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HYBRID FINITE DIFFERENCE METHODS FOR SOLVING MODIFIED BURGERS AND BURGERS-HUXLEY EQUATIONS

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ABSTRACT. Most phenomena in the real world are described through non-linear equations. One of the most fascinating extensions of the Burgers' equation in the description of non-linear phenomena is the modified Burgers' equation and the Burgers-Huxley equation. Modified Burgers equation has varied applications in the field of Physics and particularly wherein dissipation is a significant aspect of wave propagation. On the other hand, Burgers-Huxley equation, under special choice of parameters, namely the Hodgkin-Huxley equation, describes how action potentials in neurons are initiated and propagated. This equation also shows the interplay between non-linear reaction and diffusive transport.

Our objective in this paper is to devise and analyze robust numerical methods for numerically solving the modified Burgers' equation and the Burgers-Huxley equation. The methods are primarily based on monotone hybrid finite difference methods with piecewise uniform layer adaptive mesh. A rigorous analysis of the proposed methods for uniform convergence is given and the error estimates are derived. Several numerical experiments on benchmark problems are carried out and comparison of the numerical results made with the existing methods demonstrate the improvement and efficiency of the proposed methods.

Keywords: Singular perturbation; Modified Burgers' and Burgers-Huxley equation; Implicit Euler method; Quasilinearization; Upwind and central difference; Convergence

1. INTRODUCTION

The present study deals with the following two classes of one-dimensional non-linear parabolic problems of Burgers' type on the domain $D = (0,1) \times (0,T]$, with the smooth boundary $\partial D = \bar{D} \backslash D$:

$$L_{\varepsilon,1}u(x,t) = -\varepsilon\frac{\partial^2 u}{\partial x^2} + u^2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0, \tag{1.1a}$$

$$L_{\varepsilon,2}u(x,t) = -\varepsilon\frac{\partial^2 u}{\partial x^2} + \alpha u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} - \beta(1-u)(u-\gamma)u = 0, \tag{1.1b}$$

$$(x,t) \in D \equiv \Omega_x \times \Omega_t \equiv (0,1) \times (0,T], \tag{1.1c}$$

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with the initial-boundary conditions

$$u(x,0) = u_0(x), \qquad x \in \bar{\Omega}_x, \tag{1.1d}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \qquad t \in \bar{\Omega}_t. \tag{1.1e}$$

The operators $L_{\varepsilon,1}$ and $L_{\varepsilon,2}$ correspond to modified Burgers' and Burgers-Huxley equation. Here $\alpha > 0, \beta \geq 0$ and $\gamma \in (0,1)$ are the parameters and $\varepsilon \in (0,1]$ is known as the small singular perturbation parameter. $L_{\varepsilon,2}$ also reduces to *Hodgkin-Huxley equation* [1], which describes how action potentials in neurons are initiated and propagated. Hodgkin and Huxley received the 1963 Nobel Prize in Physiology/Medicine for this work. In fluid dynamics modified Burgers' equation describes the interplay between non-linear convection and diffusion while Burger-Huxley equation shows a prototype model for describing the interaction between non-linear convection effects, reaction mechanisms and diffusion transport. For $\partial D = \Gamma_l \cup \Gamma_i \cup \Gamma_r$, we distinguish the left lateral boundary $\Gamma_l = \{(x,t) : x = 0,\ t \in \bar{\Omega}_t\}$, right lateral boundary $\Gamma_r = \{(x,t) : x = 1,\ t \in \bar{\Omega}_t\}$ and the initial boundary $\Gamma_i = \{(x,t) : t = 0,\ x \in \bar{\Omega}_x\}$.

For small values of the parameter ε, the solutions of these problems presenting rapid variations in some narrow region called boundary layer, in the neighborhood of the right lateral surface Γ_r. In the present work, we construct a monotone finite difference operator for the problem classes (1.1), which is a natural development of monotone ε-uniformly convergent schemes for linear boundary value problems with exponential boundary layer. To resolve the boundary layer, we use piecewise uniform Shishkin mesh which is refined in the neighborhood of the boundary layer.

