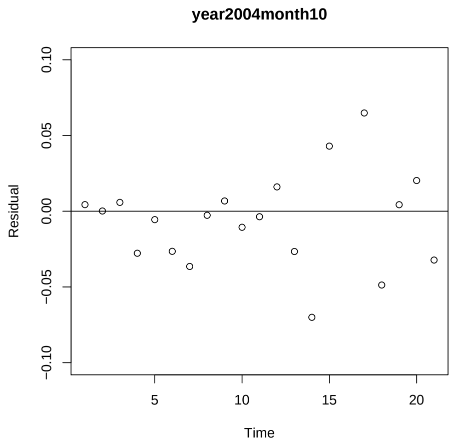

FIGURE 1. Plot1

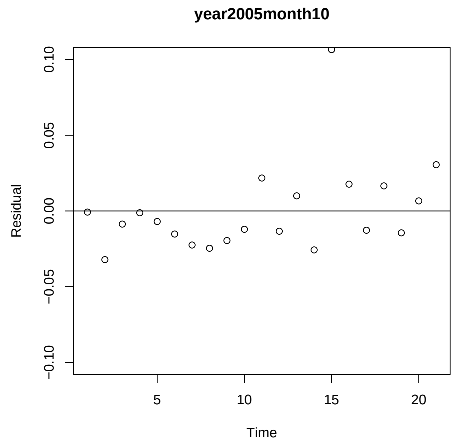

FIGURE 2. Plot2

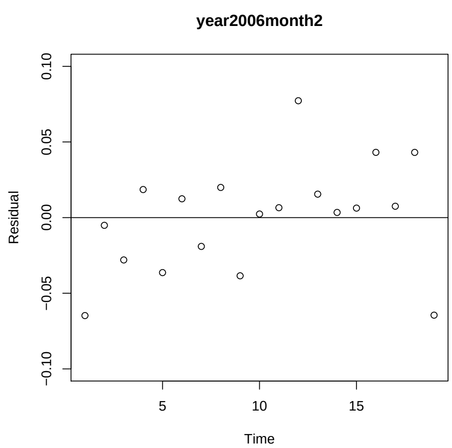

FIGURE 3. Plot3

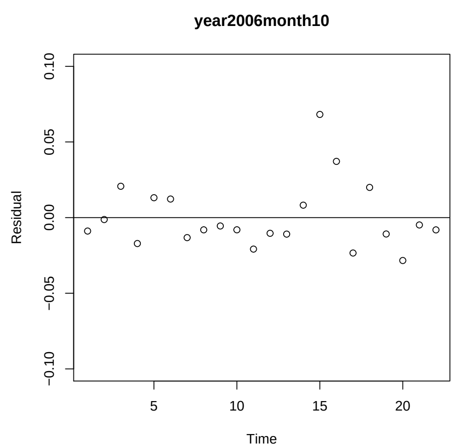

FIGURE 4. Plot4

strong trend of nonlinearity. In fact, this nonlinearity is justified by following figures and interpretations that are the representative of the entire data set. We recall [16] that $y_t = \mu - \frac{1}{2}\sigma^2 + \sigma\varepsilon_t$. The residual is

$$y_t - \hat{y}_t = lnS_t - lnS_{t-1} - (\hat{\mu} - \frac{1}{2}\hat{\sigma}^2).$$

In Figure 1, we see that the residuals start out close to zero, then spread out. In Figure 2, there are a lot of runs of many negative residuals in a row. In Figure 3, we see there is a trend of the residuals. The residuals get bigger as time goes on. And in Figure 2 and 4, we see the number of positive points is much different from the number of negative points. All these indicate that the linear model is not good enough to fit the data set. To build more precise models for competitive business, where even a not large difference is important, in this work, we continue our classical model building approach to develop nonlinear stochastic models.

2.1. **Nonlinear Stochastic Model 1 (Black-Karasinski Model).** Black-Karasinski stochastic model [12] that describes a short-term interest rate process, has the following form

$$dS_t = (\alpha lnS_t + \beta + \frac{\sigma^2}{2})S_t dt + \sigma S_t dW_t \tag{2.1}$$

where,α, β, and σ are parameters and W_t is Brownian motion. It is easy to check that equation (2.1) satisfies the conditions for existence and uniqueness of solution [5]. We note that the volatility function and drift function are linear and nonlinear, respectively. In order to derive the regression equation, we use the following transformation $V_t = lnS_t$ and apply Itô-Doob differential formula to obtain

$$\begin{aligned} dV_t &= \frac{\partial}{\partial S_t}(lnS_t)dS_t + \frac{1}{2}\big(\frac{\partial^2}{\partial S_t^2}(lnS_t)\big)(dS_t)^2 \\ &= \frac{1}{S_t}\big((\alpha lnS_t + \beta + \frac{\sigma^2}{2})S_t dt + \sigma S_t dW_t\big) - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2 (dW_t)^2. \end{aligned}$$

Then,

$$dV_t = (\alpha V_t + \beta)dt + \sigma dW_t. \tag{2.2}$$

By using the Euler type discretization process [15], stochastic differential equation (2.2) can be reduced to

$$V_t - V_{t-1} = (\alpha V_{t-1} + \beta)\Delta t + \sigma(W_t - W_{t-1}) \tag{2.3}$$

From $\varepsilon_t = W_t - W_{t-1}$ and $\Delta t = 1$, equation (2.3) can be rewritten as

$$V_t = (\alpha + 1)V_{t-1} + \beta + \sigma\varepsilon_t \tag{2.4}$$

where α, β and σ are as defined in (2.1). By following applying the least square regression method [11] and using above cited data sets, we can estimate these parameters.

