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RANDOM NETWORKS WITH INTERACTING NODES

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ABSTRACT. We study the network systems in which the nodes and their interconnections are subject to random dynamics. Both static and moving nodes models are considered in the context of network reliability in the steady-state. Potential applications range from transport to communication networks.

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1. INTRODUCTION

In this article we introduce two distinctively different models of random network dynamics in which both the nodes and their interconnections are subject to random perturbations. First kind, a *static model*, is concerned with a set of nodes at fixed locations and the corresponding fixed (rigid) interconnections. Examples and terminology for these types of networks are application specific: transportation (distribution centers/routes), water utilities (towers/pipes), power grids (transformers/lines), electric or electronic systems (relay switches/wiring). In a *mobile model*, aside from an internal random dynamics of nodes and interconnections, the nodes themselves are subject to random displacements. Typical applications include telecommunication, satellite navigational systems and wireless mobile networks. This is a continuation of our work initiated in [9], where a single node random dynamics for stationary nodes was studied in the context of stability and steady-state network reliability, for the case where the node interconnections were assumed to be operational at all times. Before defining the *static model*, the next section starts with an example to set forth the underlying ideas.

2. PRELIMINARIES

We begin with a prototype network, illustrated by several figures, whose formulation within a general framework will be given in the following section. Let's consider a network flow in Fig. 1 with *input nodes* $= \{1, 2\}$ and *output nodes* $= \{6\}$, which in Fig. 2 constitute the *boundary nodes* along with the *internal nodes* $= \{3, 4, 5\}$, and can be viewed as a cluster of a larger network configuration with boundary nodes serving as interconnecting bridges. The naming *input-output* is immaterial and can

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be interchanged (it indicates the direction of the flow, e.g., in electrical networks from + to -). The nodes within the input set and output set are assumed to have no interconnections between themselves while being connected to the internal nodes. A *directed chain* from an input node to an output node (with non-repeating internal nodes) is called a *path*, shown in Fig. 3 in a parallel-serial paths decomposition.

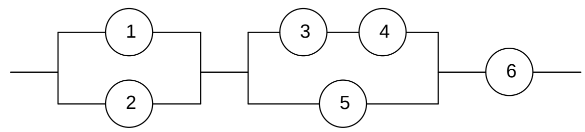

FIGURE 1. standard network flow diagram

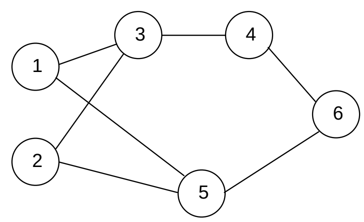

FIGURE 2. graph representation (nodes-edges)

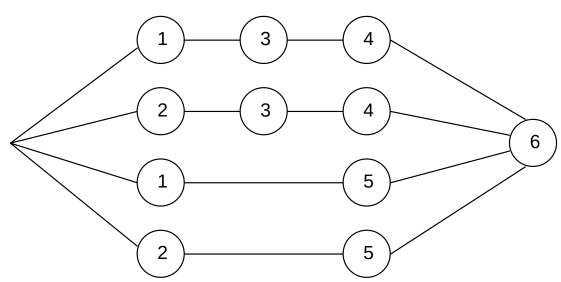

FIGURE 3. parallel-serial paths representation

Fig. 4 below will be considered a prototype of a generic network with interacting nodes. Such representation will allow us to employ $0-1$ valued structure functions corresponding to *Off-On* states, which in classical setting are utilized exclusively for nodes with interconnections being always *On*, as done in [9]. Here, the set of nodes $\{1,...,6\}$ and corresponding connections $\{7,...,16\}$ are subject to their own, time dependent, random dynamics which determine the network evolution. Formal definitions, assumptions and notational details are given in the next section.

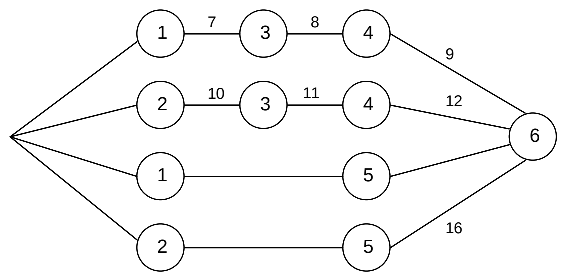

FIGURE 4. generic node interacting network

3. STATIC MODEL

We now define a network whose nodes and interconnections alternate between *On-Off* states and derive explicit formulas for network reliability in the steady-state. A convenient way to describe these networks is to use a graph-theoretical notation of nodes-edges (random processes themselves) and express the underlying dynamics in terms of random graphs (Fig. 2 and Fig. 4 may serve as a visualization in the case of a simple random network).

