Neural, Parallel, and Scientific Computations 18 (2010) 385-390

ON THE LEVEL OF CONVERGENCE OF SEQUENCE OF FUZZY RANDOM VARIABLES

J. EARNEST LAZARUS PIRIYAKUMAR[1] AND M.VASUKY[2]
[1]Department of Mathematics, Tranquebar Bishop Manickam Lutheran College, Porayar – 609 307, S. India.
[2] Department of Mathematics, Seethalakshmi Achi College for Women, Pallathur – 630107, S.India
e-mail :jelpriyakumar@yahoo.com

ABSTRACT: In this paper we introduce the notion of levelwise convergence of sequence of fuzzy random variables. We derive equivalent conditions for left quasi uniform convergence of fuzzy random variables. Conditions are obtained for eventually equi-right continuity for the sequence of fuzzy random variables. A necessary and sufficient condition is obtained under which for a sequence of fuzzy random variables $\{X_n\}$ satisfying lim $n \to \infty$ $X^L_{n,\alpha}$ = V_α and lim $n\to \infty$ $X^U_{n,\alpha}$ = $W\alpha$ where V_α and W_α are the pair of random variables for $\alpha \in [0,1]$ that determines a fuzzy random variable.
Keywords : Fuzzy random variables, Fuzzy numbers, Level convergence.

1. INTRODUCTION

The theory of fuzzy random variables is a natural extension of general real valued random variables or random vectors. The notion of fuzzy random variable was introduced by Kwakernaak [1,4] and Puri and Ralescu [2]. In order to make fuzzy random variables relevant to statistical analysis for imprecise data we introduce a variety of notions on convergence of fuzzy random variables. Since the α-level set of a closed fuzzy number is a compact interval, inorder to make the end points of the α-level set of a fuzzy random variable to be the random variables the concept of strong measurability is introduced for fuzzy random variables. In this paper the concept of level wise convergence of sequence of fuzzy random variables, is introduced. Equivalent conditions for left quasi uniform convergence of fuzzy random variables are introduced. conditions are obtained for eventually equi-left continuity and eventually equi-right continuity of fuzzy random variables. A necessary and sufficient condition is obtained under which for a sequence of fuzzy random variables $\{X_n\}$ satisfying lim $n\to \infty$ $X^L_{n,\alpha}$ = V_α and lim $n\to \infty$ $X^U_{n,\alpha}$ = W_α where V_α and W_α are the pair of random variables for $\alpha \in [0,1]$ that determines a fuzzy random variables.

In section 2, basic definitions and results about fuzzy numbers and fuzzy random variables are recalled. In section 3, the concepts of level wise convergence of sequence of fuzzy random variables, and the equivalent conditions which manifest the left quasi uniform convergence of fuzzy random variables are disclosed. In this section conditions are obtained for eventually equi-left continuity and eventually equi-right continuity for the sequence of fuzzy random variables. A necessary and sufficient conditions is obtained for the existence of fuzzy random variables interms of convergence of fuzzy random variables.

2. PRELIMINARIES

In this section, we introduce some basic concepts for fuzzy numbers and fuzzy random variables.

Definition : 2.1

Let R be the real number field, N the set of all positive integers and F(R) denote the set of all fuzzy subsets of R. A fuzzy set u on R is called a fuzzy number if it has the following properties.

Received April 25, 2010 1061-5369 $15.00

1. u is normal, i.e., there exists an $x_0 \in R$ such that $u(x_0) = 1$
2. u is convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for $x, y \in R$ and $\lambda \in [0,1]$.
3. u is upper semi continuous
4. $u_0 = cl\{x \in R; u(x) > 0\}$ is a compact set,

A real number r can be regarded as a fuzzy number $\tilde{r}$ defined as

$$\tilde{r}(t) = \begin{cases} 1 & \text{if } t = r \\ 0 & t \neq r \end{cases}$$

If $u \in F(R)$, then u is a fuzzy number if and only if u_λ is a non-empty bounded and closed interval for each $\lambda \in [0,1]$.

We denote $u_\lambda = \left[u_\lambda^L, u_\lambda^U\right]$, $\lambda \in [0,1]$.