2.2. **Nonlinear Stochastic Model 2.** This nonlinear stochastic model 2 [14] is described by the following Itô-Doob differential equation

$$dS_t = (\alpha S_t + \beta S_t^N + \frac{N}{2}\sigma^2 S_t^{2N-1})dt + \sigma S_t^N dW_t \tag{2.5}$$

where α, β, σ and N are parameters; moreover $0 < N < 1.2, N \neq 1$, and W_t is Brownian motion. It is easy to check that equation (2.5) satisfies the conditions for existence and uniqueness of solution [5]. We note that the volatility and drift function are nonlinear functions of S. In order to derive the regression equation, we use the following transformation $V_t = \frac{S_t^{1-N}}{1-N}$ and apply Itô-Doob differential formula to obtain

$$\begin{aligned} dV_t &= \frac{\partial}{\partial S_t}(\frac{S_t^{1-N}}{1-N})dS_t + \frac{1}{2}\big(\frac{\partial^2}{\partial S_t^2}(\frac{S_t^{1-N}}{1-N})\big)(dS_t)^2 \\ &= S_t^{-N}\big((\alpha S_t + \beta S_t^N + \frac{N}{2}\sigma^2 S_t^{2N-1})dt + \sigma S_t^N dW_t\big) - \frac{N}{2}S_t^{-N-1}\sigma^2 S_t^{2N}(dW_t)^2. \end{aligned}$$

Then,

$$dV_t = \big(\alpha(1-N)V_t + \beta\big)dt + \sigma dW_t. \tag{2.6}$$

Again by using the Euler type discretization process [15], stochastic differential equation (2.6) can be reduced to

$$V_t - V_{t-1} = \big(\alpha(1-N)V_{t-1} + \beta\big)\Delta t + \sigma(W_t - W_{t-1}). \tag{2.7}$$

From $\varepsilon_t = W_t - W_{t-1}$ and $\Delta t = 1$, equation (2.7) can be rewritten as

$$V_t = \big((1+\alpha(1-N))V_{t-1} + \beta\big) + \sigma\varepsilon_t \tag{2.8}$$

where α, β, N and σ are as defined in (2.5). For given N, by applying the least square regression method [11] and using above cited data sets, we can estimate these parameters.

2.3. **Nonlinear Stochastic Model 3.** This nonlinear stochastic model 3 [14] is described by the following Itô-Doob differential equation

$$dS_t = (\alpha S_t + \beta S_t^2 + \sigma^2 S_t)dt + \sigma S_t dW_t \tag{2.9}$$

where,α, β, and σ are parameters and W_t is Brownian motion. It is easy to check that equation (2.9) satisfies the conditions for existence and uniqueness of solution [5]. In order to derive the regression equation, we use the following transformation $V_t = \frac{-1}{S_t}$ and apply Itô-Doob differential formula to obtain

$$\begin{aligned} dV_t &= \frac{\partial}{\partial S_t}(\frac{-1}{S_t})dS_t + \frac{1}{2}\big(\frac{\partial^2}{\partial S_t^2}(\frac{-1}{S_t})\big)(dS_t)^2 \\ &= S_t^{-2}\big((\alpha S_t + \beta S_t^2 + \sigma^2 S_t)dt + \sigma S_t dW_t\big) - \frac{1}{2}S_t^{-3}\sigma^2 S_t^2 (dW_t)^2. \end{aligned}$$

Then,

$$dV_t = (-\alpha V_t + \beta)dt - \sigma V_t dW_t. \tag{2.10}$$

Again, the Euler type discretized version of (2.10) is as follows

$$V_t - V_{t-1} = (-\alpha V_{t-1} + \beta)\Delta t - \sigma V_{t-1}(W_t - W_{t-1}) \tag{2.11}$$

From the definition of V, we note that $y_t = \frac{V_t - V_{t-1}}{V_{t-1}} = \frac{S_{t-1}}{S_t} - 1, \varepsilon_t = W_t - W_{t-1}$ and $\Delta t = 1$. With these notations, equation (2.11) can be rewritten as

$$y_t = (-\alpha + \beta\frac{1}{V_{t-1}}) - \sigma\varepsilon_t \tag{2.12}$$

Then, parameters α, β and σ can be estimated using least square regression method [11].

3. NONLINEAR STOCHASTIC MODELS WITH EQUAL INTERVALS

Based upon our study of data partitioning [16], it is enough to consider a study of monthly data partitioning with jumps.

Monthly Nonlinear Models 1,2 and 3 Suppose $[0, t_1), [t_1, t_2), [t_2, t_3), ..., [t_{m-1}, t_m)$ represent the m monthly time intervals. Using monthly data partitioning with jumps, the nonlinear models 1,2 and 3 take the following forms, respectively, for each $t, t_{i-1} \leq t < t_i$ and $i = 1, 2, ..., m$

$$dS_t^{M_i} = \big(\alpha^{M_i} lnS_t^{M_i} + \beta^{M_i} + \frac{(\sigma^{M_i})^2}{2}\big)S_t^{M_i}dt + \sigma^{M_i}S_t^{M_i}dW_t, \tag{3.1}$$

