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A THEORETIC STOCHASTIC DYNAMIC CONTROL APPROACH FOR THE LENDING RATE POLICY

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Abstract: Normally, different financial institutions, i.e. banks, offer a variety of loans with different lending rates, according to a basic interest rate and the experience of the repayment patterns. In this paper, we construct and present a theoretic linear stochastic control model in order to evaluate the associated credit risk and obtain the optimal strategy for the determination of the level of the lending interest rates by optimizing the accumulated profit. Each sub-portfolio of loans is treated separately during a unit interval while at the end of the each time period there is some kind of solvency interaction. We assume that the repayment pattern follows a Brownian motion and using advanced optimization techniques, the optimal solutions are derived.

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Key words: Linear Stochastic Optimal Control; Brownian Motion; Matrix Riccati Differential Equation; Lending Rate Policy.

1. Introduction

In the banking system, the determination of the lending rate policy is one of the most attractive and intrusive problems, as well. Analytically, loan pricing is the determination of the lending rate policy for different banking products of loans (e.g., personal loans, business loans, mortgages, overdrafts etc) which are offered to customers, according to their risk exposure, see Saunders [17]. The respective literature is very rich although some of the approaches and the concluding results are not linked intuitively to common lending practices, see for instance, Sealey [18], Ho and Saunders [7], Slovin and Sushka [19], Allen [1], Zarruk [27], Zarruk and Madura [28], Petersen and Rajan [15], Angbazo [2], Wong [25], Saunders [17], Nakamura, Qian, Samdoh, and Nakagawa [13], Stanhouse and Stock [20] and Stein [21].

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Quite recently, the lending rate policy via an appropriate investment strategy for an interacted portfolio of different loans into a continuous-time and discrete-time stochastic framework has been examined by Zimbidis, Pantelous and Kalogeropoulos [29], and Pantelous [14], respectively. Two interesting bank optimization models with several variables, stochastic inputs and smoothness criteria described by a quadratic functional have been proposed for managing the task. In both cases, the state variable of the systems correspond to the accumulated surplus profit/loss, which can be oscillated deliberately, absorbing fluctuations in the different parameters involved.

Furthermore, the bank institution managers should be compensated with the risk that borrowers are not always consistent with the repayments. It is obvious that loans are priced according to the involved risk, and the capital profits that the management desires. However, throughout the last decades, a contradictory question is being raised about whether the financial institutions should provide cheap loans in order to attract more customers for other profitable business or not. Something really interesting has been shown by Fried and Howitt [6], and Petersen and Rajan [15]. They showed that *the welfare is enhanced by smoothing of lending rates in relation to borrower risk and market interest rates*.

In this paper, the main object is to extend further the research works proposed by Zimbidis, Pantelous and Kalogeropoulos [29] and Pantelous [14] by developing a more complicated (and more realistic) stochastic model for smoothing the lending rate policy for the different sub-portfolios of loans in a way that banking managers are seeking for. Moreover, we consider this problem in a more general framework, letting some kind of interaction between the different sub-portfolios. This paper connects the lending rate policy of an interacted portfolio of different loans with stochastic continuous-time control theory. Although optimal control theory was developed by engineers in order to investigate the properties of dynamic systems difference/differential equations, it has also been applied to financial problems. Tustin [23] was the first to spot a possible analogy between the industrial and engineering processes and post-war macroeconomic policy-making (see Holly and Hallett [8], for further historical details). More recently, in the vast literature of banking, Jobst, Mitra and Zenios [10], Topaloglou, Vladimirou and Zenios [22] develop modelling paradigms which integrate credit risk and market risk in random dynamical framework and use multistage stochastic programming tools. From this point of view, a method of controlling over time some major variables is introduced buffering any kind of fluctuations, in order to absorb partially or completely the probable unexpectedness in micro- and/or macroeconomic conditions, in external factors as competition, legal and regulatory requirements or other worsen random events. Moreover, we will assume that the financial institution managers desire to keep the profit for the bank close to a *specified* trajectory.

From this point of view, the financial institution has a certain total capacity for providing loans equal to $\Delta(t)$, at time t. We take in mind that the bank's customers are not always consistent with their repayments. So, at each time t, a different amount say $\Delta'(t)$ is repaid through the installments paid by the customers.

Actually, $\Delta'(t)$ is normally smaller than $\Delta(t)$ ($\Delta'(t) < \Delta(t)$) but at some exceptions may take values greater than $\Delta(t)$, whenever the policy holders pay with some time delay two, three or more installments to the bank or even negative when we expect bankruptcy or withdrawal of some installments.

Hence, we may visualize the situation above with the following theoretical **Figure 1**.

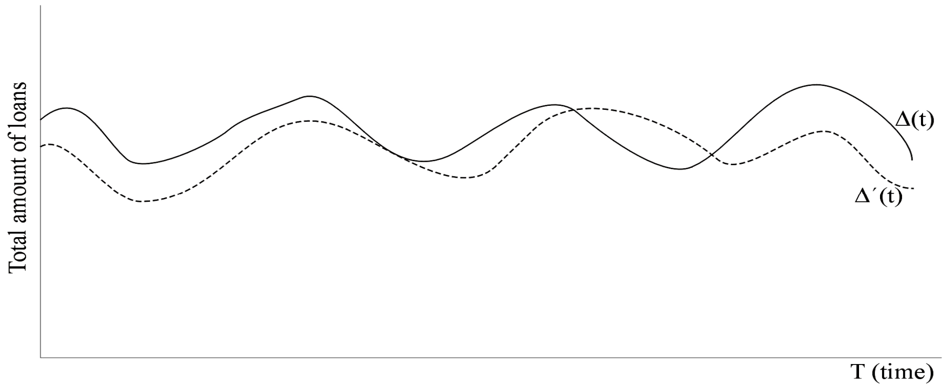

Figure 1: The financial institution has a certain total capacity for providing loans equal to $\Delta(t)$, at time t. Moreover, a different amount say $\Delta'(t)$ is repaid through the installments paid by the customers.

A brief outline of the paper is as follows. Section 2 provides the incentives and the typical modelling features of the problem. Moreover, it is devoted to the results of stochastic calculus and control theory for standard Brownian motion and the respective linear systems driven by such a process. Section 3 provides the approximation solution for the matrix Riccati differential equation. In section 4, we provide an interesting numerical example with two sub-portfolios of loans with some interesting and insightful diagrams, while section 5 concludes the whole paper.

2. Lending rate policy model into a continuous-time stochastic framework

In this part of the paper, we will start to construct our stochastic control model. It should be pointed out that we consider a portfolio of n sub-portfolios of loans. First, we define the necessary symbols and the respective notation; see **Table A**.

We assume that the *ratio* $p_i(t)$ for the i^{th} sub-portfolio of loans which is consistently repaid is driven by a standard Brownian motion. This uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The flow of information is given by the natural filtration $\{\mathcal{F}\}_{t\in[0,T]}$, i.e. the $\mathcal{P}$-augmentation of a one dimensional Brownian filtration. Without loss of generality, let us assume that the $\{\mathcal{F}\}_{t\in[0,T]} = \mathcal{F}$, i.e. the observable events are eventually known. Hence,

$$dp_i(t) = m_i(t)\,dt + \sigma_i(t)\,dW_i(t), \tag{2.1}$$

where $m_i(t)$ and $\sigma_i(t)$ represent the drift and the volatility respectively of the ratio of the total amount of loan which corresponds to the installments paid at time t over the total amount of loan which has been placed into the i^{th} sub-portfolio of loans, for $i = 1,2,...,n$. Note that we should be very careful with the choice of parameters, $m_i(t)$ and $\sigma_i(t)$. This stochastic ratio $p_i(t)$ can take both positive and negative values. Under the negative values, we assume that the bank can withdraw some of the unpaid installments.

Table A	
$\Pi_i(t)$	*Accumulated profit* or *loss* (state variable) at time t for the i^{th} sub-portfolio of loans, for $i = 1, 2, ..., n$.
$a_i(t)$	*Rate of return* (input variable) for the accumulated profit at time t for the i^{th} sub-portfolio of loans, for $i = 1, 2, ..., n$. In this paper, we assume that the rate of return is a *deterministic* function. Following this assumption, the managerial team should decide to invest conservatively the instant surpluses at time t in a pool of different secure products, for instance different kinds of bank accounts, short-time government T-bond or something equivalent.
$p_i(t)$	*Ratio* (input variable) of the total amount of loan which corresponds to the installments paid at time t over the total amount of loan which has been placed into the i^{th} sub-portfolio of loans, for $i = 1, 2, ..., n$.
$\rho_i(t)$	*Ratio* (input variable) of the total amount placed to the i^{th} sub-portfolio of loans, for $i = 1, 2, ..., n$, over the total amount of loans, $\sum_{i=1}^{n} \rho_i(t) = 1$ (see below why).
$\Delta(t)$	*Total amount* for the *whole portfolio of loans* at time t, see also **figure 1**. This amount will be invested for loans by the financial institution at time t. It is profound that the managerial team would be very happy to receive back the whole capital plus the required profit (let us say a profit margin).
$\Delta_i(t)$	*Total amount of loans* (input variable) at time t for the i^{th} *sub-portfolio of loans*, $i = 1, 2, ..., n$, where $\Delta_i(t) = \rho_i(t)\Delta(t)$. For that reason, we assume that $\sum_{i=1}^{n} \rho_i(t) = 1$.
$\Delta_i'(t)$	*Total amount of loans* (input variable) that corresponds to *customers who consistently repay their installments* at time t for the i^{th} *sub-portfolio*, of loan, for $i = 1, 2, ..., n$, where $\Delta_i'(t) = p_i(t)\Delta(t)$.
$c_i(t)$	*Capital cost* (input variable), which includes expenses, operational costs, rate of return paid to customers due to bank deposits and the desirable profit for the bank at time t for the i^{th} sub-portfolio of loan, for $i = 1, 2, ..., n$.
$\varepsilon_i(t)$	*Lending rate* (control variable) at time t for the i^{th} sub-portfolio of loans, for $i = 1, 2, ..., n$.
$\lambda_{ij}(t)$	*Percentage of solvency* (input variable) transferred from the i^{th} sub-portfolio to j^{th} sub-portfolio of loans at time t. $\sum_{j=1}^{n} \lambda_{ij}(t) = 1$ $\forall i = 1, 2, ..., n$.