Let E be the set of all fuzzy numbers. A partial ordering $\leq$ in E is defined as

$u \leq v$ iff $u_\lambda^L \leq v_\lambda^L$, $u_\lambda^U \leq v_\lambda^U$ for all $\lambda \in (0,1]$

where $u_\lambda = \left[u_\lambda^L, u_\lambda^U\right]$ and

$v_\lambda = \left[v_\lambda^L, v_\lambda^U\right]$

Give a real number x, we can induce a fuzzy number $\tilde{x}$ with membership function $\mu_{\tilde{x}}(r) < 1$ for $r \neq x$. We call $\tilde{x}$ as the fuzzy real number induced by the real number x. let F_R be a set of all fuzzy real numbers induced by the real number system R. We define the relation ~ on F_R as $\tilde{x}^1 \sim \tilde{x}^2$ iff $\tilde{x}^1$ and $\tilde{x}^2$ are induced by the same real number x. Then ~ is an equivalence relation inducing the equivalence classes $[\tilde{x}] = \{\tilde{a}\,;\ \tilde{a} \sim \tilde{x}\}$. The quotient set $F_R/\sim$ is the set of all equivalence classes then the cardinality of $F_R/\sim$ is equal to the cardinality of the real number system R. We call $F_R/\sim$ as fuzzy real number system. In practice we take only one element $\tilde{x}$ from each equivalence class $[\tilde{x}]$ to form the fuzzy real number system $(F_R/\sim)_R$, that is

$(F_R/\sim)_R = \{\tilde{x}\,;\ \tilde{x} \in [\tilde{x}], \tilde{x}$ is the only element from $[\tilde{x}]\}$

Definition : 2.2 [2]

Let (X, M) be a measurable space and (R,B) be a Borel measurable space. Let P(R) denote the power set of R, and $f : X \to P(R)$ be a set valued function. Then f is called measurable if and only if $\{(x,y)\,;\ y \in f(x)\}$ is a measurable subset of M x B.

Definition : 2.3

$\tilde{f}(x)$ is called a fuzzy valued function of $\tilde{f} : X \to E$. If $\tilde{f}$ is a fuzzy valued function then $\tilde{f}_\alpha(x)$ is the α-level set of the fuzzy number $\tilde{f}(x)$; $x \in X$. $\tilde{f}$ is called measurable if $\tilde{f}_\alpha$ is measurable for each α, $\alpha \in (0,1]$.

Let E_{cl} denote the set of all closed fuzzy numbers. $\tilde{f}(x)$ is called closed fuzzy-valued function if $f : X \to E_{cl}$ and we have

$\tilde{f}_\alpha(x) = \left[f_\alpha^L(x), f_\alpha^U(x)\right]$ for any $x \in X$.

Definition : 2.4 [5]

Let $\tilde{f}(x)$ be a closed fuzzy valued function, $\tilde{f}(x)$ is called strongly measurable if one of the following conditions is satisfied.

1) $\tilde{f}_\alpha^L(x)$ and $\tilde{f}_\alpha^U(x)$ are measurable for all $\alpha \in (0,1]$

2) $\tilde{f}(x)$ is measurable and one of $\tilde{f}_\alpha^L(x)$ and $\tilde{f}_\alpha^U(x)$ is measurable for all α.

Definition : 2.5 [5]

Let $(F_R/\sim)_R$ be a fuzzy real number system and $\tilde{X} : \Omega \to (F_R/\sim)_R$ be a closed fuzzy valued function. Then $\tilde{X}$ is called a fuzzy random variables if $\tilde{X}$ is strongly measurable.

Theorem : 2.1 [5]

Let $(F_R/\sim)_R$ be a fuzzy real number system and $\tilde{X} : \Omega \to (F_R/\sim)_R$ be a closed fuzzy valued function. $\tilde{X}$ is a fuzzy random variable if and only if $\tilde{X}_\alpha^L$ and $\tilde{X}_\alpha^U$ are random variables for all α.

Theorem : 2.2 [6]

Let u be a fuzzy number and $u_\lambda = \left[u_\lambda^L, u_\lambda^U\right]$ then the following conditions are satisfied.

1. u_λ^L is a bounded left continuous non-decreasing function on (0,1].
2. U_λ^U is bounded left continuous non-increasing function on (0,1].
3. u_λ^L and U_λ^U are right continuous at $\lambda = 0$
4. $U_1^L \leq u_1^U$

Conversely of the pair of functions α(λ) and β(λ) satisfies conditions (1)-(4) then there exists a unique fuzzy number u such that $u_\lambda = [\alpha(\lambda), \beta(\lambda)]$ for each $\lambda \in [0,1]$.