(3.2)
$$dS_t^{M_i} = \left(\alpha^{M_i} S_t^{M_i} + \beta^{M_i}(S_t^{M_i})^{N^{M_i}} + \frac{N^{M_i}}{2}(\sigma^{M_i})^2 (S_t^{M_i})^{2N^{M_i}-1}\right)dt + \sigma^{M_i}(S_t^{M_i})^{N^{M_i}} dW_t,$$

and

$$dS_t^{M_i} = \left(\alpha^{M_i} S_t^{M_i} + \beta^{M_i}(S_t^{M_i})^2 + (\sigma^{M_i})^2 S_t^{M_i}\right)dt + \sigma^{M_i} S_t^{M_i} dW_t, \tag{3.3}$$

where, $\alpha^{M_i}, \beta^{M_i}, \sigma^{M_i}$ and $n^{M_i}, i = 1, 2, ..., m$ are parameters.
By following definition [8, 9, 14], the solution process of any one of the three models (3.1) (3.2) and (3.3) can be described by
(3.4)
$$S_t = \begin{cases} S_1(t, t_0, S_0), & t_0 \le t < t_1, S_0 = S_{t_0} \\ \phi_1 S_2(t, t_1, S_1), & t_1 \le t < t_2, S_1 = \lim_{t \to t_1^-} S_1(t, t_0, S_0) \\ ... & ... \\ \phi_{m-1} S_m(t, t_{m-1}, S_{m-1}), & t_{m-1} \le t < t_m, S_{m-1} = \lim_{t \to t_{n-1}^-} S_{m-1}(t, t_{m-2}, S_{m-2}) \end{cases}$$

S_0 is the initial value of the stock price. $\phi_1, \phi_2, ..., \phi_{m-1}$ are jumps and can be estimated as

$$\hat{\phi}_1 = \frac{S_{t_1}}{\lim_{t \to t_1^-} \hat{S}_1}, \hat{\phi}_2 = \frac{S_{t_2}}{\lim_{t \to t_2^-} \hat{S}_2}, ..., \hat{\phi_{m-1}} = \frac{S_{t_{m-1}}}{\lim_{t \to t_{m-1}^-} \hat{S_{m-1}}}.$$

All estimations and results are summarized in Section 5.

4. NONLINEAR STOCHASTIC MODELS WITH UNEQUAL INTERVALS

Under the monthly data partitioning approach, jumps are assumed to occur at the end of each month. This is not always the case.

Let $[0, t_1), [t_1, t_2), [t_2, t_3), ..., [t_{k-1}, t_n)$ represent the n unequal time intervals. We suppose the jumps are at time $t_1, t_2, ..., t_{n-1}$. The given data set is decomposed into n data subsets. The i^{th} data subset is observed on the i^{th} time interval $[t_{i-1}, t_i)$ for each $i = 1, 2, ..., n$.

Nonlinear Models 1,2 and 3 with Unequal Intervals By employing the above described unequal interval data partitioning process with jumps, the nonlinear models 1,2 and 3 take the following forms respectively, for each $t, t_{i-1} \le t < t_i$ and $i = 1, 2, ..., n$

$$dS_t^{I_i} = \left(\alpha^{I_i} ln S_t^{I_i} + \beta^{I_i} + \frac{(\sigma^{I_i})^2}{2}\right) S_t^{I_i} dt + \sigma^{I_i} S_t^{I_i} dW_t, \tag{4.1}$$

$$dS_t^{I_i} = \left(\alpha^{I_i} S_t^{I_i} + \beta^{I_i}(S_t^{I_i})^{N^{I_i}} + \frac{N^{I_i}}{2}(\sigma^{I_i})^2 (S_t^{I_i})^{2N^{I_i}-1}\right)dt + \sigma^{I_i}(S_t^{I_i})^{N^{I_i}} dW_t, \tag{4.2}$$

and

$$dS_t^{I_i} = \left(\alpha^{I_i} S_t^{I_i} + \beta^{I_i}(S_t^{I_i})^2 + (\sigma^{I_i})^2 S_t^{I_i}\right)dt + \sigma^{I_i} S_t^{I_i} dW_t, \tag{4.3}$$

where, $\alpha^{I_i}, \beta^{I_i}, \sigma^{I_i}$ and $N^{I_i}, i = 1, 2, ..., n$ are parameters.
By following definition [8, 9, 14], the solution process of any one of the three models (4.1) (4.2) and (4.3) can also be described in (3.4). All estimations and results are summarized in Section 5.

5. ESTIMATIONS AND RESULTS

In the following, by using the nonlinear models, equal and unequal time intervals with jumps and solution representation, we outline an algorithm to develop the nonlinear models for stock price process as follows:

i Equal Intervals: Divide the whole stock price process into monthly sub data sets.
ii Unequal Intervals: Divide the whole stock price process into sub data sets using Unequal Interval data partitioning process.
- Compute the daily relative difference of all observations.
- Define the threshold such that the number of observation in time intervals is not too large or too small. At this stage, suppose $[0, t_1), [t_1, t_2), ..., [t_{n-1}, t_n)$ represent the n intervals.
- Divide the whole stock price data into sub data sets.

iii Assign values $0 \leq N \leq 1.2$.
iv Corresponding to 4 models (three nonlinear models and GBM model), we transform the given observed data into corresponding modeling transformations:

$$V_t^{Non1} = lnS_t$$

$$V_t^{Non2} = \frac{S_t^{1-N}}{1-N}$$

$$y_t^{Non3} = \frac{S_{t-1}}{S_t} - 1$$

$$y_t^{GBM} = lnS_t$$

v For every interval, run linear regression to estimate parameters for nonlinear 1, 2, 3 and GBM models, respectively.
vi Compute the jump coefficients by applying

$$\hat{\phi}_i = \frac{S_{t_i}}{\lim_{t \to t_i^-} \hat{S}_t}, i = 1, 2, 3, ...$$

vii For every interval, compute the residual for each model.
viii Compute the basic statistic of the models.