2. TEMPORAL SEMI-DISCRETIZATION

At the first stage, we discretize temporal variable in the operators $L_{\varepsilon,1}$, and $L_{\varepsilon,2}$ by means of the implicit Euler method with constant step size Δt. Such a semi-discretization yields the following system of non-linear elliptic differential equations:

$$u^0 = u(x,0) = u_0(x), \quad x \in \bar{\Omega}_x, \tag{2.1a}$$

$$(I + \Delta t L_{x,\varepsilon})u^{n+1} = u^n, \quad x \in \Omega_x, \quad n \geq 0, \tag{2.1b}$$

$$u^{n+1}(0) = 0, \quad u^{n+1}(1) = 0, \quad n \geq 0, \tag{2.1c}$$

where

$$(I + \Delta t L_{x,\varepsilon})u^{n+1} \equiv \begin{cases} -\varepsilon\Delta t\frac{\partial^2 u^{n+1}}{\partial x^2} + \Delta t(u^{n+1})^2\frac{\partial u^{n+1}}{\partial x} + u^{n+1}, & \text{for } L_{\varepsilon,1}, \\ -\varepsilon\Delta t\frac{\partial^2 u^{n+1}}{\partial x^2} + \alpha\Delta t u^{n+1}\frac{\partial u^{n+1}}{\partial x} & \\ \qquad -\beta\Delta t(1-u^{n+1})(u^{n+1}-\gamma)u^{n+1} + u^{n+1}, & \text{for } L_{\varepsilon,2}. \end{cases} \tag{2.2}$$

where, u^{n+1} is the solution of the Eq. (2.1), at the $(n+1)$th time level. Here $u^n \equiv u(x, t^n)$, and Δt is the uniform time step. The local truncation error of the time semi-discretization is given by $\mu_{n+1} \equiv u^{n+1} - \hat{u}^{n+1}$, where $\hat{u}^{n+1}$ is the computed solution of the following boundary value problem

$$(I + \Delta t L_{x,\varepsilon})\hat{u}^{n+1} = u^n, \qquad x \in \Omega_x, \qquad n \geq 0, \tag{2.3a}$$

$$\hat{u}^{n+1}(0) = 0, \qquad \hat{u}^{n+1}(1) = 0, \qquad n \geq 0. \tag{2.3b}$$

Local error estimates of each time step contributes to the global error of the temporal semi-discretization which is defined, at the instant t^n, as $E_n \equiv u(x, t^n) - u^n(x)$. Then, the following consistency result holds.

Lemma 2.1. (Local error estimate). *If*

$$\left| \frac{\partial^j}{\partial t^j} u(x,t) \right| \leq C, \qquad \forall (x,t) \in \bar{D}, \qquad 0 \leq j \leq 2, \tag{2.4}$$

then the local error estimates in the temporal direction is given by

$$\|\mu_{n+1}\|_\infty \leq C(\Delta t)^2. \tag{2.5}$$

Now combining the stability and consistency of the temporal semi-discretization process, we lead to the following global error estimate.

Lemma 2.2. (Global error estimate). *Under the hypotheses of Lemma* 2.1*, we have*

$$\|E_n\|_\infty \leq C\Delta t, \qquad \forall n \leq T/\Delta t.$$

Therefore, the temporal semi-discretization process is of uniformly convergent of first order.

3. QUASILINEARIZATION

In this section, we use the quasilinearization process to linearize the above non-linear ordinary differential equations. An application of the quasilinearization process [2] to the non-linear problem (2.1) introduce a sequence $\langle u_{(k)} \rangle_{k=0}^{\infty}$ of linear equations determined by the following recurrence relation

$$\bar{u}^0 = u_0(x), \qquad x \in \bar{\Omega}_x, \tag{3.1a}$$

$$\begin{aligned}(I + \Delta t \tilde{L}_{x,\varepsilon})\bar{u}^{n+1} &\equiv -\varepsilon \Delta t \frac{\partial^2 \bar{u}^{n+1}}{\partial x^2} + a(x)\Delta t \frac{\partial \bar{u}^{n+1}}{\partial x} + (1 + \Delta t b(x))\bar{u}^{n+1} \\ &= \bar{u}^n + \Delta t f(x), \quad x \in \Omega_x, \quad n \geq 0,\end{aligned} \tag{3.1b}$$

$$\bar{u}^{n+1}(0) = 0, \qquad \bar{u}^{n+1}(1) = 0, \qquad n \geq 0, \tag{3.1c}$$

where for the sake of convenience, we let $u_{(k+1)} = \bar{u}$. For the modified Burgers' equation