Then the accumulated profit of the total portfolio for the bank at time t is given by

$$\Pi(t) = \sum_{i=1}^{n} \Pi_i(t)\,. \tag{2.2}$$

We assume that the *ratio* $p_i(t)$ for the i^{th} sub-portfolio of loans which is consistently repaid is driven by a standard Brownian motion. This uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The flow of information is given by the natural filtration $\{\mathcal{F}\}_{t\in[0,T]}$, i.e. the $\mathcal{P}$-augmentation of a one dimensional Brownian filtration. Without loss of generality, let us assume that the $\{\mathcal{F}\}_{t\in[0,T]} = \mathcal{F}$, i.e. the observable events are eventually known. Hence,

$$dp_i(t) = m_i(t)\,dt + \sigma_i(t)\,dW_i(t)\,, \tag{2.1}$$

where $m_i(t)$ and $\sigma_i(t)$ represent the drift and the volatility respectively of the ratio of the total amount of loan which corresponds to the installments paid at time t over the total amount of loan which has been placed into the i^{th} sub-portfolio of loans, for $i = 1, 2, ..., n$. Note that we should be very careful with the choice of parameters, $m_i(t)$ and $\sigma_i(t)$. This stochastic ratio $p_i(t)$ can take both positive and negative values. Under the negative values, we assume that the bank can withdraw some of the unpaid installments.

Then the accumulated profit of the total portfolio for the bank at time t is given by

$$\Pi(t) = \sum_{i=1}^{n} \Pi_i(t)\,. \tag{2.2}$$

Moreover, under the above notation, we may describe the system by the following stochastic differential equation:

$$\begin{aligned}
d\Pi_i(t) &= a_i(t)\mathbf{1}_{\Pi_i(t)\in[0,\infty)}\Pi_i(t)\,dt && (\textit{Investment income} \text{ from the fund})\\
&+\varepsilon_i(t)\Delta_i(t)\,dp_i(t) - c_i(t)\Delta_i(t)\,dt && (\textit{Profit/loss} \text{ of the } i^{th} \text{ sub-portfolio of loans at time } t)\\
&+\lambda_{1i}(t)\Pi_1(t)\,dt + \ldots + \lambda_{i-1i}(t)\Pi_{i-1}(t)\,dt + \lambda_{ii}(t)\Pi_i(t)\,dt \\
&+\lambda_{i+1i}(t)\Pi_{i+1}(t)\,dt + \ldots + \lambda_{ni}(t)\Pi_n(t)\,dt && (\textit{solvency interaction})
\end{aligned} \tag{2.3}$$

or equivalently, substituting equation (2.1) into (2.3) we obtain

$$\begin{aligned}
d\Pi_i(t) &= a_i(t)\mathbf{1}_{\Pi_i(t)\in[0,\infty)}\Pi_i(t)\,dt \\
&\quad + \varepsilon_i(t)\rho_i(t)\Delta(t)\{m_i(t)\,dt + \sigma_i(t)\,dW_i(t)\} - c_i(t)\rho_i(t)\Delta(t)\,dt \\
&\quad + \lambda_{1i}(t)\Pi_1(t)\,dt + \ldots + \lambda_{i-1i}(t)\Pi_{i-1}(t)\,dt + \lambda_{ii}(t)\Pi_i(t)\,dt \\
&\quad + \lambda_{i+1i}(t)\Pi_{i+1}(t)\,dt + \ldots + \lambda_{ni}(t)\Pi_n(t)\,dt
\end{aligned} \tag{2.4}$$

for $i = 1, 2, 3, \ldots, n$.

Before we go further, it would be very helpful to draw and analyze more the following figure.

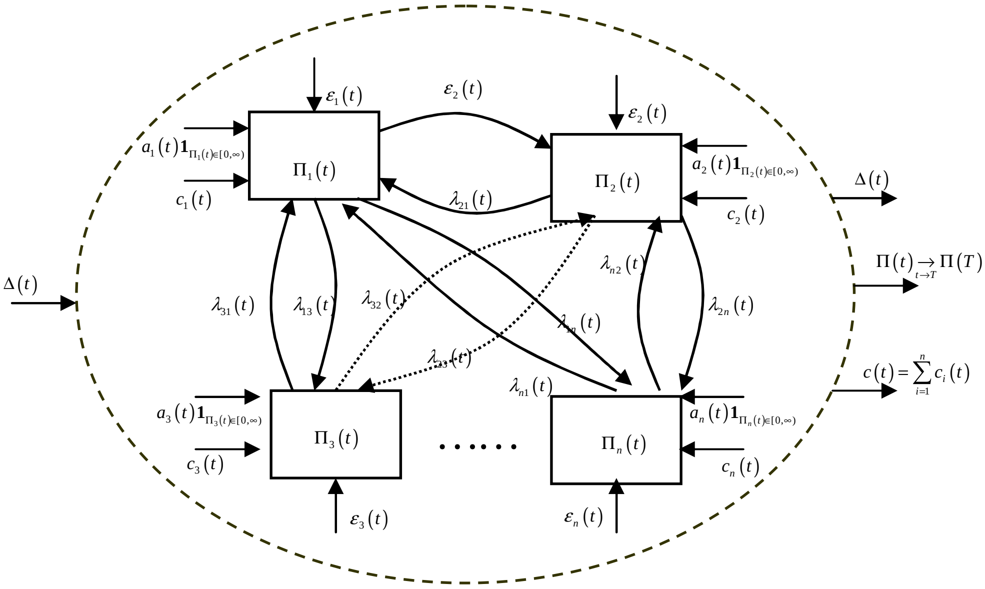

Figure 2: The described process for portfolio of *n* different loans.

As we can see descriptively above, the financial institution invests a predetermined amount of $\Delta(t)$ (money-units) into a predefined portfolio of *n* different loans. Into this pool of different products (for different clients and risks), it is provided also the opportunity to transfer funds (solvency interaction) from one product to the other, see also introduction. At the end of the process, three things are highly been desired by the managerial team:

a) To receive back the whole invested capital $\Delta(t)$.

b) To obtain the desired capital cost, which includes the different kind of *expenses*, and *operational costs*; the *rate of return* paid to customers due to bank deposits and the *desirable profi*t for the bank.

c) Under the above conditions, at the end of the predetermined time-period, the financial institution would like to have $\lim_{t\to T}\Pi(t)=\lim_{t\to T}\sum_{i=1}^{n}\Pi_i(t)=\Pi(T)$. Here, we would like to underline that since $\Pi_i(t)$ can be either *positive* (profit) or *negative* (loss), we are interested about the limit ($t\to T$) of the $\sum_{i=1}^{n}\Pi_i(t)$ to tend to $\Pi(T)$ (in some cases, this fund can be equal to zero).

In order to obtain (a)-(c), we should determine a particular **L**inear **Q**uadratic **R**egulator (LQR) controller, which can transfer instantly the state variable into the desired-tracking path. In this paper, we will not pay attention to the *percentage of solvency* ($\lambda_{ij}(t)$) transferred from the i^{th} sub-portfolio to j^{th} sub-portfolio of loans at time t. Thus, naturally speaking, the managerial team should predetermine the policy for the solvency interaction, i.e. which product can be benefited more or less. As a further extension of this research work, we are thinking to determine this policy for the interaction factors among the different portfolio by using another optimization algorithm.

In this paper, as we will discuss again later, we assume that the optimization algorithm has been already applied and the derived numbers for the $\lambda_{ij}(t)$ parameters are the optimal ones; see numerical example in Section 4.

Consequently, see also Zimbidis, Pantelous and Kalogeropoulos [29], we take in matrix form the (non homogeneous) linear stochastic differential equation of type

$$\begin{cases} d\bar{\Pi}(t)=\left\{A(t)\bar{\Pi}(t)+B(t)\bar{\varepsilon}(t)+C(t)\right\}dt+\sum_{j=1}^{n}H_j(t)\bar{\varepsilon}(t)dW_j(t) \\ \bar{\Pi}(0)=\Pi_0 \end{cases}, \qquad (2.5)$$

where

$$A(t)=\begin{bmatrix} a_1(t)\mathbf{1}_{\Pi_1(t)\in[0,\infty)}+\lambda_{11}(t) & \lambda_{21}(t) & \cdots & \lambda_{n1}(t) \\ \lambda_{12}(t) & a_2(t)\mathbf{1}_{\Pi_2(t)\in[0,\infty)}+\lambda_{22}(t) & \cdots & \lambda_{n2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1n}(t) & \lambda_{2n}(t) & \cdots & a_n(t)\mathbf{1}_{\Pi_n(t)\in[0,\infty)}+\lambda_{nn}(t) \end{bmatrix},$$

$$\bar{\Pi}(t)=\begin{bmatrix} \Pi_1(t) \\ \Pi_2(t) \\ \vdots \\ \Pi_n(t) \end{bmatrix}, \qquad \bar{\varepsilon}(t)=\begin{bmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \\ \vdots \\ \varepsilon_n(t) \end{bmatrix},$$

$$B(t)=diag\left\{m_1(t)\rho_1(t)\Delta(t);m_2(t)\rho_2(t)\Delta(t);\cdots;m_n(t)\rho_n(t)\Delta(t)\right\},$$

$$C(t)=\begin{bmatrix} -c_1(t)\rho_1(t)\Delta(t) \\ -c_2(t)\rho_2(t)\Delta(t) \\ \vdots \\ -c_n(t)\rho_n(t)\Delta(t) \end{bmatrix} \text{ and } H_j(t)=\begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \sigma_j(t)\rho_i(t)\Delta(t) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The functions appearing in stochastic linear equation (2.5) satisfy

$$A(t),B(t),H_j(t)\in L^{\infty}\left(0,T;\mathbb{R}^{n\times n}\right) \text{ and } C(t)\in L^{\infty}\left(0,T;\mathbb{R}^{n}\right).$$

For any $t\in[0,T]$, we denote $U^{\omega}[0,T]$ the set of all 5-tuples $\left(\Omega,\mathcal{F},\mathcal{P},W(\cdot),\bar{\varepsilon}(\cdot)\right)$ satisfying the following mathematical conditions

- $(\Omega,\mathcal{F},\mathcal{P})$ is a complete probability space.
- $\{W(t)\}_{t\geq 0}$ is a n-dimensional standard Brownian motion defined on $(\Omega,\mathcal{F},\mathcal{P})$ over $[0,T]$ (with $W(0)=0$ almost surely), and $\mathcal{F}_t=\sigma\{W(r):0\leq r\leq t\}$ augmented by all the $\mathcal{P}$-null sets in $\mathcal{F}$.
- $\bar{\varepsilon}(\cdot)\in L_F^2\left(0,T;\mathbb{R}^n\right)$.