All the Estimations and results are summarized as follows.

The first data set of stock X is collected over $3\frac{1}{2}$ years, and selected from Fortune 500 companies. There are 848 observations. The least square estimation method [11] is used to estimate the parameters of linear and nonlinear stochastic models. Due to the large data set, our presented prediction results are limited to the part of data set 300 - 600 observations. This segment of sub data set is good representative of the overall data set. Figures 5-8 are the predictions of Stock X. Table 1 contains the basic statistics of linear and nonlinear models with equal and unequal data partitioning. More details will appear in [10].

From Table 1 we can see that overall, the Nonlinear Model 1 with Unequal Interval has less variance among all Models (Monthly and Unequal Intervals) . With Monthly data partitioning, Nonlinear Model 3 with Jumps has the least mean and variance of residual. With Unequal Interval data partitioning, all Nonlinear Models 1, 2 and 3 have less mean and variance of residual than GBM (linear) model.

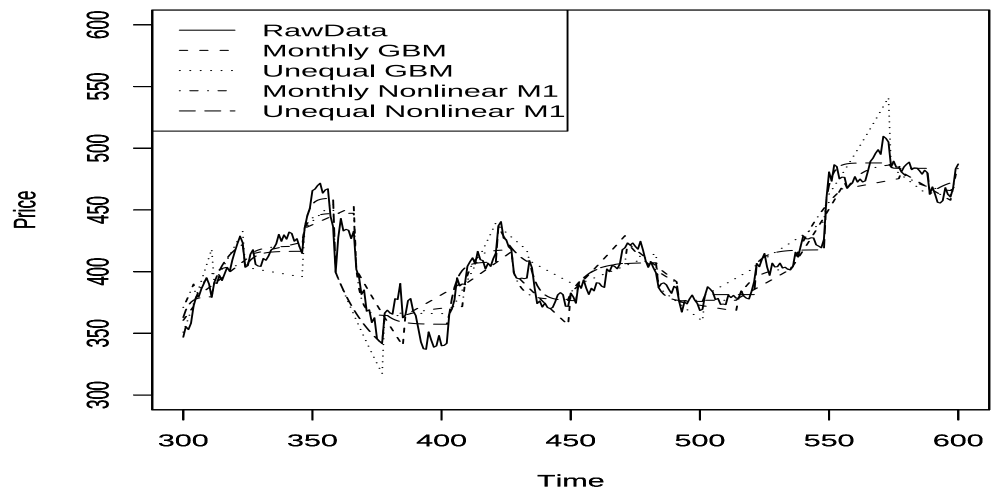

FIGURE 5. Prediction of Stock X for the Observations. 300 - 600 Using Nonlinear Model 1

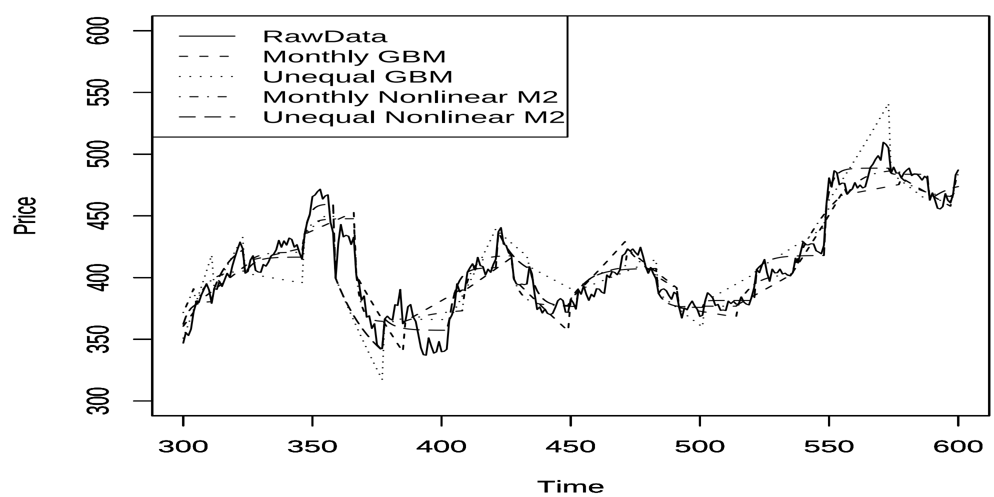

FIGURE 6. Prediction of Stock X for the Observations. 300 - 600 Using Nonlinear Model 2

TABLE 1. Basic Statistics for Models of Stock X

	$\bar{r}$	S_r^2	S_r	No. of Intervals
Monthly GBM	-1.22683	207.3278	14.3989	41
Monthly Nonlinear Model 1	-1.928296	141.1754	11.88173	41
Monthly Nonlinear Model 2	-2.090806	143.2248	11.96765	41
Monthly Nonlinear Model 3	-1.731151	139.2792	11.80166	41
Unequal Interval GBM	-1.962899	258.1040	16.06562	39
Unequal Interval Model 1	-0.5315015	131.2354	11.4558	39
Unequal Interval Model 2	-0.6097021	131.3068	11.45892	39
Unequal Interval Model 3	-0.4023679	132.16	11.49609	39