3. CONVERGENCE OF FUZZY RANDOM VARIABLES

In this section we introduce various notions of convergence of fuzzy random variables.

Definition : 3.1

Let $\{\tilde{X}_n\}$ be a sequence of fuzzy random variables and $\tilde{X}$ be a fuzzy random variable. We say $\{\tilde{X}_n\}$ level converges to $\tilde{X}$ denoted as $\tilde{X}_n \xrightarrow{l} \tilde{X}$ if $\lim_{n\to\infty} X_{n,\lambda}^L = X_\lambda^L$ and lim $n\to\infty$ $\tilde{X}_{n,\lambda}^U = \tilde{X}_\lambda^U$ for all $\lambda \in [0,1]$.

Definition : 3.2

A sequence of fuzzy random variables $X_m(\lambda) : [a,b] \to R$ (m = 1,2,....) is said to converge left quasi uniformly to X(λ) at $\lambda_0 \in (a,b)$ if for any $\in > 0$, there exist $\delta > 0$ $(\delta < \lambda_0)$ and $m_0 \in N$ such that $\left| X_{m,\alpha}^U(\lambda) - X_\alpha^U(\lambda) \right| < \in$ for each $\alpha \in (0,1]$ and $\lambda \in (\lambda_0 - \delta, \lambda_0]$.

Definition : 3.3

A sequence of fuzzy random variables $X_m(\lambda) : [a, b] \to R$ (m = 1,2...) is said to converge right quasi uniformly to X(λ) at $\lambda_0 \in [a,b)$ if for any $\in > 0$ there exist $\delta > 0$, $\delta > \lambda_0$ and m0 ∈ N ∍

$\left| X_{m,\alpha}^U(\lambda) - X_\alpha^U(\lambda) \right| < \in$ for each $\alpha \in (0,1]$ and $\lambda \in [\lambda_0, \lambda_0 + \delta)$

Theorem : 3.1

Let $\{X_n\}$ be a sequence of fuzzy random variables. Then the following two statements are equivalent

(a) $\inf_{r<\lambda} \sup_n X_{n,r}^U = \sup_n X_{n,\lambda}^U$

(b) The sequence of functions $\left\{\max 1 \leq n \leq m \; X_{n,\lambda}^U\right\}_{m=1}^{\infty}$

Converges left quasi uniformly to $\sup_n X^U_{n,\lambda}$ at $\lambda \in (0,1]$.

Proof (a) $\Rightarrow$ (b)

By stipulation of (a) for any $\in > 0$, there exists $r_0 \in (0, \lambda)$ such that $\sup_n X^U_{n,r_0} < \sup_n X^U_{n,\lambda} + \in$. So there exists $m_0 \in N$ such that $\sup_n X^U_{n,r_0} < X^U_{m_0,\lambda} + \in$.

Thus when $r \in (r_0, \lambda]$ we have

$$\begin{aligned}\max_{1\le n\le m} X^U_{n,r} &\le \sup_n X^U_{n,r} \le \sup_n X^U_{n,r_0}\\ &\le X^U_{m_0,\lambda} + \in \le X^U_{m_0,r} + \in\\ &\le \max_{1\le n\le m} X^U_{n,r} + \in \qquad \forall\, m \ge m_0.\end{aligned}$$

This establishes (b)

(b) $\Rightarrow$ (a) since $X^U_{n,\lambda}$ is non-increasing about λ, we have

$$\sup_n X^U_{n,\lambda} \le \inf_{r<\lambda} \sup_n X^U_{n,r}$$

By stipulation of (b) for any $\in > 0$, there exist $m_0 \in N$ and $\delta > 0$ $(\delta < \lambda)$ such that

$$\sup_n X^U_{n,r} < \max_{1\le n\le m_0} X^U_{n,r} + \in \tag{1}$$

for $r \in (\lambda - \delta, \lambda]$.