Definition 3.1. (*general*) Random network is a pair $\{\eta(t), \varepsilon(t)\}$ describing the states of the *nodes* and *edges*, where

$$\eta(t) = (\eta_1(t), ..., \eta_N(t)), \quad \eta_i(t) \in \{0, 1\} \tag{3.1}$$

is a random process whose components govern the dynamics of nodes $\{1, ..., N\}$, whereas a *random adjacency matrix*

$$\varepsilon(t) = \begin{pmatrix} \varepsilon_{11}(t) & \cdot & \cdot & \cdot & \varepsilon_{1N}(t) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \varepsilon_{N1}(t) & \cdot & \cdot & \cdot & \varepsilon_{NN}(t) \end{pmatrix}, \quad \varepsilon_{ij}(t) \in \{0, 1\} \tag{3.2}$$

incorporates the dynamics of the existing (*non-empty*) interconnections (edges) between the nodes. It is assumed that the edges of $\{\eta(t), \varepsilon(t)\}$ are *Alternating Renewal Processes* [10], each having a stationary *On-Off* distribution, to which the corresponding longtime frequencies settle down when approaching the steady-state.

In order to study longtime behavior of random networks one takes advantage of paths decomposition presented in Fig. 4. Namely, the alternating node-edge nature of the *paths*, arranged in parallel and clamped to input and output nodes, makes the analysis tractable by exploiting closed form representations of the corresponding structure functions which are suitable for reliability calculations. To this end we replace $\eta(t)$ by

$$x(t) = (x_1(t), ..., x_N(t)) \tag{3.3}$$

arrange all non-empty entries of $\varepsilon(t)$ linearly by introducing

$$y(t) = (y_1(t), ..., y_M(t)), \quad 1 \leq M = \# \textit{ of interconnections} \leq \binom{N}{2} \tag{3.4}$$

which can be merged into $N + M$ component system

$$z(t) = (x(t), y(t)) \equiv (z_1(t), ..., z_{N+M}(t)) \tag{3.5}$$

Definition 3.2. (*path representation*) Let us label $N = N_1 + N_2 + N_3$ nodes of the network by $\{1, ..., N_1\} \cup \{N_1 + 1, ..., N_1 + N_2\} \cup \{N_1 + N_2 + 1, ..., N_1 + N_2 + N_3\} = \{1, ..., N\}$ which correspond to input, internal and output nodes respectively. Then by (3.3)-(3.4) the network has the following path representation

$$\{(x_i(t), y_{k_1}(t), x_{l_1}(t), \ ... \ x_{l_{ij}}(t), y_{k_{ij}}(t), x_j(t))\} \tag{3.6}$$

with the indices of input, internal and output nodes satisfying

$$i \in \{1, ..., N_1\}, \ \{l_1, ..., l_{ij}\} \subset \{N_1 + 1, ..., N_1 + N_2\}, \ j \in \{N_1 + N_2 + 1, ..., N\} \tag{3.7}$$

whereas the indices of node interconnections satisfy

$$\{k_1, ..., k_{ij}\} \subset \{1, ..., M\} \tag{3.8}$$

For computational purposes, which will become clear later on, we use (3.5) and combine the notation of nodes and their interconnections to represent the network as follows

$$\mathcal{N}(t) = \{(z_i(t), z_{k_1}(t), z_{l_1}(t), \ ... \ z_{l_{ij}}(t), z_{k_{ij}}(t), z_j(t)) \, , \ \mathcal{I}\} \tag{3.9}$$

where the index set $\mathcal{I} = \{(i, j, l_1, ..., l_{ij}, k_1, ..., k_{ij})\}$ satisfies (3.7) with (3.8) replaced by $\{k_1, ..., k_{ij}\} \subset \{N + 1, ..., N + M\}$.