- Under $\bar{\varepsilon}(\cdot)$, for any $\bar{\Pi}(0)=\Pi_0\in\mathbb{R}^n$ equation (2.3) admits a unique solution $\bar{\Pi}(\cdot)$ on $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathcal{P}\right)$.

In control theory, the appropriate choice of the optimizing criterion is more than important. In our case, we minimize the expression (2.6) which is a *quadratic cost criterion*. In the literature such kind of problems are called *(stochastic) linear quadratic problems*. **S**tochastic **L**inear **Q**uadratic (SLQ) problems have been studied by many authors, among them we mention Wonham [24], McLane [12], Davis [5], Ichikawa [9], Chen and Yong [4] etc. In many recent works on mathematical finance (see option pricing, utility optimization), as well as, in engineering problems (note that, here, it is sometimes called *energy cost function*) this criterion has been applied. Analytically,

$$J^{\bar{\varepsilon}}\left(t,\bar{\Pi}\right)=\mathbb{E}^{t,\bar{\Pi}}\left[\beta\int_0^T\left(\bar{\varepsilon}(t)-\bar{\varepsilon}_\tau\right)^T\left(\bar{\varepsilon}(t)-\bar{\varepsilon}_\tau\right)dt+(1-\beta)\left(\bar{\Pi}(\mathrm{T})-\bar{\Pi}_\tau\right)^T\left(\bar{\Pi}(\mathrm{T})-\bar{\Pi}_\tau\right)\right] \tag{2.6}$$

where, $T>0$, $R=\beta I_n$ and $G=(1-\beta)I_n$, β is a weighting factor i.e. $0\leq\beta\leq 1$.

This criterion requires a lending rate policy $\bar{\varepsilon}(t)$, near to the target rate $\bar{\varepsilon}_\tau$ (desired-tracking path) which is fully acceptable by the managers of the banking system and it is affordable by the customers while also a small final value for the surplus fund $\bar{\Pi}(T)$ obtained from this operation. The weight β (or $1-\beta$) measures the impact that occurs when the control variable and the surplus respectively are changed. This parameter would be obtained after an insightful research and negotiations with all parties involved in the banking and financial system (i.e. financial institution managers, customers, international banking authorities etc.). Since this work is based on a more abstract framework, the exact determination of this weight is beyond its scope.

Now, the above SLQ problem at $(0,G)\in[0,T]\times\mathbb{R}^n$, where G (a bounded condition, see expression (2.10)) is solvable if there exists a control $\left(\Omega,\mathcal{F},\mathcal{P},W(\cdot),\bar{\varepsilon}^*(\cdot)\right)$ $\in U^\omega[0,T]$ such that

$$J\left(0,G;\bar{\varepsilon}^*(\cdot)\right)=\inf_{\bar{\varepsilon}(\cdot)\in U^\omega[0,T]}J\left(0,G;\bar{\varepsilon}(\cdot)\right)\stackrel{\Delta}{=}V(0,G). \tag{2.7}$$

We should stress that in the case where $\bar{\varepsilon}^*(\cdot)$, see expression (2.12), is an optimal control; the corresponding $\bar{\Pi}^*(\cdot)$ and $(\bar{\Pi}^*(\cdot),\bar{\varepsilon}^*(\cdot))$ are called an optimal state process and an optimal pair, respectively, to our problem. Finally, closing this section, we provide the optimal controller and the solution of the (non-homogeneous) linear stochastic differential equation. Analytically, suppose that $P(\cdot)\in C\left([0,T];\mathbb{R}^n\right)$ is symmetric and $\phi(\cdot)\in C\left([0,T];\mathbb{R}^n\right)$ are matrix-stochastic equations of the following form

$$\begin{cases}\dot{P}(t)+P(t)A(t)+A^T(t)P(t)-P(t)B(t)\left(R+\sum_{j=1}^n H_j^T(t)P(t)H_j(t)\right)^{-1}B(t)P(t)=0\\ P(T)=G,\quad a.e.\ t\in[0,T]\end{cases} \tag{2.8}$$

and

$$\begin{cases} \dot{\phi}(t) + \left\{ A(t) - B(t)\left(R + \sum_{j=1}^{n} H_j^T(t)P(t)H_j(t) \right)^{-1} B(t)P(t) \right\}^T \phi(t) + P(t)C(t) = 0 \\ \phi(T) = 0, \quad a.e.\ t \in [0,T] \end{cases}$$

(2.9)

The following three theorems are practically very important.

Theorem 2.1 (Yong and Zhou [26]) *Let* $P(\cdot) \in C([0,T];\mathbb{R}^n)$ *and* $\phi(\cdot) \in C([0,T];\mathbb{R}^n)$ *be the solution of (2.8) and (2.9) respectively, then*

$$\begin{cases} B\Psi(t), D\Psi(t) \in L^\infty(0,T;\mathbb{R}^{n\times n}), where \\ \Psi(t) \triangleq \left(R + \sum_{j=1}^{n} H_j^T(t)P(t)H_j(t) \right)^{-1} B^T(t)P(t) \end{cases} \tag{2.10}$$

and

$$\begin{cases} B\psi(t), D\psi(t) \in L^2(0,T;\mathbb{R}^n), where \\ \psi(t) \triangleq \left(R + \sum_{j=1}^{n} H_j^T(t)P(t)H_j(t) \right)^{-1} B^T(t)\phi(t). \end{cases} \tag{2.11}$$

The SLQ-problem (see expressions (2.5) and (2.6)) is solvable with the optimal control $\overline{\varepsilon}^*(\cdot)$ *being of a state feedback form*

$$\overline{\varepsilon}^*(t) = \overline{\varepsilon}_\tau - \left[\Psi(t)\left(\overline{\Pi}(t) - \overline{\Pi}_\tau\right) + \psi(t) \right], \ for\ t \in [0,T]. \ \square \tag{2.12}$$

To find the optimal control (2.12) for $t \in [0,T]$, we need the solution of the non linear differential matrix equation (2.10), $P(t)$, which is discussed extensively in the next section. Moreover, the solution of equation (2.9) has the following form

$$\phi(t) = \zeta(t,0)\phi_0 + \int_0^t \zeta(t,r)P(r)C(r)\,dr.$$

Now, we define

$$\tilde{A}(s) = \left[A(s) - B(s)\left(R + \sum_{j=1}^{n} H_j^T(s)P(s)H_j(s) \right)^{-1} B(s)P(s) \right]^T,$$

where the state transition matrix is given by the following expression, which is called the Peano-Baker series (see Antsaklis and Michel [3]).

$$\zeta(t,r) = I + \int_r^t \tilde{A}(s)\,ds + \int_r^t \tilde{A}(s_1)\int_r^{s_1} \tilde{A}(s_2)\,ds_2 ds_1 + \ldots + \int_r^t \tilde{A}(s_1)\ldots\int_r^{s_n} \tilde{A}(s_n)\,ds_n \ldots ds_1 + \ldots$$

(2.13)

Theorem 2.2 (see Yong and Zhou [26]) *For the time-varying matrices* $A(t)$, $C(t) \in L^{\infty}\left(0,T;\mathbb{R}^{n\times n}\right)$ *and* $b(t)$, $d(t) \in L^{\infty}\left(0,T;\mathbb{R}^{n}\right)$ *the linear equation*

$$\begin{cases} dX(t) = \left\{A(t)X(t) + b(t)\right\}dt - \sum_{j=1}^{n}\left[C_j(t)X(t) + d(t)\right]dW_j(t) \\ X(0) = X_0 \end{cases}, \tag{2.14}$$

and $\Phi(t)$ *is the solution of the following*

$$\begin{cases} d\Phi(t) = A^*(t)\Phi(t)dt - \sum_{j=1}^{n}C_j^*(t)\Phi(t)dW_j(t) \\ \Phi(0) = \mathrm{I} \end{cases} \tag{2.15}$$

then the strong solution of X *can be represented as*

$$\begin{aligned} X(t) = \Phi(t)X_0 + \Phi(t)\int_0^t \Phi(s)^{-1}\left[b^*(s) - \sum_{j=1}^{n}C_j^*(s)d_j^*(s)\right]ds \\ + \sum_{j=1}^{n}\Phi(t)\int_0^t \Phi(s)^{-1}d_j^*(s)dW_j(s), \end{aligned} \tag{2.16}$$

where

$$\begin{cases} d\left(\Phi(t)^{-1}\right) = \Phi(t)^{-1}\left[-A^*(t) + \sum_{j=1}^{n}\left(C_j^*(t)\right)^2\right]dt - \sum_{j=1}^{n}\Phi(t)^{-1}C_j^*(t)dW_j(t) \\ \Phi(0)^{-1} = \mathrm{I} \end{cases}. \tag{2.17}$$

□

Now, we can conclude the whole discussion of this section by presenting the following Theorem. Under the following result, we can determine our controller (2.12), i.e. the lending rate policy for the portfolio of *n* loans.

Theorem 2.3 *Finally, the surplus is given*

$$\begin{aligned} \bar{\Pi}(t) = \Phi(t)\bar{\Pi}_0 + \Phi(t)\int_0^t \Phi(s)^{-1}\left[b^*(s) - \sum_{j=1}^{n}C_j^*(s)d_j^*(s)\right]ds \\ + \sum_{j=1}^{n}\Phi(t)\int_0^t \Phi(s)^{-1}d_j^*(s)dW_j(s), \end{aligned} \tag{2.18}$$

where $\Phi(t), \Phi(t)^{-1}$ *are the solution of the stochastic equations (2.15) and (2.17) respectively for the following matrices,*

$$A^*(t) = A(t) - B(t)\Psi(t),\ b^*(t) = B(t)\left[\Psi(t)\bar{\Pi}_\tau - \psi(t)\right] + C(t),$$

$$C^*(t) = H_j(t)\Psi(t) \text{ and } d^*(t) = H_j(t)\left(\Psi(t)\bar{\Pi}_\tau - \psi(t)\right), \tag{2.19}$$

where

$$\begin{cases} d\Phi(t) = A^*(t)\Phi(t)\,dt - \sum_{j=1}^{n} C_j^*(t)\Phi(t)dW_j(t) \\ \Phi(0) = \mathrm{I} \end{cases} \tag{2.20}$$

and

$$\begin{cases} d\left(\Phi(t)^{-1}\right) = \Phi(t)^{-1}\left[-A^*(t) + \sum_{j=1}^{n}\left(C_j^*(t)\right)^2\right]dt - \sum_{j=1}^{n}\Phi(t)^{-1}C_j^*(t)\,dW_j(t) \\ \Phi(0)^{-1} = \mathrm{I} \end{cases}. \tag{2.21}$$

□

In the next section, we describe an analytic method for expressing the general solution of the nonlinear differential matrix equation $P(t)$.