$$a(x) = a_{(k)}(x, t^{n+1}) = (u_{(k)}^{n+1})^2, \quad b(x) = b_{(k)}(x, t^{n+1}) = 2u_{(k)}^{n+1}\frac{\partial u_{(k)}^{n+1}}{\partial x},$$
$$f(x) = f_{(k)}(x, t^{n+1}) = \left(2(u_{(k)}^{n+1})^2\frac{\partial u_{(k)}^{n+1}}{\partial x}\right),$$

and for Burgers-Huxley equation

$$\begin{aligned} a(x) &= a_{(k)}(x, t^{n+1}) = \alpha u_{(k)}^{n+1}, \\ b(x) &= b_{(k)}(x, t^{n+1}) = \alpha u_{(k)}^{n+1}\frac{\partial u_{(k)}^{n+1}}{\partial x} + \beta\Big((u_{(k)}^{n+1} - \gamma)u_{(k)}^{n+1} - (1 - u_{(k)}^{n+1})u_{(k)}^{n+1} \\ &\quad - (1 - u_{(k)}^{n+1})(u_{(k)}^{n+1} - \gamma)\Big), \\ f(x) &= f_{(k)}(x, t^{n+1}) = \alpha u_{(k)}^{n+1}\frac{\partial u_{(k)}^{n+1}}{\partial x} + \beta\left((1 - u_{(k)}^{n+1})(u_{(k)}^{n+1} - \gamma)u_{(k)}^{n+1} + (u_{(k)}^{n+1} - \gamma)u_{(k)}^{n+1}\right) \\ &\quad + \beta\left(-(1 - u_{(k)}^{n+1})u_{(k)}^{n+1} - (1 - u_{(k)}^{n+1})(u_{(k)}^{n+1} - \gamma)\right). \end{aligned}$$

Further, we assume that the functions $a(x), b(x)$ and $f(x)$ are sufficiently smooth functions in the spatial direction with

$$a(x) \geq \eta > 0, \qquad x \in \bar{\Omega}_x, \tag{3.2a}$$

$$b(x) \geq \delta > 0, \qquad x \in \bar{\Omega}_x. \tag{3.2b}$$

These conditions ensure that the boundary layer is located at $x = 1$ and also ensure the uniqueness of the solution [3]. Thus, by using quasilinearization process, we get the linear boundary value problem (3.1) for the function $\bar{u}^{n+1} = u_{(k+1)}^{n+1}$ and in lieu of solving the original non-linear problem (2.1), we will solve the sequence of second order singularly perturbed linear elliptic problems (3.1), for $k = 0, 1, 2, \ldots$ and $n \geq 0$ by using monotone finite difference operator, which is introduced in the next section.

Theorem 3.1. (Convergence of quasilinearization process). *Let $\langle u_{(k)}^{n+1}\rangle_{k=0}^{\infty}$ be the sequence produced by quasilinearization technique at $(n+1)$th time level. Then there exists a constant $C > 0$, independent of k, such that*

$$\left\|u_{(k+1)}^{n+1} - u_{(k)}^{n+1}\right\|_{\bar{\Omega}_x} \leq C\left\|u_{(k)}^{n+1} - u_{(k-1)}^{n+1}\right\|_{\bar{\Omega}_x}^2,$$

i.e., the quasilinearization process converges quadratically.

4. A PRIORI ESTIMATES FOR SPATIAL DISCRETIZATION

In this section, bounds for the solution of the semi-discretized problem (3.1) and its derivatives are derived. Further, we analyse the asymptotic behavior of the solution and obtain bounds for the smooth and singular components of the solution separately. These bounds are used in the convergence analysis of the totally discrete scheme.

Lemma 4.1. *If $\bar{u}^{n+1}(x)$ is the solution of the problem* (3.1)*, then $\forall\varepsilon > 0$, there exists a constant C such that*

$$\left\| \frac{\partial^i \bar{u}^{n+1}}{\partial x^i} \right\|_{\bar{\Omega}_x} \leq C\left(1 + \varepsilon^{-i} \exp\left(\frac{-\eta(1-x)}{\varepsilon}\right)\right), \qquad 0 \leq i \leq 4.$$

Proof. The proof follows inductively by differentiating the problem (3.1) with respect to x up to $i = 4$ and using the technique of Kellogg and Tsan [4].

In order to obtain more precise error estimates, we decompose the solution $\bar{u}^{n+1}(x)$ into smooth and singular components at the $(n+1)$-th time step.