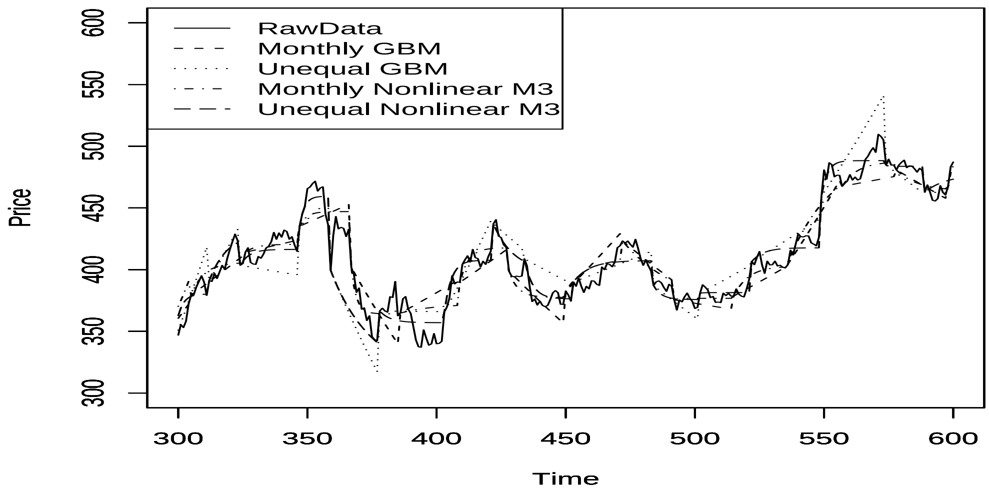

FIGURE 7. Prediction of Stock X for the Observations. 300 - 600 Using Nonlinear Model 3

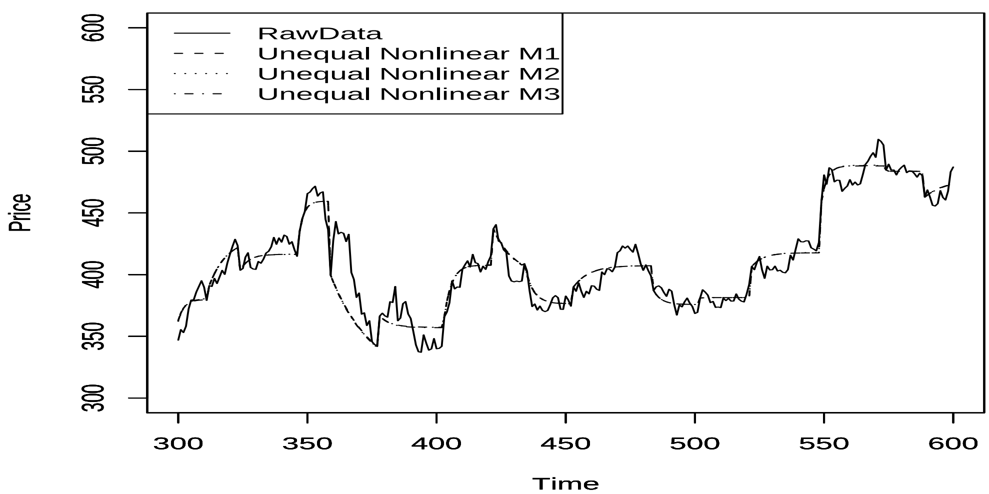

FIGURE 8. Prediction of Stock X for the Observations. 300 - 600 Using Three Nonlinear Models

The second data set we applied is the stock Y. It lasts more than 22 years and has 5630 observations. It is also selected from Fortune 500 companies. Due to the large data set, our prediction results are limited to the part of data set 5000 - 5300 observations. This segment of sub data set is good representative of the overall data set. Figures 9-12 are the predictions of Stock Y for 5000 - 5300 observations. Table 2 contains the basic statistics of linear and nonlinear models with different data partitioning. Details will also appear in [10].

Table 2 shows overall basic statistics of Stock Y applying all Monthly and Unequal Interval Nonlinear Models. With Monthly data partitioning, Nonlinear Model 2 has

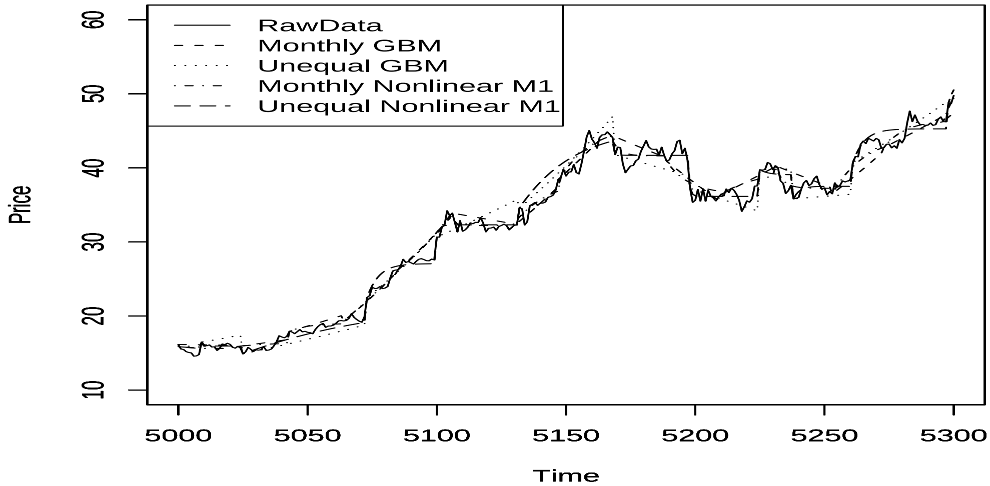

FIGURE 9. Prediction of Stock Y for the Observations. 5000 - 5300 Using Nonlinear Model 1