3. The general solution of the nonlinear differential matrix equation $P(t)$.

In this section, we present the solution of the nonlinear differential matrix equation of $P(\cdot) \in C\left([0,T];\mathbb{R}^n\right)$ (note that $P(\cdot)$ is symmetric; this assumption is really very mild), see also Zimbidis, Pantelous and Kalogeropoulos [29]. We define

$$P(t) = \left(P_{ij}(t)\right)_{i,j=1,2,\ldots,n} \tag{3.1}$$

where $P_{ij}(t)$ are scalars t-continuous functions.

Moreover, in order to simplify our calculations (the full extension requires quite *cumbersome* calculations), we assume that the matrix $A(t) \in \mathbb{R}^{k\times k}$ is also symmetric, i.e. $\lambda_{ij}(t) = \lambda_{ji}(t)$, for $i \neq j$ and we also assume:

$a_i(t) = a(t)$, the same rate of return for the accumulated profit or loss at time t for the each sub-portfolio of loans, $i = 1, 2, ..., n$,

$\lambda_{ij}(t) = \lambda(t)$, the percentage of profit or loss transferred from the i sub-portfolio to j sub-portfolio of loans at time t, $\lambda_{ii}(t) = 1-(n-1)\lambda(t)$, and

$\rho_i(t) = \rho(t)$, the ratio of the total amount placed to the i loan sub-portfolio, $i = 1, 2, ..., n$, over the total amount of loans.

The expression of (2.8) can be rewritten as follows.

$$\dot{P}(t) + A(t)P(t) + P(t)A(t) - P(t)B(t)\left(R(t) + \sum_{j=1}^{n} H_j^T(t)P(t)H_j(t)\right)^{-1} B(t)P(t) = 0, \tag{3.2}$$

where the symmetric matrix $A(t)$ takes the following format

$$A(t)=\begin{bmatrix} a(t)\mathbf{1}_{\Pi_1(t)\in[0,\infty)}+\left(1-(n-1)\lambda(t)\right) & \lambda(t) & \cdots & \lambda(t) \\ \lambda(t) & a(t)\mathbf{1}_{\Pi_2(t)\in[0,\infty)}+\left(1-(n-1)\lambda(t)\right) & \cdots & \lambda(t) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda(t) & \lambda(t) & \cdots & a(t)\mathbf{1}_{\Pi_n(t)\in[0,\infty)}+\left(1-(n-1)\lambda(t)\right) \end{bmatrix}. \tag{3.3}$$

Now each element of (3.3) can be written as:

- $$A(t)P(t)+P(t)A(t)=\left(Q_{ij}(t)\right)_{ij=1,2,\ldots,n} \tag{3.4}$$

where, the above matrix is symmetric, i.e.

$$\left(A(t)P(t)+P(t)A(t)\right)^T=A(t)P(t)+P(t)A(t),$$

as $A(t)$, $P(t)$ are $n\times n$-symmetric matrices. Thus,

$$\underset{i\le j}{Q_{ij}}=2\left[a+\left(1-(k-1)\lambda\right)\right]P_{ij}+\lambda\left[\sum_{l=1}^{i}P_{li}+\sum_{l=i}^{k}P_{il}-P_{ij}-P_{ii}\right]+\lambda\left[\sum_{l=1}^{j}P_{lj}+\sum_{l=j}^{k}P_{jl}-P_{ij}-P_{jj}\right]. \tag{3.5}$$

Before we go further, we calculate the

$$B(t)\left(R+\sum_{j=1}^{n}H_j^T(t)P(t)H_j(t)\right)^{-1}B(t)$$

$$=\begin{bmatrix} \dfrac{\left(m_1(t)\rho(t)\Delta(t)\right)^2}{\beta+\left(\sigma_1(t)\rho(t)\Delta(t)\right)^2P_{11}(t)} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{\left(m_n(t)\rho(t)\Delta(t)\right)^2}{\beta+\left(\sigma_n(t)\rho(t)\Delta(t)\right)^2P_{nn}(t)} \end{bmatrix}. \tag{3.6}$$

- $$P(t)B(t)\left(R+\sum_{j=1}^{n}H_j^T(t)P(t)H_j(t)\right)^{-1}B(t)P(t)=\left(S_{ij}(t)\right)_{i,j=1,2,\ldots,n} \tag{3.7}$$

We easily prove that (3.7) is also symmetric, as P is symmetric and B, R and H_j, for $j=1,2,\ldots,n$ are diagonal matrices. Thus, for $i=1,2,\ldots,n$,

$$\underset{i\le j}{S_{ij}}=\begin{bmatrix} \sum_{l=1}^{i}\dfrac{(m_l\rho\Delta)^2}{\beta+(\sigma_l\rho\Delta)^2P_{ll}}P_{li}P_{lj}+\sum_{l=i}^{j}\dfrac{(m_l\rho\Delta)^2}{\beta+(\sigma_l\rho\Delta)^2P_{ll}}P_{il}P_{lj}+\sum_{l=j}^{k}\dfrac{(m_l\rho\Delta)^2}{\beta+(\sigma_l\rho\Delta)^2P_{ll}}P_{il}P_{jl} \\ -\dfrac{(m_i\rho\Delta)^2}{\beta+(\sigma_i\rho\Delta)^2P_{ii}}P_{ii}P_{ij}-\dfrac{(m_j\rho\Delta)^2}{\beta+(\sigma_j\rho\Delta)^2P_{jj}}P_{ij}P_{jj} \end{bmatrix}. \tag{3.8}$$

Substituting the above expressions to (3.2), we obtain the family of the following ordinary nonlinear differential equations:

$$\dot{P}_{ij}+2\left[a+\left(1-\left(k-1\right)\lambda\right)\right]P_{ij}+\lambda\left[\sum_{l=1}^{i}P_{li}+\sum_{l=i}^{k}P_{il}-P_{ij}-P_{ii}\right]+\lambda\left[\sum_{l=1}^{j}P_{lj}+\sum_{l=j}^{k}P_{jl}-P_{ij}-P_{jj}\right]$$
$$+\sum_{l=1}^{i}\frac{\left(m_l\rho\Delta\right)^2}{\beta+\left(\sigma_l\rho\Delta\right)^2P_{ll}}P_{li}P_{lj}+\sum_{l=i}^{j}\frac{\left(m_l\rho\Delta\right)^2}{\beta+\left(\sigma_l\rho\Delta\right)^2P_{ll}}P_{il}P_{lj}+\sum_{l=j}^{k}\frac{\left(m_l\rho\Delta\right)^2}{\beta+\left(\sigma_l\rho\Delta\right)^2P_{ll}}P_{il}P_{jl}$$
$$-\frac{\left(m_i\rho\Delta\right)^2}{\beta+\left(\sigma_i\rho\Delta\right)^2P_{ii}}P_{ii}P_{ij}-\frac{\left(m_j\rho\Delta\right)^2}{\beta+\left(\sigma_j\rho\Delta\right)^2P_{jj}}P_{ij}P_{jj}=0. \tag{3.9}$$

With the expression (3.9), we succeed in transferring the non homogeneous matrix (Riccati) differential equation (2.8), into a Cauchy problem for a system of first order differential equations, where $P\left(T\right)=G,\ a.e.\ t\in\left[0,T\right]$.

Consider the Cauchy problem of the first-order differential equation:

$$\dot{P}_k = f_k\left(t,P_{ij}\right),\ \text{for}\ i\le j,\ i,j,k=1,2,...,n \tag{3.10}$$

or equivalently,

$$\dot{P} = f\left(t,P\right) \tag{3.11}$$

where

$$P=\left(P_{11},P_{12},...,P_{1n},...,P_{ij},...,P_{nn}\right)$$

and

$$f\left(t,P_{11},P_{12},...,P_{1n},...,P_{ij},...,P_{nn}\right)$$
$$=\left(f_1\left(t,P_{11},P_{12},...,P_{1n},...,P_{ij},...,P_{nn}\right),...,f_n\left(t,P_{11},P_{12},...,P_{1n},...,P_{ij},...,P_{nn}\right)\right) \tag{3.12}$$

with the initial condition, after a change of variable,

$$P\left(t\right)=P\left(T-t\right) \tag{3.13}$$

so, $$P_0=P\left(0\right)=G,\qquad a.e.\ t\in\left[0,T\right]$$
(3.14)

where $G=\left(1-\beta\right)I_n$, β is a weighting factor i.e. $0\le\beta\le1$.

The method of successive approximations obtains the solution $P\left(T-t\right)$ as the limit of a sequence of functions $P^{(k)}\left(T-t\right)$ which are determined by the recurrence formula.

$$P^{(k)}\left(T-t\right)=P^{(0)}+\int_{T-t}^{T}f\left(r,P^{(k-1)}\left(T-r\right)\right)dr \tag{3.15}$$

It has been shown by Petrovsky [16] that, if the right-hand term in the domain $Q\in\mathbb{R}^{k+1}\left\{\left|t\right|\le k_1,\left|P-P_0\right|\le k_2\right\}$ satisfies the Lipschitz condition with respect to P

$$\left|f\left(t,P^{(1)}\right)-f\left(t,P^{(2)}\right)\right|\le K\left|P^{(1)}-P^{(2)}\right|,\ K>0 \tag{3.16}$$

then, irrespective of the choice of the initial function, the consecutive approximations $P^{(k)}(T-t)$ converge on some interval $[0,h]$ to the solution of this Cauchy problem.