Theorem 4.1. *Assume that the solution $\bar{u}^{n+1}(x)$ of the semi-discretized boundary value problem* (3.1) *is decomposed into regular and singular components as*

$$\bar{u}^{n+1}(x) = v^{n+1}(x) + w^{n+1}(x), \qquad \forall x \in \bar{\Omega}_x.$$

Then for all non-negative integer i such that $0 \leq i \leq 4$, the regular component $v^{n+1}(x)$ satisfies

$$\left\| \frac{\partial^i v^{n+1}}{\partial x^i} \right\|_{\bar{\Omega}_x} \leq C\left(1 + \varepsilon^{(3-i)} \exp\left(\frac{-\eta(1-x)}{\varepsilon}\right)\right),$$

and the singular component $w^{n+1}(x)$ satisfies

$$\left\| \frac{\partial^i w^{n+1}}{\partial x^i} \right\|_{\bar{\Omega}_x} \leq C\left(\varepsilon^{-i} \exp\left(\frac{-\eta(1-x)}{\varepsilon}\right)\right).$$

Proof. Follows by the same way given in Kadalbajoo & Gupta [5].

5. SPATIAL DISCRETIZATION

In this section, we construct the totally discrete scheme using a monotone difference operator on Shishkin mesh in the spatial direction. Shishkin mesh condense large number of mesh points in the boundary layer region as $\varepsilon \to 0$. Shishkin mesh is defined as follows:

5.1. **Shishkin Mesh.** For $N \geq 2^r$, where $r \geq 2$ is an integer, the piecewise uniform Shishkin mesh $\bar{\Omega}_x^N$ is designed by partitioning the spatial domain $\bar{\Omega}_x$ into two subintervals $\Omega_1 = [0, 1-\tau]$ and $\Omega_2 = (1-\tau, 1]$ such that $\bar{\Omega}_x = \Omega_1 \cup \Omega_2$. Here, transition parameter τ is defined by

$$\tau = \min\left\{\frac{1}{2}, \frac{2\varepsilon}{\eta}\log N\right\}.$$

Moreover, mesh spacing $\tilde{h}$ in spatial direction is given by

$$\tilde{h} = \begin{cases} \tilde{h}_1 = h_i = (2(1-\tau))/N, & \text{if } i = 1, 2, \ldots, N/2, \\ \tilde{h}_2 = h_i = 2\tau/N, & \text{if } i = N/2+1, \ldots, N. \end{cases} \tag{5.1}$$

Therefore, set of mesh points $\bar{\Omega}_x^N = \{x_i\}_{i=0}^N$ is given by

$$x_i = \begin{cases} (2(1-\tau)/N)i, & \text{if } i = 0, 1, 2, \ldots, N/2, \\ (1-\tau) + (2\tau/N)(i - N/2) & \text{if } i = N/2+1, \ldots, N. \end{cases} \tag{5.2}$$

Thus, when $\tau = 1/2$, the mesh is uniform, otherwise mesh condenses near the right part Γ_r of the lateral surface.

5.2. **Hybrid Finite Difference Scheme.** The monotone hybrid difference scheme is a composition of upwinding and central differencing on a special piecewise equidistant mesh in the spatial domain $\bar{\Omega}_x$. We employ the upwind finite difference operator on the coarse mesh region Ω_1 and central difference operator on the fine mesh region Ω_2, whenever the local mesh size allows us to do this without losing stability. The totally discrete approximation is considered as

$$\bar{u}_i^0 = \bar{u}^0(x_i), \qquad i = 0, 1, \ldots, N, \tag{5.3a}$$

$$\left(I + \Delta t \tilde{L}_{x,\varepsilon}^N\right) \bar{u}_i^{n+1} = g_i^n, \quad i = 1, 2, \ldots, N-1, \tag{5.3b}$$

$$\bar{u}_0^{n+1} = 0, \qquad \bar{u}_N^{n+1} = 0 \qquad n \geq 0, \tag{5.3c}$$

where, discrete linear operator $\tilde{L}_{x,\varepsilon}^N$ is defined as

$$\tilde{L}_{x,\varepsilon}^N \bar{u}_i^{n+1} = \begin{cases} \tilde{L}_{x,\varepsilon,\mathrm{up}}^N \bar{u}_i^{n+1} = \left(-\varepsilon\delta_x^2 + a_i D_x^{-1} + b_i I\right)\bar{u}_i^{n+1}, & i = 1, 2, \ldots N/2, \\ \tilde{L}_{x,\varepsilon,\mathrm{c}}^N \bar{u}_i^{n+1} = \left(-\varepsilon\delta_x^2 + a_i D_x^{0} + b_i I\right)\bar{u}_i^{n+1}, & i = N/2+1, \ldots N-1, \end{cases} \tag{5.4}$$