Let ϕ be the structure function for the network $\mathcal{N}(t)$

$$\phi(z(t)) = \phi(z_1(t), ..., z_{N+M}(t)) \, , \quad z_1(t) \in \{0, 1\} \tag{3.10}$$

We recall that by definition a structure function $\phi(u_1, ..., u_n) : \{0, 1\}^n \to \{0, 1\}$ takes value 1 when the system is *On* and becomes 0 when the system is *Off*. We assume that ϕ is increasing, i.e., $\phi(u_1, ..., u_n) \leq \phi(v_1, ..., v_n)$, whenever $u_i \leq v_i, \ i = 1, ..., n$. This is intuitively clear in the sense that replacing any *Off* component by *On* component will make the system more likely to be *On* (if the system was *Off*) and the system will continue to stay *On* when already *On*. In the case of our network, $\phi = 1$ means that there is a flow from the input nodes to the output nodes through at least one path, whereas $\phi = 0$ means that there is no such path. It turns out that for the networks under consideration the structure functions are polynomials in the variables $\{u_1, ..., u_n\}$ with ± 1 coefficients. This allows for calculations of expected values $E\phi(U_1, ..., U_n)$ whenever u_i is a realization of a random variable U_i, as is always the case for random networks. We remark that the pair $\{\mathcal{N}(t), \phi(\cdot)\}$ completely determines the network.

By stationarity of alternating renewal processes of nodes and interconnections we have by (3.5)

$$\lim_{t\to\infty} P(\, z_i(t) = 1\,) = p_i\,, \qquad i = 1, ..., N+M \tag{3.11}$$

To distinguish usually different characteristics of nodes and interconnections, we divide the steady-state probabilities p_i into two groups as follows

$$p_i = \frac{E\,\text{Service}_i}{E\,\text{Service}_i + E\,\text{Repair}_i} = \frac{1}{1+\rho_i}\,, \quad \rho_i = \frac{E\,\text{Repair}_i}{E\,\text{Service}_i}\,, \quad i = 1, ..., N \tag{3.12}$$

with ρ_i called Repair to Service *frequency ratio* for nodes. Similarly,

$$p_i = \frac{E\,\text{Up}_i}{E\,\text{Up}_i + E\,\text{Down}_i} = \frac{1}{1+\delta_i}\,, \quad \delta_i = \frac{E\,\text{Down}_i}{E\,\text{Up}_i}\,, \quad i = N+1, ..., N+M \tag{3.13}$$

and δ_i called Down to Up *frequency ratio* for interconnections.

By an alternating renewal process we mean a sequence of i.i.d. $(X_i, Y_i) \sim (X, Y)$ whose sums $X_i + Y_i$ constitute a renewal cycle (X_i and Y_i may be dependent!). In our case (X, Y) correspond to (Service, Repair) and (Up, Down) periods, for nodes and interconnections respectively. The idea behind the introduced characteristics ρ_i and δ_i is that if they are sufficiently small then the network components probabilities to be *On* will be close to 1, and will ultimately lead to a high reliability network.

Definition 3.3. (*dynamic reliability*) Given a random network $\{\mathcal{N}(t), \phi(\cdot)\}$

$$r(t) = P(\phi(z(t)) = 1) = E\phi(z(t)) \tag{3.14}$$

with $\phi(z(t))$ in (3.10) is called a *network reliability* at time t.

Lemma 3.4. (*structure function representation*) *The function* $\phi(\cdot)$ *in* $\{\mathcal{N}(t), \phi(\cdot)\}$ *has the following form*

$$\begin{aligned} \phi(z(t)) &= max\{z_i(t) z_{k_1}(t) z_{l_1}(t)\ ...\ z_{l_{ij}}(t) z_{k_{ij}}(t) z_j(t)\} \\ &= 1 - \prod (1 - z_i(t) z_{k_1}(t) z_{l_1}(t)\ ...\ z_{l_{ij}}(t) z_{k_{ij}}(t) z_j(t)) \end{aligned} \tag{3.15}$$

where the max and the product is over all indices from $\mathcal{I}$ *described by* (3.9).