Moreover, if $f(t,P)$ is continuous in a rectangle $Q\in\mathbb{R}^{k+1}\{|t|\le k_1, |P-P_0|\le k_2\}$, then the error of the approximate solution $P^{(k)}(T-t)$ on the interval $[0,h]$ is estimated by the inequality:

$$\varepsilon_k = \left|P(T-t)-P^{(k)}(T-t)\right| \le MK^k \frac{(T-t)^{k+1}}{(k+1)!}, \tag{3.17}$$

where $M = \max\limits_{(t,P)\in\mathbb{R}^{k+1}} |f(t,P)|$ and h is determined by $h=\min\left(k_1,\dfrac{k_2}{M}\right)$.

4. An numerical application for a portfolio of two loans

To derive the loan rate by the method proposed in section 2, we consider the two dimensional stochastic system, i.e. a portfolio composed by two sub-portfolios of loans indexed 1 and 2, see **figure 3**.

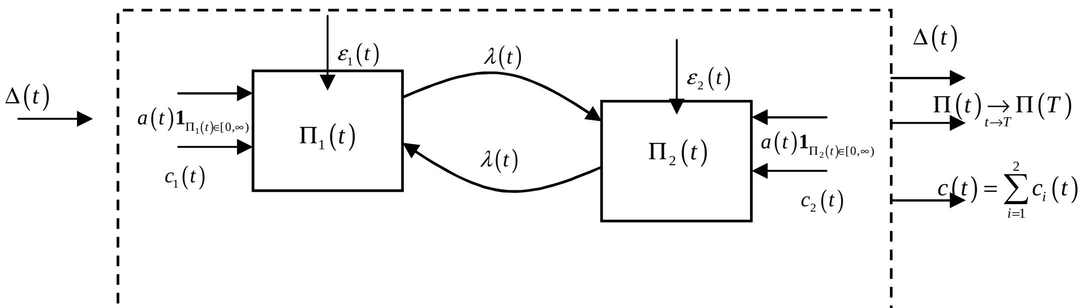

Figure 3 The described process for portfolio of *2* different loans.

Thus, we obtain the following expression,

$$\begin{cases} d\bar{\Pi}(t)=\{A(t)\bar{\Pi}(t)+B(t)\bar{\varepsilon}(t)+C(t)\}dt+\sum\limits_{j=1}^{2}H_j(t)\bar{\varepsilon}(t)dW_j(t) \\ \bar{\Pi}(0)=\Pi_0 \end{cases} \tag{4.1}$$

where

$$A(t)=\begin{bmatrix} a(t)\mathbf{1}_{\Pi_1(t)\in[0,\infty)}+1-\lambda(t) & \lambda(t) \\ \lambda(t) & a(t)\mathbf{1}_{\Pi_2(t)\in[0,\infty)}+1-\lambda(t) \end{bmatrix},$$

$$\bar{\Pi}(t)=\begin{bmatrix}\Pi_1(t)\\ \Pi_2(t)\end{bmatrix},\quad \bar{\varepsilon}(t)=\begin{bmatrix}\varepsilon_1(t)\\ \varepsilon_2(t)\end{bmatrix},$$

$$B(t)=diag\{m_1(t);m_2(t)\}\rho(t)\Delta(t),$$

$$C(t)=-\begin{bmatrix}c_1(t)\\ c_2(t)\end{bmatrix}\rho(t)\Delta(t) \text{ and } H_j(t)=\begin{bmatrix}0 & 0\\ 0 & \sigma_j(t)\end{bmatrix}\rho(t)\Delta(t).$$

We define $P(t)$ as follows:

$$P(t) = \left(P_{ij}(t)\right)_{i,j=1,2} \tag{4.2}$$

where $P_{ij}(t),\ i \le j,\ i, j = 1,2$ are scalars continuous functions and

$$P(T) = G \Leftrightarrow P_{11}(T) = P_{22}(T) = 1-\beta \text{ and } P_{12}(T) = 0\,. \tag{4.3}$$

Although the parameters, such as $a(t),\ \lambda(t)$ etc, are time variant functions of t, for the simplicity of calculations, we define $a(t) = a$ and $\lambda(t) = \lambda$ to be constant, $a,\ \lambda > 0$. Moreover, we obtain the ratio of the total amount placed to the i^{th} sub-portfolio of loans, for $i = 1, 2, ..., n$, over the total amount of loans, $\rho(t)$ and the total amount for the whole portfolio of loans at time t, $\Delta(t)$ to be also constants and equal to ρ and Δ, respectively. Additionally, we compute the drift and the volatility of the percentage for the two sub-portfolios of loans, $m_1(t) = m_1$, $m_2(t) = m_2$ and $\sigma_1(t) = \sigma_1$, $\sigma_2(t) = \sigma_2$ respectively.

Consequently, the basic parameters *α*, *λ*, *ρ* and *Δ* set out in the following tables and the other subsidiary variables m_1, m_2, σ_1, σ_2, c_1, c_2, T, β also shown as below (**Table B**).

Table B							
Application parameters							
α	5%		ρ	0.5		T	2
λ	0,2		Δ	1		c1	0,035
m1	0,85		σ1	0,055		c2	0.030
m2	0.90		σ2	0,045		β	0.5

The matrix $P(\cdot)$ (see Appendix **A**) has the following elements

$$P_{11}(t) = 0.0570 + 1.7945t - 1.2009t^2\,,$$

$$P_{12}(t) = -0.0828 + 0.8400t - 0.5981t^2 \text{ and } P_{22}(t) = 0.2348 + 1.6300t - 1.1602t^2\,. \tag{4.4}$$

For convenience, the other coefficients are equivalently small, for instance

$$k_{11,4} = 0.1345 << k_{11,3}\,, \text{ and } k_{11,5} = -0.00012 << k_{11,3}\,.$$

Moreover,

$$\zeta(t,0) = I + \Lambda_1 t + \Lambda_2 t^2 + \Lambda_3 t^3\,,$$

where the desirable *capital cost* is $c_1 = 0.03$ and $c_2 = 0.02$, respectively

$$\Lambda_1 = \begin{bmatrix} 0.8294 & 0.2335 \\ 0.2299 & 0.7549 \end{bmatrix},\ \Lambda_2 = \begin{bmatrix} -0.3241 & -0.1698 \\ -0.1518 & -0.3299 \end{bmatrix} \text{ and } \Lambda_2 = \begin{bmatrix} 0.1452 & 0.0809 \\ 0.0723 & 0.1569 \end{bmatrix}. \tag{4.5}$$

Now, the matrix equation $\phi(\cdot)$ is given by the expression

$$\phi(t)=\left[I+\Lambda_1 t+\Lambda_2 t^2+\Lambda_3 t^3\right]\left\{\phi_0-\begin{bmatrix}-0.00024t+0.0440t^2-0.0300t^3\\ 0.0021t+0.0391t^2-0.0279t^3\end{bmatrix}\right\}$$
$$+\begin{bmatrix}-0.00024t+0.0443t^2+0.0154t^3-0.0522t^4-0.0523t^5-0.0066t^6\\ 0.0021t+0.0407t^2+0.0112t^3-0.04724t^4-0.0475t^5-0.0065t^6\end{bmatrix} \tag{4.6}$$

and

$$\phi_0=\begin{bmatrix}0.0563\\0.0537\end{bmatrix}. \tag{4.7}$$

Using the expressions (2.8) and (2.12), the optimal control $\bar{\varepsilon}^*(\cdot)$ (being of a state feedback form) has the following format, where $S.R_1=\Pi_1/\Pi_{\tau 1}$ and $S.R_2=\Pi_2/\Pi_{\tau 2}$ is the solvency ratio for the 1st and the 2nd sub-portfolio of loans, respectively. Note that $\Pi_{\tau 1}$ and $\Pi_{\tau 2}$ are the required (target) surplus for the 1st and the 2nd sub-portfolio, respectively.

$$\bar{\varepsilon}^*(t)=\bar{\varepsilon}_\tau-\Psi(t)\begin{bmatrix}\Pi_{\tau 1} & 0\\ 0 & \Pi_{\tau 2}\end{bmatrix}\begin{bmatrix}\dfrac{\Pi_1}{\Pi_{\tau 1}}-1\\ \dfrac{\Pi_2}{\Pi_{\tau 2}}-1\end{bmatrix}-\psi(t), \tag{4.8}$$

$\bar{\varepsilon}_\tau=\begin{bmatrix}0.055\\0.045\end{bmatrix}$ is the base lending rate for each sub-portfolio of loans, respectively.

Remark In order to make more insightful implementations, we point out two significant parameters: a) the *capital cost* (including expenses, operational cost, rate of return paid to customers due to bank deposits and the desirable profit for the bank etc) is below the earned from (risk-free) investments interest rate, i.e. a (see **Table B**) at time t for both sub-portfolios of loans. Consequently, since the sub-portfolios are profitable, there is an opportunity of further diminishing of lending rate policy and b) the borrowers are not consistent, but not very heavily, with their repayments, as it is also clear form **Table B**. Moreover, the first sub-portfolio is most vulnerable to changes, in all the following case studies. Analytically,

- For surpluses $\Pi_{\tau 1}=0.01$ and $\Pi_{\tau 2}=0.02$, and solvency ratio 0.9 (i.e. the actual surplus is $\Pi_1=0.009<0.01$) and 1.1 ($\Pi_1=0.022>0.02$) for each sub-portfolio of loans, respectively (**figure 4 (a), (b)**).

First Case Study: Both sub-portfolios of loans have positive solvency ratios and target surpluses. For the 1st sub-portfolio of loans, the solvency ratio is below 1 and consequently the actual surplus is below the target, i.e. $\Pi_1<\Pi_{\tau 1}$. On the other hand, the 2nd sub-portfolio is more profitable, since the solvency ratio is above 1 and its actual surplus is greater than the target one, i.e. $\Pi_2<\Pi_{\tau 2}$. Now, considering also the above

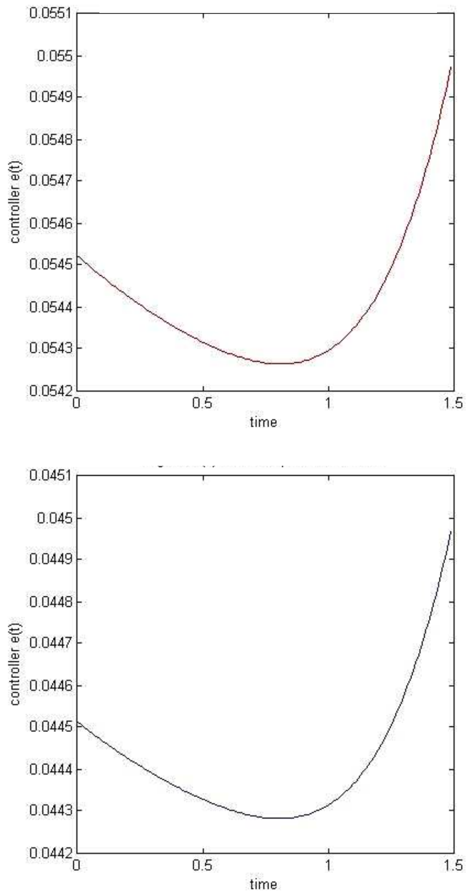

Figure 4 (a), (b): For surpluses $\Pi_{\tau 1} = 0.01$ and $\Pi_{\tau 2} = 0.02$, and solvency ratio 0.9 and 1.1 for each sub-portfolio of loans, respectively.