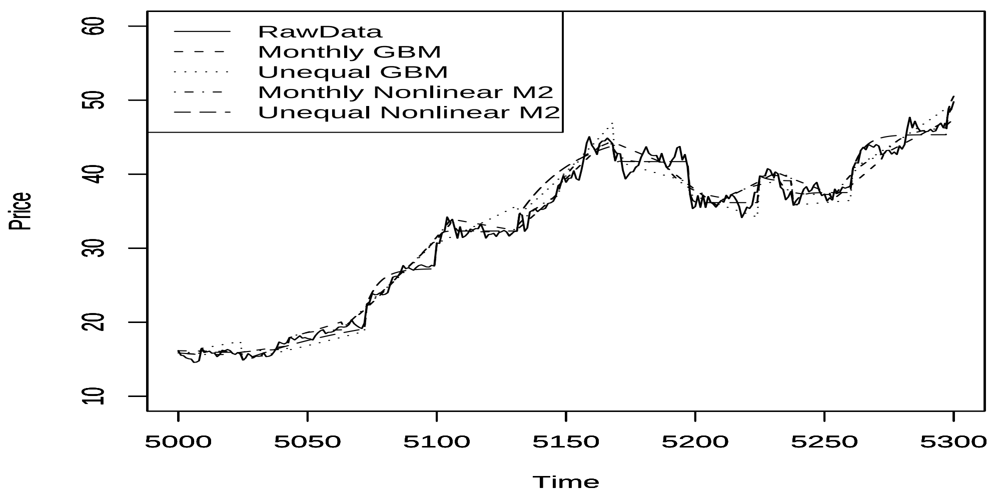

FIGURE 10. Prediction of Stock Y for the Observations. 5000 - 5300 Using Nonlinear Model 2

the least mean and variance of residual. With Unequal Interval data partitioning, all Nonlinear Models 1, 2 and 3 have less mean and variance of residual than GBM model. Nonlinear Model 2 with Unequal Interval has the least mean and variance of residual among all models, moreover it has less number of subintervals.

The third data set we applied is the S&P 500 Index. It lasts more than 59 years and has 14844 observations, from1/1/1950 to 12/31/2008. Due to the large data set, our prediction results are limited to the part of data set 14000-14300 observations. This segment of sub data set is good representative of the overall data set. Figures 13-16 are the predictions of S&P 500 Index for 14000 - 14300 observations. Table

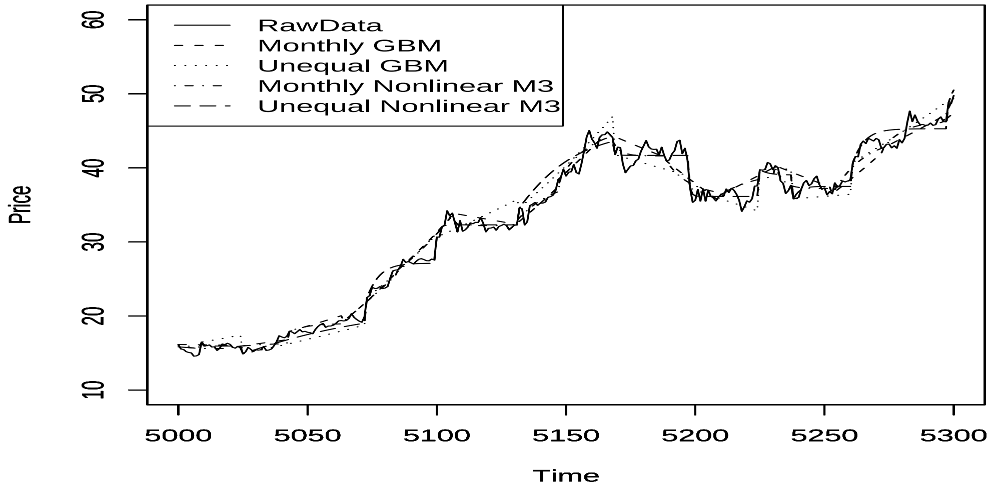

Figure 11. Prediction of Stock Y for the Observations. 5000 - 5300 Using Nonlinear Model 3

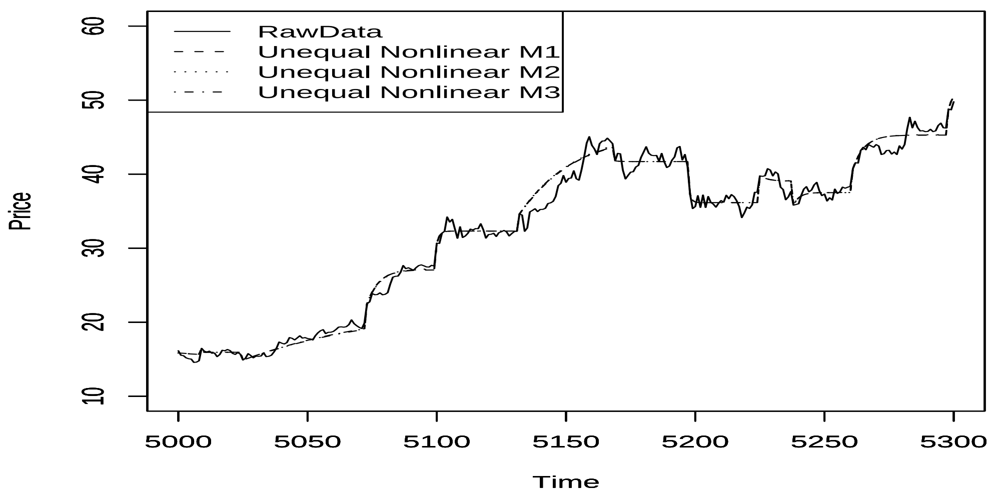

Figure 12. Prediction of Stock Y for the Observations. 5000 - 5300 Using Three Nonlinear Models

3 contains the basic statistics of linear and nonlinear models with different data partitioning. More details will appear in [10].