Since $\max 1 \le n \le m_0\ X^U_{n,\lambda}$ is non-increasing and left continuous, we have

$$\inf_{r<\lambda} \max_{1\le n\le m_0} X^U_{n,r} = \max_{1\le n\le m_0} X^U_{n,\lambda}$$

then it follows from (3.1) that

$$\inf_{r<\lambda} \sup_n X^U_{n,r} \le \max_{1\le n\le m_0} X^U_{n,\lambda} + \in \le \sup_n X^U_{n,\lambda} + \in$$

This establishes (a)

Definition : 3.4

Let $\{X_n\}$ be a sequence of fuzzy random variables defined as [a,b] and $\lambda_0 \in [a,b]$. $\{X_n\}$ is said to be eventually equi-left continuous at λ_0 if for any $\in > 0$ there exists $N \in \mathbb{N}$ and $\delta > 0$ such that

$$\left| X^U_{n,\lambda} - X^U_{n,\lambda_0} \right| \in \text{ when ever } \lambda \in (\lambda_0 - \delta, \lambda_0] \text{ and } n \ge N.$$

Definition : 3.5

Let $\{X_n\}$ be a sequence of fuzzy random variables defined on [a,b] and $\lambda_0 \in (a,b]$. $\{X_n\}$ is said to be eventually equi-right continuous at λ_0 if for any $\in > 0$ there exists $N \in \mathbb{N}$ and $\delta > 0$ such that

$$\left| X^L_{n,\lambda} - X^L_{n,\lambda_0} \right| < \in \text{ whenever } \lambda \in [\lambda_0, \lambda_0+\delta) \text{ and } n \ge N.$$

Theorem : 3.2

Let $\{X_n\}$ be a sequence of fuzzy random variables and X be a fuzzy random variables. If $\lim_{n\to\infty} X^L_{n,\lambda} = X^L_\lambda$ ($\lim_{n\to\infty} X^U_{n,\lambda} = X^U_\lambda$) for each $\lambda \in (0,1]$, then $\{X^L_{n,\lambda}\}$ (resp $\{X^L_{n,\lambda}\}$) is eventually equi-left continuous at $\lambda \in (0,1]$. Further more if $\lim_{n\to\infty} u^L_{n,0} = u^L_0$ ($\lim n\to\infty\ u^U_{n,0} = u^U_0$) then $\{u^L_{n,\lambda}\}$ (resp $\{u^U_{n,\lambda}\}$) is equally equi right continuous at $\lambda = 0$.

Proof

Suppose that $\lim_{n\to\infty} X^L_{n,\lambda} = X^L_\lambda$ for each $\lambda \in (0,]$

Since X^L_λ is left continuous at $\lambda \in (0,1]$ for any $\in > 0$ there exists $0<\lambda_1<\lambda$

$$\left| X^L_\mu - X^L_\lambda \right| < \in/3 \text{ for all } \mu \in (\lambda_1, \lambda] \tag{2}$$

Taking $0 < \delta < \lambda - \lambda_1$ we have $\lambda - \delta \in (\lambda_1, \lambda)$.

Since $\lim_{n\to\infty} X^L_{n,\lambda} = X^L_\lambda$ and $\lim_{n\to\infty} X^L_{n,\lambda-\delta} = X^L_{\lambda-\delta}$ there exists $N \in \mathbb{N}$ such that

$$\left| X^L_{n,\lambda} - X^L_\lambda \right| < \in/3, \left| X^L_{n,\lambda-\delta} - X^L_{\lambda-\delta} \right| < \in/3 \text{ for all } n \geq N. \tag{3}$$

$X^L_{n,\lambda}$ and X^L_λ are non-decreasing for λ. It follows from (2) and (3) that

$$X^L_{n,\lambda-\delta} \leq X^L_{n,\mu} \leq X^L_{n,\lambda} < X^L_\lambda + \in/3 \quad < X^L_{\lambda-\delta} + 2\in/3$$

whenever $\mu \in (\lambda - \delta, \lambda]$

and $n \geq N$. so we have

$$0 \leq X^L_{n,\lambda} - X^L_{n,\mu} \leq X^L_{\lambda-\delta} - X^L_{n,\lambda-\delta} + 2\in/3 < \in$$

whenever $\mu \in (\lambda - \delta, \lambda]$ and $n \geq N$.

This shows that $\{ X^L_{n,\lambda} \}$ is eventually equi-left continuous at λ_0.