Remark, the financial institution managers may decrease further the lending rates for both of the sub-portfolios of loans at the beginning of the time-period. However, as the sight horizon obtain the above limit of 1.5 time-unit (or three time-unit periods), both rates tend to follow the same increasing pattern, as figure **4 (a), (b)** clearly show. Furthermore, since the profit margin between the lending rate and the capital cost is smaller for the 2nd sub-portfolio of loans than the 1st one, it is obtained that the lending rate policy of the 2nd is more conservative, see figure **4 (a), (b)**.

- For surpluses $\Pi_{\tau 1} = 0.03$ and $\Pi_{\tau 2} = 0.01$, and solvency ratio -0.4 (i.e. the actual surplus is $\Pi_1 = -0.012$ and $\Pi_2 = -0.004$) for both sub-portfolio of loans, respectively (**figure 5 (a), (b)**).

Second Case Study: Both sub-portfolios of loans have negative solvency ratios and positive target surpluses. For the 1st sub-portfolio of loans, the solvency ratio is negative and consequently, the actual surplus is far below the target, i.e. $\Pi_1 < 0 < \Pi_{\tau 1}$ and $|\Pi_1 - \Pi_{\tau 1}| = 0.042$ (in absolute values). Simultaneously, the 2nd sub-portfolio is less non-profitable, since the actual surplus is quite close to the target, i.e. $|\Pi_2 - \Pi_{\tau 2}| = 0.014$ (in absolute values). Considering also the above **Remark**, the

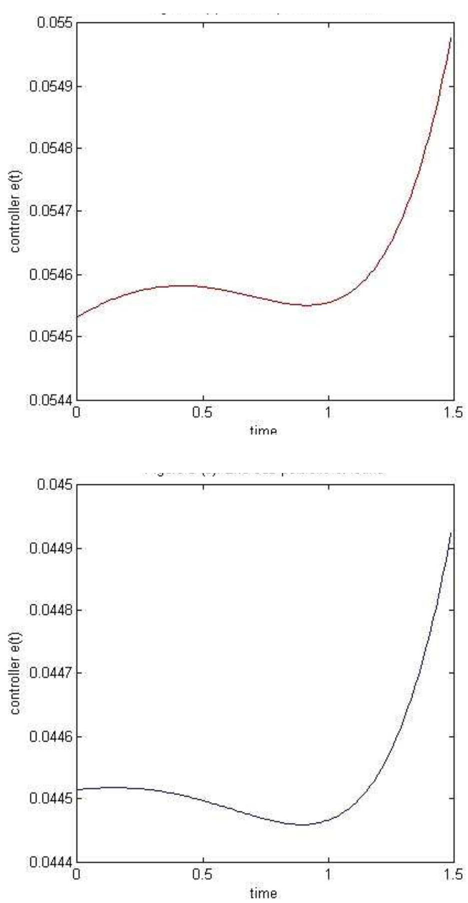

Figure 5 (a), (b): For surpluses $\Pi_{\tau 1} = 0.03$ and $\Pi_{\tau 2} = 0.01$, and solvency ratio -0.4 f for each sub-portfolio of loans, respectively.

financial institution managers may decrease further the lending rate policy for both of the sub-portfolios of loans at the beginning of the time-period. Afterwards, the lending rate policy follows an increasing pattern for both of the sub-portfolios, as a coincidence of the desired profitable target surpluses; see figure **5 (a), (b)**. Note that the lending rate policy is milder for the 2nd sub-portfolio that the other one. This is a result of both the fact that the borrowers are not consistent, but not very heavily, with their repayments and that $|\Pi_2 - \Pi_{\tau 2}| < |\Pi_1 - \Pi_{\tau 1}|$. Furthermore, as the sight horizon obtain the above limit of 1.5 time-unit (or three time-unit periods), both rates tend to follow the same pattern, as figure **5 (a), (b)** clearly show.

- For surpluses $\Pi_{\tau 1} = 0.03$ and $\Pi_{\tau 2} = 0.01$, and solvency ratio 2.0 and 1 for each sub-portfolio of loans, respectively (**figure 6 (a), (b)**).

Third Case Study: Both sub-portfolios of loans have positive solvency ratios and positive target surpluses. For the 1st sub-portfolio of loans, the solvency ratio is far above 1 and consequently the actual surplus is greater that the target, i.e. $\Pi_1 > \Pi_{\tau 1}$. On the other hand, the 2nd sub-portfolio is less profitable, since the solvency ratio is 1 and its actual surplus is equal to the target one, i.e. $\Pi_2 = \Pi_{\tau 2}$.

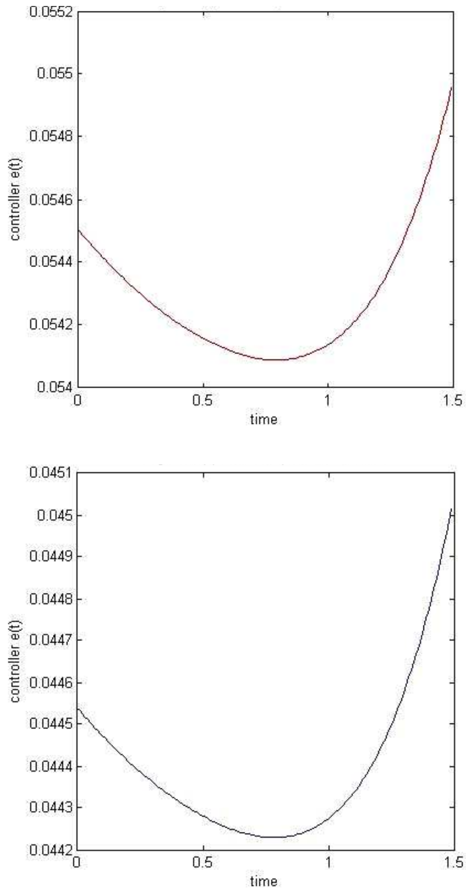

Figure 6 (a), (b): For surpluses $\Pi_{\tau 1} = 0.03$ and $\Pi_{\tau 2} = 0.01$, and solvency ratio 2.0 and 0.8 for each sub-portfolio of loans, respectively.

In this last interesting case study, by considering also the above **Remark**, the financial institution managers may decrease more highly the lending rates for both of the sub-portfolios of loans in order to reward their customers and possibly attract them for other more profitable business at the beginning of the time-period. However, as in the first case study where the solvency ratios and the surpluses are also positive, the sight horizon obtain the above limit of 1.5 time-unit (or three time-unit periods), both rates tend to follow the same increasing pattern; see figure **6 (a), (b)**. Furthermore, since the solvency ratio for the 1st sub-portfolio is higher than the 2nd one, their customers may be rewarded more by the lending rate policy.

5. Conclusions

The paper provides a theoretical model for the lending rate policy using a stochastic dynamic framework. The assumption that the repayment pattern (i.e. the proportion of persons who properly repay their loans) follows a Brownian motion also upgrades the realism of the model.

At the end, the full model is proved to be quite complicated but using advanced optimization techniques of stochastic control theory we manage to obtain the solution of the stochastic differential equations in closed form. The solution is actually an automatic controller which determines the level of lending rate policy for each sub-portfolio of loans. Then standard approximation procedures (as the method of successive approximations of Picard) are employed in order to obtain analytical solution in open form.

Furthermore, by applying the optimal controller in a certain banking system with two sub-portfolios of loans we gain some insightful experience, by answering the question of whether the banks should provide cheap loans, (i.e. with smaller lending rates), in order to attract more customers for other profitable business or not. It is evident from the numerical example that solvency interaction smooth out the lending rates and the respective results.

REFERENCES

[1] Allen, L., 1988. The determinants of bank interest margins: a note. *The Journal of Financial and Quantitative Analysis* Vol. 23, pp. 231-235.

[2] Angbazo, L., 1996. Commercial bank net interest margins, default risk, interest-rate risk, and off-balance sheet banking. *Journal of Banking and Finance* Vol. 21, pp. 55-87.

[3] Antsaklis, P.J. and Michel, A.N., 1996. *Linear systems*, The McGraw-Hill Companies, INC. USA.

[4] Chen, S. and Yong, J., 2001. Stochastic Linear Quadratic Optimal Control Problems. *Applied Mathematics and Optimization* Vol. 43, pp. 21-45.

[5] Davis, M.H.A., 1977. *Linear Estimation and Stochastic Control*, Chapman and Hall, London.

[6] Fried, J. and Howitt, P., 1980. Credit rationing and implicit contract theory. *Journal of Money, Credit, and Banking* Vol. 12, pp. 471-487.

[7] Ho, T.S.Y. and Saunders, A., 1981. The determinants of bank interest margins: theory and empirical evidence. *The Journal of Financial and Quantitative Analysis* Vol. 16, pp. 581-600.

[8] Holly, S. and Hallett, H.A., 1989. *Optimal Control, Expectations and Uncertainty*. Cambridge University Press, UK.

[9] Ichikawa, A., 1979. Dynamic programming approach to stochastic evolution equation. *SIAM Journal of Control Optimization* Vol. 17, pp. 152-174.

[10] Jobst, N.J., Mitra, G., Zenios, S.A., 2006. Integrating market and credit risk: a simulation and optimization perspective. *Journal of Banking and Finance* Vol. 30, pp. 717-742.

[11] Allen, L., 1988. The determinants of bank interest margins: a note. *The Journal of Financial and Quantitative Analysis* Vol. 23, pp. 231-235.