Table 3 shows overall basic statistics of S&P 500 Index applying all Monthly and Unequal Interval Nonlinear Models. With Monthly data partitioning, Nonlinear Model 2 has the least mean and variance of residual. With Unequal Interval data partitioning, all Nonlinear Models 1, 2 and 3 have less mean and variance of residual than GBM model. Nonlinear Model 2 with Unequal Interval has the least mean and the variance of residual among all models, moreover the least variance of residual and the Number of Intervals.

Table 2. Basic Statistics for Models of Stock Y

	$\bar{r}$	S_r^2	S_r	No. of Intervals
Monthly GBM	-0.00982612	1.206479	1.098399	268
Monthly Nonlinear Model 1	0.02006753	1.469688	1.212307	268
Monthly Nonlinear Model 2	-0.002057085	1.155363	1.074878	268
Monthly Nonlinear Model 3	0.02663191	1.295051	1.138003	268
Unequal Interval GBM	-0.01248156	1.199703	1.095310	256
Unequal Interval Model 1	-0.004257979	0.6064698	0.7787617	256
Unequal Interval Model 2	-0.01147757	0.6033852	0.7767788	256
Unequal Interval Model 3	0.006576893	0.6125127	0.7826319	256

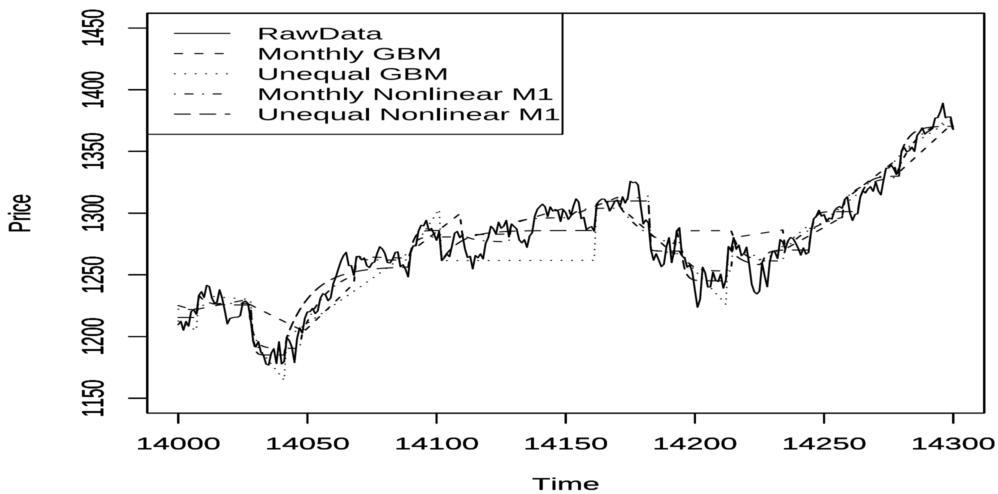

Figure 13. Prediction of S&P 500 Index for the Observations. 14000 - 14300 Using Nonlinear Model 1

Table 3. Basic Statistics for Models of S&P500 Index

	$\bar{r}$	S_r^2	S_r	No. of Intervals
Monthly GBM	58.30902	486.4579	22.05579	708
Monthly Nonlinear Model 1	4.027517	281.7153	16.78438	708
Monthly Nonlinear Model 2	4.084275	281.7003	16.78393	708
Monthly Nonlinear Model 3	4.274907	282.7780	16.81600	708
Unequal Interval GBM	2.471564	210.2159	14.49882	570
Unequal Interval Model 1	0.6186245	79.46592	8.914366	570
Unequal Interval Model 2	0.5835638	78.5180	8.861039	570
Unequal Interval Model 3	0.6590607	79.5725	8.920342	570

All results are summarized in Table 4 with regard to Stock X, Y and S&P 500 Index. The Table 4 shows that nonlinear model 2 ranks No.1 in both monthly and unequal interval data partitioning of 2 out 3 data sets.

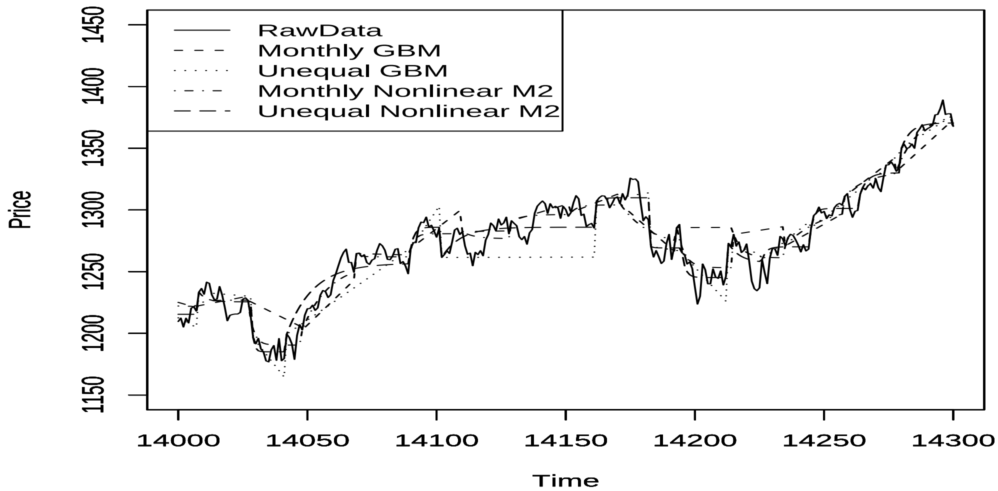

Figure 14. Prediction of S&P 500 Index for the Observations. 14000 - 14300 Using Nonlinear Model 2