Proof. Observe that the network is $On \Leftrightarrow \exists$ a path from the set of input nodes to the set of output nodes $\Leftrightarrow \exists\ (i, j, l_1, ..., l_{ij}, k_1, ..., k_{ij}) \in \mathcal{I}$ with all $z_\cdot(t) = 1 \Leftrightarrow \phi(z(t)) = 1$. This verifies the first equality. To show the second equality, notice that if $\exists$ a path $(z_i(t), z_{k_1}(t), z_{l_1}(t),\ ...\ z_{l_{ij}}(t), z_{k_{ij}}(t), z_j(t))$ with all $z_\cdot(t) = 1$ then $(1 - z_i(t)z_{k_1}(t)z_{l_1}(t)\ ...\ z_{l_{ij}}(t)z_{k_{ij}}(t)z_j(t)) = 0$ turns the $\prod$ into 0 and the last expression in (3.15) is reduced to 1. □

To derive a closed form representation for reliability function $r(t)$ we need to rearrange the network linearly and identify the paths with the product of their elements as follows

$$\mathcal{N}(t) = \{a_k\,,\ k = 1, ..., m\}\,, \quad m = Card\,[\,\mathcal{I}\,] \tag{3.16}$$

where

$$a_k = z_i(t)z_{k_1}(t)z_{l_1}(t)\ ...\ z_{l_{ij}}(t)z_{k_{ij}}(t)z_j(t) \tag{3.17}$$

for some index $(i, j, l_1, ..., l_{ij}, k_1, ..., k_{ij}) \in \mathcal{I}$. Then we have

$$\begin{aligned} 1 - \prod_{k=1}^{m}(1 - a_i) &= \sum_i a_i - \sum_{i<j} a_i a_j + ... + (-1)^{m+1} a_1 a_2 ... a_m \\ &= \sum_i b_i - \sum_{i<j} b_{ij} + ... + (-1)^{m+1} b_{12...m} \end{aligned} \tag{3.18}$$

with the blocks

$$b_\cdot = z_{\alpha_1}(t)z_{\alpha_2}(t)\ ...\ z_{\alpha_\cdot}(t)\,, \quad \alpha_1 < \alpha_2 ... < \alpha_\cdot\,,\ \alpha_i \in \{1, ..., N + M\} \tag{3.19}$$

because $z_{\alpha_i}(t)^n = z_{\alpha_i}(t)$ for any power n, thanks to $z_{\alpha_i}(t) \in \{0, 1\}$.

Theorem 3.5. *Random Network* $\{\mathcal{N}(t), \phi(\cdot)\}$ *with mutually independent nodes and interconnections has the following reliability function in the limit as* $t \to \infty$

$$\begin{aligned} r &= \sum_i b_i - \sum_{i<j} b_{ij} + ... + (-1)^{m+1} b_{12...m} \\ b_\cdot &= p_{\alpha_1} p_{\alpha_2}\ ...\ p_{\alpha_\cdot}\,, \quad \alpha_1 < \alpha_2 ... < \alpha_\cdot\,, \quad \alpha_i \in \{1, ..., N + M\} \end{aligned} \tag{3.20}$$

Proof. Taking the expected value in (3.19) and using independence, $E\, z_{\alpha_i}(t) \to p_{\alpha_i}$ by (3.11) and continuity of $\phi(\cdot)$ in variables z_i completes the proof. □

Example 1. Consider the network illustrated in Fig. 4 with $N = 6$ nodes, $M = 10$ interconnections and $m = 4$ paths for which (3.17) reads

$$a_1 = z_1(t)z_7(t)z_3(t)z_8(t)z_4(t)z_9(t)z_6(t)$$

$$a_2 = z_2(t)z_{10}(t)z_3(t)z_{11}(t)z_4(t)z_{12}(t)z_6(t)$$

$$a_3 = z_1(t)z_{13}(t)z_5(t)z_{14}(t)z_6(t)$$

$$a_4 = z_2(t)z_{15}(t)z_5(t)z_{16}(t)z_6(t)$$

Assuming the nodes and interconnections have the same characteristics $\rho_i = \rho$, $\delta_i = \delta$ and calculating $b.$'s defined in (3.19), Theoreom 3.5 yields the following steady-state reliability

$$r = 2(\tfrac{1}{1+\rho})^4(\tfrac{1}{1+\delta})^3 + 2(\tfrac{1}{1+\rho})^3(\tfrac{1}{1+\delta})^2 - (\tfrac{1}{1+\rho})^8(\tfrac{1}{1+\delta})^6 - 4(\tfrac{1}{1+\rho})^7(\tfrac{1}{1+\delta})^5$$

$$-(\tfrac{1}{1+\rho})^6(\tfrac{1}{1+\delta})^4 + 2(\tfrac{1}{1+\rho})^{11}(\tfrac{1}{1+\delta})^8 + 2(\tfrac{1}{1+\rho})^{10}(\tfrac{1}{1+\delta})^7 - (\tfrac{1}{1+\rho})^{14}(\tfrac{1}{1+\delta})^{10}$$