[12] Angbazo, L., 1996. Commercial bank net interest margins, default risk, interest-rate risk, and off-balance sheet banking. *Journal of Banking and Finance* Vol. 21, pp. 55-87.

[13] Antsaklis, P.J. and Michel, A.N., 1996. *Linear systems*, The McGraw-Hill Companies, INC. USA.

[14] Chen, S. and Yong, J., 2001. Stochastic Linear Quadratic Optimal Control Problems. *Applied Mathematics and Optimization* Vol. 43, pp. 21-45.

[15] Davis, M.H.A., 1977. *Linear Estimation and Stochastic Control*, Chapman and Hall, London.

[16] Fried, J. and Howitt, P., 1980. Credit rationing and implicit contract theory. *Journal of Money, Credit, and Banking* Vol. 12, pp. 471-487.

[17] Ho, T.S.Y. and Saunders, A., 1981. The determinants of bank interest margins: theory and empirical evidence. *The Journal of Financial and Quantitative Analysis* Vol. 16, pp. 581-600.

[18] Holly, S. and Hallett, H.A., 1989. *Optimal Control, Expectations and Uncertainty*. Cambridge University Press, UK.

[19] Ichikawa, A., 1979. Dynamic programming approach to stochastic evolution equation. *SIAM Journal of Control Optimization* Vol. 17, pp. 152-174.

[20] Jobst, N.J., Mitra, G., Zenios, S.A., 2006. Integrating market and credit risk: a simulation and optimization perspective. *Journal of Banking and Finance* Vol. 30, pp. 717-742.

[21] Machauer, A. and Weber, M., 1998. *Bank behavior based on internal credit ratings of borrowers,* CFS Working Paper No. 98/08.

[22] McLane, P.J., 1971. Optimal stochastic control of linear systems with state- and control-dependent disturbances. *IEEE Transaction on Automatic Control* Vol. 16, pp. 793-798.

[23] Nakamura, S., Qian, C., Samdoh, H. and Nakagawa, T., 2001. Determination of loan interest rate considering bankruptcy and mortgage collection costs. *International Transactions in Operational Research* Vol. 9, pp. 695-701.

[24] Pantelous, A.A., 2008. Dynamic risk management of the lending rate policy of an interacted portfolio of loans via an investment strategy into a discrete stochastic framework. *Economic Modelling* Vol. 25 (4), pp. 658-675.

[25] Petersen, M.A. and Rajan, R.G., 1995. The effect of credit market competition on lending relationships, *Quarterly Journal of Economics* Vol. 110, pp. 407-443.

[26] Petrovsky, I.G., 1970. *Lectures on theory of ordinary differential equations*, “Nauka” Moskow.

[27] Saunders, A., 1997. *Financial institutions management: a modern perspective*, 2nd edition, Irwin, Chicago, USA.

[28] Sealey, C.W. Jr., 1980. Deposit rate-setting, risk aversion, and the theory of depository financial intermediaries. *The Journal of Finance* Vol. 35, pp. 1139-1154.

[29] Slovin, M.B. and Sushka, M.E., 1983. A model of the commercial loan rate. *The Journal of Finance* Vol. 38, pp. 1583-1596.

[30] Stanhouse, B. and Stock, D., 2004. The impact of loan prepayment risk and deposit withdrawal risk on the optimal intermediation margin. *Journal of Banking and Finance* Vol. 28, pp. 1825-1843.

[31] Stein, R.M., 2005. The relationship between default prediction and lending profits: Integrating ROC analysis and loan pricing. *Journal of Banking and Finance* Vol. 29, pp. 1213-1236.

[32] Topaloglou, N., Vladimirou, H. and Zenios, S.A. (2008) A dynamic stochastic programming model for international portfolio management. *European Journal of Operational Research* Vol. 185 (3), pp. 1501-1524.

[33] Tustin, A., 1953. *The Mechanism of Economic Systems: an Approach to the Problem of Economic Stabilisation from the Point of View of Control – System Engineering*, Harvard University Press, Cambridge, Massachusetts, U.S.A..

[34] Wonham, W.M., 1968. On a matrix Riccati equation of stochastic control. *SIAM Journal of Control* Vol. 6, pp. 312-326.

[35] Wong, K.P., 1996. On the determinants of bank interest margins under credit and interest rate risks, *Journal of Banking and Finance* Vol. 21, pp. 251-271.

[36] Yong, J. and Zhou, X.Y. 1999. *Stochastic controls: Hamiltonian Systems and HJB Equations*, Springer. USA.

[37] Zarruk, E.R., 1989. Bank spread with uncertain deposit level and risk aversion, *Journal of Banking and Finance* Vol. 13, pp. 797-810.

[38] Zarruk, E.R. and Madura, J., 1992. Optimal bank interest margin under capital regulation and deposit insurance. *The journal of Financial and Quantitative Analysis* Vol. 22, pp. 143-148.

[39] Zimbidis, A.A, Pantelous, A.A. and Kalogeropoulos, G.I, 2007, Stochastic control of interest rate policy and solvency interaction within a mixed portfolio of loans, *Bulletin of the Greek Mathematical Society*, Vol. 54, pp. 115-126.

Appendix A

In this appendix, some significant details for the numerical calculations of the application of two sub-portfolios of loans are discussed. However, before we go further, we should express

$$\frac{(m_k\rho\Delta)^2}{\beta+(\sigma_k\rho\Delta)^2 P_{kk}} = \frac{(m_k\rho\Delta)^2}{\beta}\sum_{s=0}^{\infty}(-1)^s\left(\frac{(\sigma_k\rho\Delta)^2}{\beta}\right)^s (P_{kk})^s \approx \frac{(m_k\rho\Delta)^2}{\beta}\left\{1-\frac{(\sigma_k\rho\Delta)^2}{\beta}P_{kk}\right\} \quad \text{(A.1)}$$

where, $k=1,2$ and $\frac{\sigma_k}{\beta} << 1$.

By the expression (4.2), we obtain the following system of nonlinear first order ordinary differential equations.

$$\dot{P}_{11}(T-t) = -2\left[a+(1-\lambda)\right]P_{11}(T-t) - 2\lambda P_{12}(T-t) \underbrace{-\frac{(m_1\rho\Delta)^2}{\beta}\left\{P_{11}^2(T-t)-\frac{(\sigma_1\rho\Delta)^2}{\beta}P_{11}^3(T-t)\right\}-\frac{(m_2\rho\Delta)^2}{\beta}\left\{1-\frac{(\sigma_2\rho\Delta)^2}{\beta}P_{22}(T-t)\right\}P_{12}^2(T-t)}_{f_1(t,P)}, \quad \text{(A.2)}$$

$$\dot{P}_{22}(T-t) = -2\left[a+(1-\lambda)\right]P_{22}(T-t) - 2\lambda P_{12}(T-t) \underbrace{-\frac{(m_2\rho\Delta)^2}{\beta}\left\{P_{22}^2(T-t)-\frac{(\sigma_2\rho\Delta)^2}{\beta}P_{22}^3(T-t)\right\}-\frac{(m_1\rho\Delta)^2}{\beta}\left\{1-\frac{(\sigma_1\rho\Delta)^2}{\beta}P_{11}(T-t)\right\}P_{12}^2(T-t)}_{f_2(t,P)}, \quad \text{(A.3)}$$

$$\dot{P}_{12}(T-t) = -2\left[a+(1-\lambda)\right]P_{12}(T-t) - \lambda\left(P_{11}(T-t)+P_{22}(T-t)\right) \underbrace{-\frac{(m_1\rho\Delta)^2}{\beta}\left\{P_{11}(T-t)-\frac{(\sigma_1\rho\Delta)^2}{\beta}P_{11}^2(T-t)\right\}P_{12}(t)-\frac{(m_2\rho\Delta)^2}{\beta}\left\{P_{22}(t-t)-\frac{(\sigma_2\rho\Delta)^2}{\beta}P_{22}^2(T-t)\right\}P_{12}(t)}_{f_3(t,P)}. \quad \text{(A.4)}$$

The method of successive approximations of Picard has the following general form

$$P_k^{(n)}(T-t) = P_k^{(0)} + \int_{T-t}^{T} f_l\left(r, P^{(n-1)}(T-r)\right)dr,$$

for $k=11,\ 12,\ 22$ and $l=1,2,3$, respectively.

In what it follows, we assume that our time horizon is short, for instance $T<3$ (which is quite acceptable as the financial conditions change dramatically fast). We make this is assumption only for calculation's significance, since the next step of calculations is more demanding, and with that restriction we may consider the functions $P_{11}(T-t)$, $P_{12}(T-t)$ and $P_{22}(T-t)$ as 2nd order polynomials. Longer time period is regarded to higher order polynomials. Consequently, after some calculation, the analytical formulae for the 2nd step are provided, in the next lines. Thus,

$$P_{11}^{(2)}(t) = \kappa_{11,1}^{(2)} + \kappa_{11,2}^{(2)}t + \kappa_{11,3}^{(2)}t^2, \quad \text{(A.5)}$$

where the coefficients are provided as follows

$$\boldsymbol{k}_{\boldsymbol{11,1}}^{(2)} = P_{11}^{(0)} - \hat{P}_{11,1}T - \hat{P}_{11,2}T^2 - \hat{P}_{11,3}T^3 - \hat{P}_{11,4}T^4, \quad \boldsymbol{k}_{\boldsymbol{11,2}}^{(2)} = \hat{P}_{11,1} + 2\hat{P}_{11,2}T + 3\hat{P}_{11,3}T^2 + 4\hat{P}_{11,4}T^3,$$

$$\text{and} \quad \boldsymbol{k}_{\boldsymbol{11,3}}^{(2)} = -\left\{\hat{P}_{11,2} + 3\hat{P}_{11,3}T + 6\hat{P}_{11,4}T^2\right\}.$$