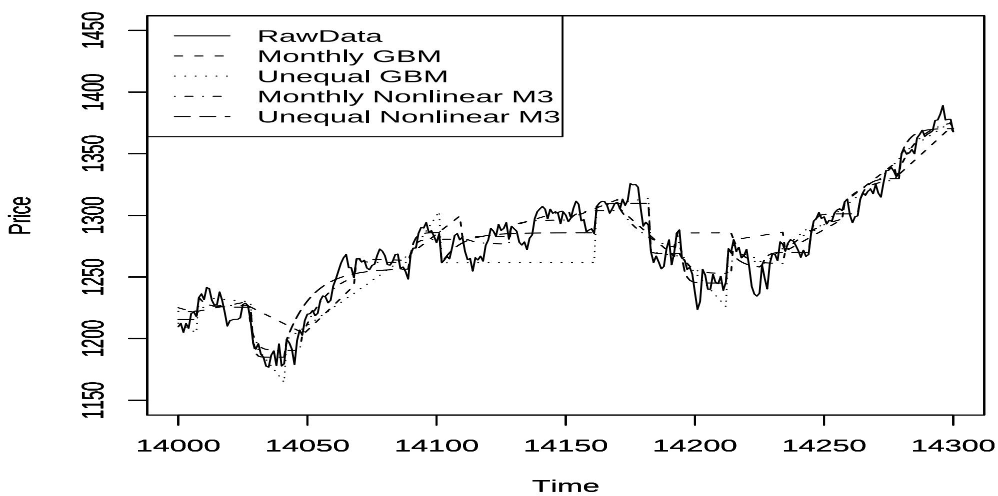

Figure 15. Prediction of S&P 500 Index for the Observations. 14000 - 14300 Using Nonlinear Model 3

Table 4. Summary of Results

	Monthly Interval				Unequal Interval			
Stock	Rank1	Rank2	Rank3	Rank4	Rank1	Rank2	Rank3	Rank4
X	Non.3	Non.1	Non.2	GBM	Non.1	Non.2	Non.3	GBM
Y	Non.2	GBM	Non.3	Non.1	Non.2	Non.1	Non.3	GBM
S&P 500	Non.2	Non.1	Non.3	GBM	Non.2	Non.1	Non.3	GBM

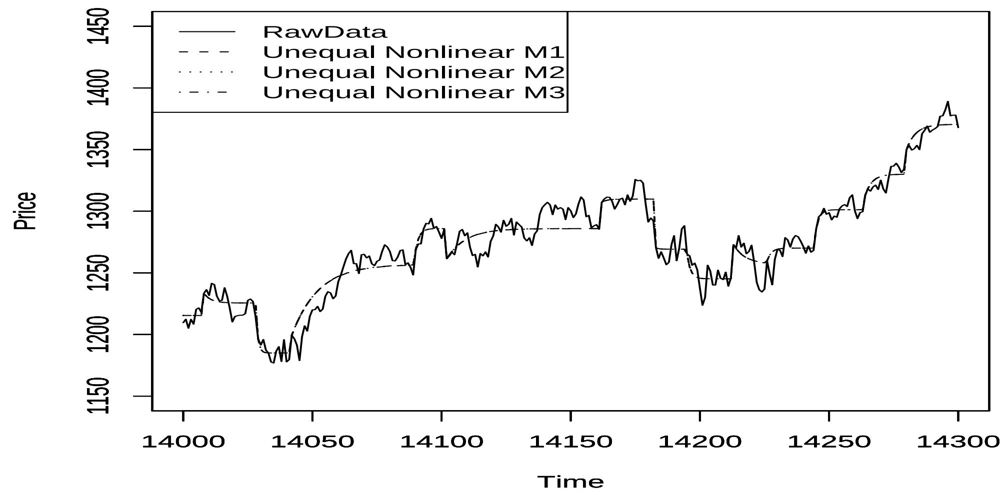

FIGURE 16. Prediction of S&P 500 Index for the Observations. 14000 - 14300 Using Three Nonlinear Models

6. CONCLUSIONS AND FUTURE WORK

In this work, we presented three nonlinear stochastic models. By using classical model building process [16], we developed the modified version of nonlinear stochastic models using equal and unequal data partitioning with jumps. Based on our study, we draw a few important conclusions.

i For monthly data partitioning, all three nonlinear models are better than GBM model for stock X and S&P 500 Index, and for Stock Y, nonlinear model 2 is better than GBM model and nonlinear models 1 and 3.
ii For unequal interval data partitioning, and for all three stock data sets, all nonlinear models are better than GBM model .
iii The unequal data partitioning approach is superior than the monthly data partitioning.
iv For both equal and unequal data partitioning, the nonlinear model 2 is the best for stock Y and S&P 500 Index, and nonlinear model 1 is best for stock X.

So far, we focused our attention to build stochastic models to fit the data sets. The important problem in modeling is to predict the future stock market price. The study of this problem is under our current investigation, and it will be published elsewhere.

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