and

$$g_i^n = \bar{u}_i^n + \Delta t f_i, \qquad i = 1, 2, \ldots, N-1. \tag{5.5}$$

Here,

$$a_i = a(x_i), \quad b_i = b(x_i), \quad f_i = f(x_i), \quad g_i^n = g(x_i, t^n),$$

First order derivatives of $u(x,t)$ with respect to the spatial variable at the point (x_i, t^n) corresponding to forward, backward and central difference operators, are given by

$$D_x^+ u_i^n = \frac{u_{i+1}^n - u_i^n}{h_{i+1}}, \quad D_x^- u_i^n = \frac{u_i^n - u_{i-1}^n}{h_i}, \quad D_x^0 u_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{h_i + h_{i+1}},$$

respectively. We shall approximate second-order derivative at (x_i, t^n) by

$$\delta_x^2 u_i^n = \frac{1}{\bar{h}_i}\left(D_x^+ u_i^n - D_x^- u_i^n\right) \quad \text{where } \bar{h}_i = \frac{h_i + h_{i+1}}{2}.$$

Finally, after simplification, the totally discrete approximation (5.3) takes the following form

$$\bar{u}_i^0 = \bar{u}^0(x_i), \qquad i = 0, 1, \ldots, N, \tag{5.6a}$$

$$\begin{cases} p_i^- \bar{u}_{i-1}^{n+1} + p_i^c \bar{u}_i^{n+1} + p_i^+ \bar{u}_{i+1}^{n+1} &= g_i^n, \quad i = 1, 2, \ldots, N/2, \\ q_i^- \bar{u}_{i-1}^{n+1} + q_i^c \bar{u}_i^{n+1} + q_i^+ \bar{u}_{i+1}^{n+1} &= g_i^n, \quad i = N/2+1, \ldots, N-1, \end{cases} \tag{5.6b}$$

$$\bar{u}_0^{n+1} = 0, \qquad \bar{u}_N^{n+1} = 0, \qquad n \geq 0, \tag{5.6c}$$

where, elements in the system matrix $(I + \Delta t \tilde{L}_{x,\varepsilon}^N)$ are as follows

$$\begin{aligned} p_i^- &= -\left(\frac{\Delta t \varepsilon}{h_i \bar{h}_i} + \frac{\Delta t a_i}{h_i}\right), & p_i^c &= \left(1 + \Delta t b_i - p_i^- - p_i^+\right), \\ p_i^+ &= -\left(\frac{\Delta t \varepsilon}{h_{i+1} \bar{h}_i}\right), & & i = 1, 2, \ldots, N/2, \\ q_i^- &= -\left(\frac{\Delta t \varepsilon}{h_i \bar{h}_i} + \frac{\Delta t a_i}{h_i + h_{i+1}}\right), & q_i^c &= \left(1 + \Delta t b_i - q_i^- - q_i^+\right) \\ q_i^+ &= -\left(\frac{\Delta t \varepsilon}{h_{i+1} \bar{h}_i} - \frac{\Delta t a_i}{h_i + h_{i+1}}\right), & & i = N/2+1, \ldots, N-1. \end{aligned}$$

6. STABILITY AND CONVERGENCE ANALYSIS

In this section, we establish the stability and ε-uniform error estimate for the totally discrete scheme by decomposing the approximate solution $\bar{u}_i^n$ in an analogous manner as that of the continuous solution $\bar{u}^n(x)$ at nth time step. For the sake of simplicity, we denote the discrete solution $\bar{u}_i^n$ by $\bar{u}^N(x_i, t^n)$ during convergence analysis. In order to attain a monotone discrete operator $(I + \Delta t \tilde{L}_{x,\varepsilon}^N)$, we impose the following mild assumption on the minimum number of mesh points

$$\frac{\tilde{h}_2 \|a\|_{\bar{\Omega}_x}}{2\varepsilon} < 1, \quad \text{i.e.,} \quad \frac{N}{\log N} > 2\frac{\|a\|_{\bar{\Omega}_x}}{\eta}. \tag{6.1}$$

We start with stating the following discrete maximum principle.