Furthermore, suppose $\lim_{n\to\infty} X^L_{n,0} = X^L_0$

Since X^L_λ is right continuous at $\lambda = 0$, for any $\in > 0$ there exists $\delta_1 > 0$ such that

$$0 \leq X^L_\mu - X^L_0 < \in/3 \text{ for all } \mu \in [0, \delta_1) \tag{4}$$

Take $\delta = \delta_{1/2}$

Since $\lim_{n\to\infty} X^L_{n,0} = X^L_0$ and

$\lim_{n\to\infty} X^L_{n,\delta} = X^L_\delta$, there exists $N \in \mathbb{N}$ such that

$$\left| X^L_{n,0} - X^L_0 \right| < \in/3$$

$$\left| X^L_{n,\delta} - X^L_\delta \right| < \in/3 \text{ for all } n \geq N \tag{5}$$

It follows from (4) and (5) that

$$0 \leq X^L_{n,\mu} - X^L_{n,0} < X^L_{n,\delta} - \left(X^L_0 - \in/3 \right) < X^L_\delta + \in/3 - X^L_0 + \in/3 < \in$$

whenever $\mu \in [0, \delta)$ and $n \geq N$.

Therefore $\{ X^L_{n,\lambda} \}$ is eventually equi-right continuous at $\lambda = 0$. Similarly we can prove the other results.

Theorem : 3.6

Let $\{X_n\}$ be a sequence of fuzzy random variables such that $\lim\limits_{n\to\infty} X^L_{n,\lambda} = V_\lambda$ and $\lim_{n\to\infty} X^U_{n,\lambda} = w_\lambda$ for each $\lambda \in [0,1]$. Then the pair of functions V_λ and W_λ determine a fuzzy random variable if and only if the sequence of functions $\{ X^L_{n,\lambda} \}$ and $\{ X^U_{n,\lambda} \}$ are eventually equi-left continuous at each $\lambda \in (0,1]$ and eventually equi-right continuous at $\lambda=0$.

Proof. Suppose that there exists a fuzzy random variable X such that

$\lim_{n\to\infty} X^L_{n,\lambda} = V_\lambda = X^L_\lambda$ $\lim_{n\to\infty} X^U_{n,\lambda} = W_\lambda = X^U_\lambda$ for all $\lambda \in [0,1]$

By theorem 3.5 we know that { $X^L_{n,\lambda}$ } and { $X^U_{n,\lambda}$ } are eventually equi- left continuous at each $\lambda \in (0,1]$ and eventually right continuous at $\lambda = 0$.

Conversely, since { $X^L_{n,\lambda}$ } and { $X^U_{n,\lambda}$ } are eventually equi-left continuous at each $\lambda \in (0,1]$ for any $\in > 0$, there exists $N \in \mathbb{N}$ and $\delta > 0$ such that

$$0 \le X^L_{n,\lambda} - X^L_{n,\mu} < \in, \; 0 \le X^U_{n,\mu} - X^U_{n,\lambda} < \in \tag{6}$$

whenever $\mu \in (\lambda - \delta, \lambda]$ and $n \ge N$.

Letting $n \to \infty$ in (6) we obtain

$$0 \le V_\lambda - V_\mu \le \in, \; 0 \le W_\mu - W_\lambda \le \in \text{ whenever } \mu \in (\lambda - \delta, \lambda]$$

This shows that V_λ and W_λ are left continuous at $\lambda \in (0,1]$.

Similarly by using the eventual equi-right continuity of { $X^L_{n,\lambda}$ } and { $X^U_{n,\lambda}$ } at $\lambda = 0$ it can be proved that V_λ and W_λ are right continuous at $\lambda = 0$.

It is easy to see that V_λ is non-decreasing and W_λ is non-increasing and $V_1 \le W_1$. This shows that V_λ and W_λ, satisfy the conditions (1)-(4) in theorem 2.2. Hence there exists a fuzzy random variable X, such that $X^L_\lambda = V_\lambda$ and $X^U_\lambda = W_\lambda$ for each $\lambda \in [0,1]$. This concludes the proof.

REFERENCES

1. Aumann. R.J. (1965), Integrals of set valued functions, J. Math. Anal. Appl. 12, pp. 1-12.
2. Kwakernaak. H (1978), Fuzzy random variables I : Definitions and Theorems, Inform. Sci. 15, pp. 1-29.
3. Kwakernaak H. (1979) Fuzzy random variables II : Algorithms and Examples for the discrete data, Inform. Sci. 17, pp. 253-278.
4. Goetschel R. W.Voxman (1986), Elementary fuzzy calculus, Fuzzy sets and systems, 18, pp. 31-42.
5. Puri. M.L., & Ralescu. D.A., (1986) Fuzzy random variables, J.Math. Anal. Appl. 114, pp. 409-422.
6. Wu. H.C., (1999) Probability density functions of fuzzy random variables, Fuzzy sets and systems, 105, pp. 139-158.