Remark 1. Notice that for $\rho = \delta = 0$ we have $r = 1$. By continuity, any desired reliablity can be achieved by choosing sufficiently small Repair to Service *frequency ratio* ρ for nodes and Down to Up *frequency ratio* δ for interconnections. In fact, since $\phi(\cdot)$ is increasing in $z = (z_1, ..., z_n)$, by conditioning on Z_i and independence of network components, one checks that $r(p_1, ..., p_n)$ is increasing in p_i for each i.

4. **MOBILE MODEL**

We shall consider the nodes moving freely till their steady-state spatial location distribution is reached. Some special cases in one and two dimensions were treated in [2, 3] and [6], however from our standpoint (due to intractability of spatial nodes distribution) we concentrate on uniform distribution resulting from Brownian motion with reflecting boundary in bounded regions of R^2. Our considerations will be based on *Binomial Random Graphs*, for which one may consult the monograph [1] or [7, 8] for *Eulerian Graphs* and related representations. We begin by stating a classical result due to Gilbert [5], which provides a recursive formula for a probability $P_N(p)$ that a graph G with N vertices is *connected*, given a connection between any two vertices (nodes) is made at random (independently) with probability p. Namely, the following recursion formula holds

$$P_N(p) = 1 - \sum_{k=1}^{N-1} \binom{N-1}{k-1} (1-p)^{k(N-k)} P_k(p)\,, \quad N = 2, 3, ..., \quad P_1(p) = 1 \tag{4.1}$$

and in particular we have for large N

$$P_N(p) \sim 1 - N(1-p)^{N-1}\,, \qquad 1 - (N+1)(1-p)^{N-1} \le P_N(p) \tag{4.2}$$

To develop a better understanding of the quantitative behavior of $P_N(p)$, using (4.1) we have calculated the probability that all N nodes inter-communicate as follows

$$P_2(p) = p$$

$$P_3(p) = 3p^2 - 2p^3$$

$$P_4(p) = 16p^3 - 33p^4 + 24p^5 - 6p^6$$

$$P_5(p) = 125p^4 - 528p^5 + 970p^6 - 980p^7 + 570p^8 - 180p^9 + 24p^{10}$$

$$P_{10}(p) = 362880p^{45} - 14515200p^{44} + 282592800p^{43}$$

$$- 3567715200p^{42} + 32833495800p^{41} - 234748765440p^{40}$$

$$+ 1357020856800p^{39} - 6517548349200p^{38} + 26521978127400p^{37}$$

$$- 92792729053500p^{36} + 282287441908080p^{35} - 753273866698920p^{34}$$

$$+ 1775448575926410p^{33} - 3716558335019880p^{32} + 6939551178972720p^{31}$$

$$- 11596879696617600p^{30} + 17388982649046960p^{29} - 23437879996999860p^{28}$$

$$+ 28429756177413360p^{27} - 31050312703343640p^{26} + 30532209914200806p^{25}$$

$$- 27011077082801580p^{24} + 21469710851551800p^{23} - 15300758477189520p^{22}$$

$$+ 9748958193896580p^{21} - 5532426738592740p^{20} + 2782630494934920p^{19}$$

$$- 1232671556293800p^{18} + 477077447178540p^{17} - 159642667620135p^{16}$$

$$+ 45558310696800p^{15} - 10884316965480p^{14} + 2121183237600p^{13}$$

$$- 324496267200p^{12} + 36628300800p^{11} - 2719892160p^{10} + 100000000p^9$$

$$(P_{10}(\tfrac{1}{8}),\ P_{10}(\tfrac{2}{8}),\ P_{10}(\tfrac{3}{8}),\ P_{10}(\tfrac{4}{8}),\ P_{10}(\tfrac{5}{8}),\ P_{10}(\tfrac{6}{8}))$$

$$= (0.0241,\ 0.4378,\ 0.8596,\ 0.9804,\ 0.9985,\ 0.9999)$$

which means that neither p has to be close to 1 nor N has to be large for achieving high reliability. In fact, by (4.2), for $\forall\, p > 0$ $P_N \approx 1$ for sufficiently large N, whence an asymptotic approximation is a practical alternative to a computational complexity of the exact recursive formula (4.1).