Lemma 6.1. (Discrete Maximum Principle). *Under the assumption* (6.1), *the totally discrete scheme* (5.6) *satisfies a discrete maximum principle for any mesh function* ψ^N *defined on* $\bar{D}^N = \bar{\Omega}_x^N \times \bar{\Omega}_t^n$ *such that if* $\psi^N(x_i, t^n) \geq 0$, $\forall (x_i, t^n) \in \Gamma^N$ *and* $(I + \Delta t \tilde{L}_{x,\varepsilon}^N)\psi^N(x^i, t^n) \geq 0$, $\forall (x_i, t^n) \in D^N$, *then* $\psi^N(x_i, t^n) \geq 0$, $\forall (x_i, t^n) \in \bar{D}^N$.

Proof. It is easily seen that the system matrix $(I + \Delta t \tilde{L}_{x,\varepsilon}^N)$ is an $(N-1) \times (N-1)$ irreducible M-matrix and has a positive inverse. Moreover, discrete system (5.3) satisfies the desired discrete maximum principle. Discrete maximum principle ensures the stability of the spatial discretization process.

To analyse the proposed scheme in space, we split the solution into smooth and singular component and use analytical finite difference techniques consisting of truncation error bounds, discrete comparison principle and appropriate choices of discrete barrier functions.

Theorem 6.1. (Error in the Spatial Direction). *Let* $\bar{u}^N(x_i, t^n)$ *be the hybrid finite difference approximation in the spatial direction to the solution* $\bar{u}^n(x) \in C^4(\bar{\Omega}_x)$ *of the problem* (3.1) *at* n*-th time level. Then under the assumption* (6.1), *following error estimates hold for the proposed hybrid finite difference scheme in the spatial discretization process at the* n*-th time level*

$$\|(\bar{u}^N - \bar{u})(x_i, t^n)\|_{\bar{\Omega}_x^N} \leq \begin{cases} CN^{-1}, & i = 0, 1, \ldots, N/2, \quad n\Delta t \leq T, \\ CN^{-2}(\log N)^2, & i = N/2 + 1 \ldots, N, \quad n\Delta t \leq T. \end{cases}$$

Proof. follows from the error estimates of smooth and singular components.

Theorem 6.2. (Error in the Totally Discrete Scheme). *Let* $u(x, t)$ *be the continuous solution of the Burger-Huxley equation* (1.1), $\bar{u}^n(x)$ *be the solution of the semi-discrete problem* (3.1) *after the temporal discretization and quasilinearization process and* $\bar{u}^N(x_i, t^n)$ *be the solution of the totally discrete problem* (5.3), *then under the assumption* (6.1) *following error estimates satisfy for the totally discrete scheme*

$$\|(\bar{u}^N - u)(x_i, t^n)\|_{\bar{D}^N} \leq \begin{cases} C\left(\Delta t + N^{-1}\right), & i = 0, 1, \ldots, N/2, \quad n\Delta t \leq T, \\ C\left(\Delta t + N^{-2}(\log N)^2\right), & i = N/2 + 1 \ldots, N, \quad n\Delta t \leq T. \end{cases}$$

where C *is a positive constant independent of* ε *and mesh parameters.*

Proof. The proof easily follows by combining the estimates given in Lemma 2.2 and Theorem 6.1.

7. NUMERICAL EXPERIMENTS AND RESULTS

In this section, to demonstrate applicability, accuracy and the convergence order of the method presented in this paper, we report some numerical results for both the modified Burgers' and Burgers-Huxley equation. Range $(0, 1]$ of the parameter ε shows our interest in the singularly perturbrd case.

Example 1. This example corresponds to the following singularly-perturbed non-linear parabolic initial-boundary value problem:

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \qquad (x, t) \in D, \tag{7.1a}$$

with sinusoidal initial condition

$$u(x, 0) = \sin(\pi x), \qquad x \in \bar{\Omega}_x, \tag{7.1b}$$

and boundary conditions

$$u(0, t) = 0, \quad u(1, t) = 0, \quad t \in \bar{\Omega}_t. \tag{7.1c}$$

For the numerical computation, we begin with $N = 16, T = 1$ and $\Delta t = 0.1$ and we multiply N by 2 and divide Δt by 2. For small values of the parameter ε, the exact solution of the modified Burgers' equation is not available, therefore to illustrate the performance of the proposed scheme at low viscosity coefficient ε, we use the double mesh principle to estimate the pointwise error as follows

$$e_\varepsilon^{N,\Delta t}(x_i, t^n) = |u^N(x_i, t^n) - u^{2N}(x_i, t^n)|. \tag{7.2}$$

where the superscript N denotes the number of mesh points in the spatial direction, $t^n = n\Delta t$ and Δt is the time step. For each ε, the maximum nodal error is given by

$$E_\varepsilon^{N,\Delta t} = \max_{i,n} e_\varepsilon^{N,\Delta t}, \tag{7.3}$$

and, for each N and Δt, the ε-uniform maximum pointwise error is define by

$$E^{N,\Delta t} = \max_{\varepsilon} E_\varepsilon^{N,\Delta t}. \tag{7.4}$$

We also tabulate the numerical rate of convergence in the following way

$$p_\varepsilon^{N,\Delta t} = \frac{\log(E_\varepsilon^{N,\Delta t}/E_\varepsilon^{2N,\Delta t/2})}{\log 2}. \tag{7.5}$$

The numerical ε-uniform order of convergence is given by

$$p^{N,\Delta t} = \frac{\log(E^{N,\Delta t}/E^{2N,\Delta t/2})}{\log 2}. \tag{7.6}$$

Numerical results are tabulated in Table 1 with piecewise uniform Shishkin mesh for various values of ε.

Example 2. This example corresponds to the following Burgers-Huxley equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \varepsilon \frac{\partial^2 u}{\partial x^2} = (1-u)(u-0.5)u, \qquad (x, t) \in D, \tag{7.7a}$$

with sinusoidal initial condition

$$u(x, 0) = \sin(\pi x), \qquad x \in \bar{\Omega}_x, \tag{7.7b}$$

TABLE 1. Maximum pointwise errors $E_\varepsilon^{n,\Delta t}$ and numerical order of convergence $p_\varepsilon^{N,\Delta t}$ for Example 1 with Shishkin mesh

$\varepsilon \downarrow$	N=16	N=32	N=64	N=128	N=256
2^0	8.4807E-6	5.4448E-7	1.4570E-7	3.4354E-8	1.95E-8
	3.9612	1.9019	2.0844	0.8170	
2^{-2}	1.1621E-3	6.3448E-4	3.2706E-4	1.6858E-4	8.5102E-5
	0.8731	0.9560	0.9562	0.9862	
2^{-6}	1.7164E-2	5.9542E-3	2.8138E-3	8.7279E-4	3.4687E-4
	1.5274	1.0814	1.6888	1.3312	
2^{-10}	3.6458E-2	1.7454E-2	5.3819E-3	2.4288E-3	1.0921E-3
	1.0627	1.6973	1.1479	1.1531	
2^{-14}	2.9973E-2	1.4676E-2	5.7897E-3	2.6373E-3	1.1868E-3
	1.0302	1.3419	1.1344	1.1520	
2^{-18}	2.9574E-2	1.3756E-2	5.8260E-3	2.5338E-3	1.1967E-3
	1.1042	1.2395	1.2012	1.0822	
2^{-22}	2.9547E-2	1.3699E-2	5.8759E-3	2.5063E-3	1.1640E-3
	1.1090	1.2212	1.2293	1.1065	
2^{-24}	2.9546E-2	1.3696E-2	5.9676E-3	2.5047E-3	1.1623E-3
	1.1092	1.1985	1.2525	1.1076	
$\mathbf{E^{N,\Delta t}}$	**3.9275E-2**	**1.7481E-2**	**5.9676E-3**	**2.6373E-3**	**1.2999E-3**
$\mathbf{p^{N,\Delta t}}$	**1.1678**	**1.5506**	**1.1781**	**1.0207**	

and boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t \in \bar{\Omega}_t. \tag{7.7c}$$

For the numerical computation, we use the double mesh principle to estimate the error as given in Example 1. Numerical results with the parameters $\alpha = 1, \beta = 1$ and $\gamma = 0.5$, are tabulated in Table 2 with piecewise uniform Shishkin mesh for various values of ε.