Moreover, we calculate the other two parameters of matrix $P(\cdot)$

$$P_{22}^{(2)}(t) = \kappa_{22,1}^{(2)} + \kappa_{22,2}^{(2)}t + \kappa_{22,3}^{(2)}t^2 \tag{A.6}$$

$$\boldsymbol{k}_{\boldsymbol{22,1}}^{(2)} = P_{22}^{(0)} - \hat{P}_{22,1}T - \hat{P}_{22,2}T^2 - \hat{P}_{22,3}T^3 - \hat{P}_{22,4}T^4, \quad \boldsymbol{k}_{\boldsymbol{22,2}}^{(2)} = \hat{P}_{22,1} + 2\hat{P}_{22,2}T + 3\hat{P}_{22,3}T^2 + 4\hat{P}_{22,4}T^3,$$

$$\text{and} \quad \boldsymbol{k}_{\boldsymbol{22,3}}^{(2)} = -\left\{\hat{P}_{22,2} + 3\hat{P}_{22,3}T + 6\hat{P}_{22,4}T^2\right\}.$$

Finally $$P_{12}^{(2)}(t) = \kappa_{12,1}^{(2)} + \kappa_{12,2}^{(2)}t + \kappa_{12,3}^{(2)}t^2,$$
(A.7)

where the coefficients are provided as follows

$$\boldsymbol{k}_{\boldsymbol{12,1}}^{(2)} = -\hat{P}_{12,1}T - \hat{P}_{12,2}T^2 - \hat{P}_{12,3}T^3 - \hat{P}_{12,4}T^4, \quad \boldsymbol{k}_{\boldsymbol{12,2}}^{(2)} = \hat{P}_{12,1} + 2\hat{P}_{12,2}T + 3\hat{P}_{12,3}T^2 + 4\hat{P}_{12,4}T^3,$$

$$\text{and} \quad \boldsymbol{k}_{\boldsymbol{12,3}}^{(2)} = -\left\{\hat{P}_{12,2} + 3\hat{P}_{12,3}T + 6\hat{P}_{12,4}T^2\right\}.$$

Now, the solution of equation (2.12) is provided by using also the successive approximation method of Picard, where

$$\tilde{A}(s) = \left[A(s) - B(s)\left(R + \sum_{j=1}^{n} H_j^T(s)P(s)H_j(s)\right)^{-1} B(s)P(s)\right]^T.$$

Suppose that $$\zeta(t,r) \approx I + \int_r^t \tilde{A}(s)\,ds,$$

(A.8)

where $\tilde{A}(t)$ is given by (A.9)

$$\left[A - B\left(R + \sum_{j=1}^{n} H_j^T P(s) H_j\right)^{-1} BP(s)\right]^T$$

$$= \begin{bmatrix} a+1-\lambda - \dfrac{(m_1\rho\Delta)^2 P_{11}(s)}{\beta + (\sigma_1\rho\Delta)^2 P_{11}(s)} & \lambda - \dfrac{(m_2\rho\Delta)^2 P_{12}(s)}{\beta + (\sigma_2\rho\Delta)^2 P_{22}(s)} \\ \lambda - \dfrac{(m_1\rho\Delta)^2 P_{12}(s)}{\beta + (\sigma_1\rho\Delta)^2 P_{11}(s)} & a+1-\lambda - \dfrac{(m_2\rho\Delta)^2 P_{22}(s)}{\beta + (\sigma_2\rho\Delta)^2 P_{22}(s)} \end{bmatrix}. \tag{A.9}$$

Thus, after some algebraic calculations, we obtain

$$\zeta(t,r) = I + \Lambda_1(t-r) + \Lambda_2\left(t^2 - r^2\right) + \Lambda_3\left(t^3 - r^3\right) \tag{A.10}$$

where Λ_1, Λ_2 and Λ_3 are constant matrices as follows

$$\Lambda_1 \triangleq \begin{bmatrix} \Lambda_{11}^1 & \Lambda_{12}^1 \\ \Lambda_{21}^1 & \Lambda_{22}^1 \end{bmatrix} = \begin{bmatrix} (a+1-\lambda) - \frac{(m_1\rho\Delta)^2}{\beta}\tilde{a}_1 & \lambda - \frac{(m_2\rho\Delta)^2}{\beta}\tilde{d}_1 \\ \lambda - \frac{(m_1\rho\Delta)^2}{\beta}\tilde{c}_1 & (a+1-\lambda) - \frac{(m_2\rho\Delta)^2}{\beta}\tilde{b}_1 \end{bmatrix},$$

$$\Lambda_2 \triangleq \begin{bmatrix} \Lambda_{11}^2 & \Lambda_{12}^2 \\ \Lambda_{21}^2 & \Lambda_{22}^2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} \frac{(m_1\rho\Delta)^2}{\beta}\tilde{a}_2 & \frac{(m_2\rho\Delta)^2}{\beta}\tilde{d}_2 \\ \frac{(m_1\rho\Delta)^2}{\beta}\tilde{c}_2 & \frac{(m_2\rho\Delta)^2}{\beta}\tilde{b}_2 \end{bmatrix},$$

and

$$\Lambda_3 = \begin{bmatrix} \Lambda_{11}^3 & \Lambda_{12}^3 \\ \Lambda_{21}^3 & \Lambda_{22}^3 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} \frac{(m_1\rho\Delta)^2}{\beta}\tilde{a}_3 & \frac{(m_2\rho\Delta)^2}{\beta}\tilde{d}_3 \\ \frac{(m_1\rho\Delta)^2}{\beta}\tilde{c}_3 & \frac{(m_2\rho\Delta)^2}{\beta}\tilde{b}_3 \end{bmatrix},$$

while

$$\zeta(t,0) = I + \Lambda_1 t + \Lambda_2 t^2 + \Lambda_3 t^3 .$$

Finally, after some calculations, we obtain a matrix expression for $\phi(\cdot)$

$$\begin{aligned} \phi(t) = \left[I + \Lambda_1 t + \Lambda_2 t^2 + \Lambda_3 t^3\right]&\left\{\phi_0 - \begin{bmatrix} \lambda_1 t + \frac{1}{2}\lambda_2 t^2 + \frac{1}{3}\lambda_3 t^3 \\ \mu_1 t + \frac{1}{2}\mu_2 t^2 + \frac{1}{3}\mu_3 t^3 \end{bmatrix}\right\} \\ &+ \begin{bmatrix} \nu_1 t + \frac{1}{2}\nu_2 t^2 + \frac{1}{3}\nu_3 t^3 + \frac{1}{4}\nu_4 t^4 + \frac{1}{5}\nu_5 t^5 + \frac{1}{6}\nu_6 t^6 \\ \xi_1 t + \frac{1}{2}\xi_2 t^2 + \frac{1}{3}\xi_3 t^3 + \frac{1}{4}\xi_4 t^4 + \frac{1}{5}\xi_5 t^5 + \frac{1}{6}\xi_6 t^6 \end{bmatrix} \end{aligned} \tag{A.11}$$

where the respective coefficients are

$$\nu_1 = \lambda_1,\ \nu_2 = \lambda_2 + \lambda_1\Lambda_{11}^1 + \mu_1\Lambda_{12}^1,\ \nu_3 = \lambda_3 + \sum_{i=1}^{2}\lambda_i\Lambda_{11}^{3-i} + \sum_{i=1}^{2}\mu_i\Lambda_{12}^{3-i},$$

$$\nu_4 = \sum_{i=1}^{3}\lambda_i\Lambda_{11}^{4-i} + \sum_{i=1}^{3}\mu_i\Lambda_{12}^{4-i},\ \nu_5 = \sum_{i=2}^{3}\lambda_i\Lambda_{11}^{4-i} + \sum_{i=2}^{3}\mu_i\Lambda_{12}^{4-i},\ \nu_6 = \lambda_3\Lambda_{11}^3 + \mu_3\Lambda_{12}^3$$

and

$$\xi_1 = \mu_1,\ \xi_2 = \mu_2 + \mu_1\Lambda_{22}^1 + \lambda_1\Lambda_{21}^1,\ \xi_3 = \mu_3 + \sum_{i=1}^{2}\mu_i\Lambda_{22}^{3-i} + \sum_{i=1}^{2}\lambda_i\Lambda_{21}^{3-i},$$

$$\xi_4 = \sum_{i=1}^{3}\mu_i\Lambda_{22}^{4-i} + \sum_{i=1}^{3}\lambda_i\Lambda_{21}^{4-i},\ \xi_5 = \sum_{i=2}^{3}\mu_i\Lambda_{22}^{4-i} + \sum_{i=2}^{3}\lambda_i\Lambda_{21}^{4-i},\ \xi_6 = \mu_3\Lambda_{22}^3 + \lambda_3\Lambda_{21}^3$$

where, for the terminal condition of equation (2.9), $\phi(T) = 0$, it follows

$$\phi_0 = -\left[I + \Lambda_1 T + \Lambda_2 T^2 + \Lambda_3 T^3\right]^{-1} \begin{bmatrix} \nu_1 T + \frac{1}{2}\nu_2 T^2 + \frac{1}{3}\nu_3 T^3 + \frac{1}{4}\nu_4 T^4 + \frac{1}{5}\nu_5 T^5 + \frac{1}{6}\nu_6 T^6 \\ \xi_1 T + \frac{1}{2}\xi_2 T^2 + \frac{1}{3}\xi_3 T^3 + \frac{1}{4}\xi_4 T^4 + \frac{1}{5}\xi_5 T^5 + \frac{1}{6}\xi_6 T^6 \end{bmatrix} + \begin{bmatrix} \lambda_1 T + \frac{1}{2}\lambda_2 T^2 + \frac{1}{3}\lambda_3 T^3 \\ \mu_1 T + \frac{1}{2}\mu_2 T^2 + \frac{1}{3}\mu_3 T^3 \end{bmatrix}.$$

Finally, combining equations (2.9), (2.11) with (A.5) - (A.11), it is derived that

$$\Psi(t) = \begin{bmatrix} \dfrac{m_1\rho\Delta}{\beta + (\sigma_1\rho\Delta)^2 P_{11}(t)} & \dfrac{m_1\rho\Delta}{\beta + (\sigma_1\rho\Delta)^2 P_{11}(t)} \\ \dfrac{m_2\rho\Delta}{\beta + (\sigma_2\rho\Delta)^2 P_{22}(t)} & \dfrac{m_2\rho\Delta}{\beta + (\sigma_2\rho\Delta)^2 P_{22}(t)} \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix}$$

and $$\psi = \begin{bmatrix} \dfrac{m_1\rho\Delta}{\beta + (\sigma_1\rho\Delta)^2 P_{11}(t)} \\ \dfrac{m_2\rho\Delta}{\beta + (\sigma_2\rho\Delta)^2 P_{22}(t)} \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}.$$