Turning back to our mobile model, unlike in the static case, we now allow all $\binom{N}{2}$ *bi-directional connections* to be *On* whenever the respective nodes are within certain range from transmitters. Nodes are assumed to operate independently and move randomly according to a standard Brownian motion in a specified region in R^2, populated with transmitters (assumed operating and inter-connected at all times).

Definition 4.1. (*mobile network*) A network with N nodes and n transmitters is denoted by

$$\mathcal{M}(t) = \{\eta(t),\, \varepsilon(t),\, w(t),\, D,\, S\} \tag{4.3}$$

where $\eta(t)$, $\varepsilon(t)$ are as in (3.1) – (3.2)

$$w(t) = (w_1(t),\, ...,\, w_N(t))\,, \quad w_i(t) \textit{ are i.i.d. Brownian motions in } D \subset R^2 \tag{4.4}$$

with $w_i(t)$ reflected at the boundary of a *regular*, *bounded* and *connected* region D, which contains the transmitters at fixed locations from the set $S = \{v_1,\, ...,\, v_n\}$.

Consequently, given transmitters range ρ of reaching nodes, we have

$$\begin{aligned} P(\varepsilon_{ij}(t) = 1) &= P(\,\exists\, k \;\; |w_i(t) - v_k| \le \rho \;\;\cap\;\; \exists\, l \;\; |w_j(t) - v_l| \le \rho) \\ &= P(\,\exists\, k \;\; |w_i(t) - v_k| \le \rho)\, P(\,\exists\, l \;\; |w_j(t) - v_l| \le \rho) \end{aligned} \tag{4.5}$$

To take full advantage of the stationary *uniform distribution* over region D, corresponding to the node movements $w_i(t)$, we make

Symmetry Assumption. D has the following representation (*tessellation*)

$$D = D_1 \cup \,...\, \cup D_m \tag{4.6}$$

where D_i are *disjoints* (*except possibly sharing sides*)*, congruent, regular polygons* (e.g., equilateral triangles, squares, hexagons, etc.), with sides of length a satisfying $a \ge 2\rho$, and whose vertices comprise the set of all transmitters S.

Theorem 4.2. *Let $\mathcal{M}(t)$ be a mobile network such that* $\lim_{t\to\infty} P(\eta_i(t) = 1) = \beta$, $i = 1, ..., N$. *Then we have*

$$r = \lim_{t\to\infty} P(\textit{all nodes communicate with each other}) = \beta^N\, P_N(\alpha) \tag{4.7}$$

where $P_N(\alpha)$ is given by (4.1) *and*

$$\alpha = \left(\frac{\Sigma_{v \in V_1}\, Area(D_1 \,\cap\, B(v,\rho))}{Area(D_1)} \right)^2 \tag{4.8}$$

with $B(v,\rho)$ = ball of radius ρ centered at vertex v and V_1 stands for the set of all vertices of D_1.

Proof. Communication of all nodes is equivalent to all N nodes being *On* in conjunction with the corresponding graph being connected. By generalization of Gilbert's result [5] and (4.5) – (4.6) one checks that (4.7) holds for α given by (4.8), due to symmetry and uniform distribution over D of each node's location as $t \to \infty$. □

Example 2. In the case D_1 is a triangle, a square or a hexagon, we get $\alpha = \frac{4}{3}\pi^2(\frac{\rho}{a})^4 \leq \frac{\pi^2}{12}$, $\alpha = \pi^2(\frac{\rho}{a})^4 \leq \frac{\pi^2}{16}$, and $\alpha = \frac{16}{27}\pi^2(\frac{\rho}{a})^4 \leq \frac{\pi^2}{27}$ respectively.

Clearly by (4.2) and (4.7), increasing α, β, or N enhances reliability r.

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