8. DISCUSSIONS AND CONCLUSIONS

A numerical scheme has been developed to solve modified Burgers' and Burgers-Huxley equation. The qualitative aspects of the modified Burgers' equation have been studied by means of singular perturbation theory. At small values of ε, these problems produce a sharp gradient in the boundary layer region with a smooth initial data, when the Dirichlet boundary condition is employed. Taking more mesh points in the boundary layer can lead to an outstanding result for a much larger value of R, therefore we have used a special piecewise uniform Shishkin mesh for its simple structure. To tackle the non-linearity, quasilinearization process is used and shown

TABLE 2. Maximum pointwise errors $E_\varepsilon^{n,\Delta t}$ and numerical order of convergence $p_\varepsilon^{N,\Delta t}$ for Example 2 with the parameters $\alpha = 1, \beta = 1$ and $\gamma = 0.5$ on Shishkin mesh

$\varepsilon \downarrow$	N=16	N=32	N=64	N=128	N=256
2^0	6.8456E-6	4.0455E-7	1.5834E-7	2.4868E-8	1.3719E-8
	4.0808	1.3533	2.6707	0.8581	
2^{-2}	1.0777E-3	5.8450E-4	3.0298E-4	1.5405E-4	7.7650E-5
	0.8827	0.9480	0.9758	0.9883	
2^{-6}	1.7859E-2	6.4179E-3	2.4857E-3	9.9519E-4	3.9756E-4
	1.4781	1.3684	1.3206	1.3238	
2^{-10}	3.7477E-2	1.8198E-2	6.8745E-3	3.0270E-3	1.4537E-3
	1.0422	1.4045	1.1834	1.0582	
2^{-14}	3.0955E-2	1.5729E-2	7.4116E-3	3.3446E-3	1.5994E-3
	0.9768	1.0855	1.1479	1.0643	
2^{-18}	3.0527E-2	1.4822E-2	7.4581E-3	3.3622E-3	1.6105E-3
	1.0423	0.9909	1.1494	1.0619	
2^{-22}	3.0499E-2	1.5175E-2	7.4989E-3	3.3634E-3	1.6113E-3
	1.0071	1.0170	1.1568	1.0617	
2^{-24}	3.0499E-2	1.5327E-2	7.5517E-3	3.3634E-3	1.6113E-3
	0.9926	1.0213	1.1668	1.0617	
$\mathbf{E^{N,\Delta t}}$	**4.0948E-2**	**1.8548E-2**	**7.5517E-3**	**3.3634E-3**	**1.6113E-3**
$\mathbf{p^{N,\Delta t}}$	**1.1425**	**1.2964**	**1.1668**	**1.0617**	
$\mathbf{E^{N,\Delta t}}$ [6]	**4.0835E-2**	**2.2530E-2**	**1.1907E-2**	**6.128E-3**	**3.110E-3**

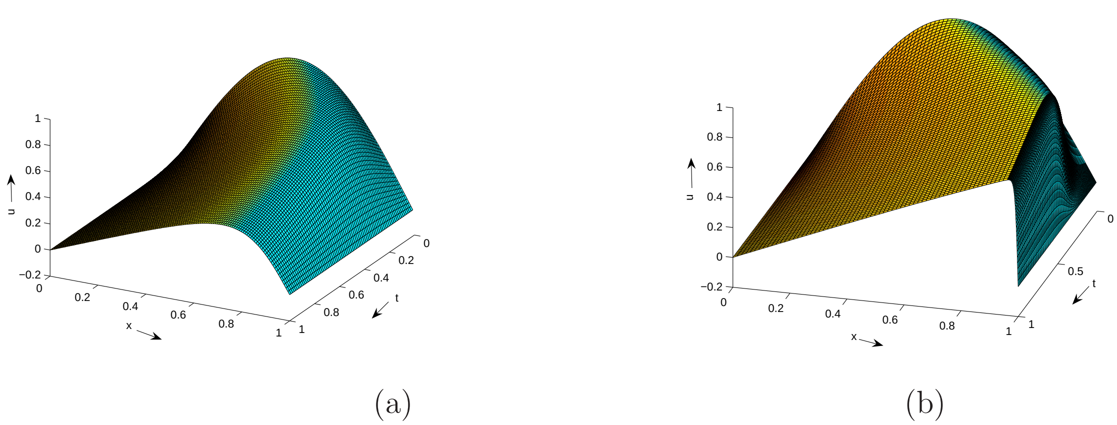

FIGURE 1. Numerical solution profiles of Example-1 with $N = 128$ and $\Delta t = 1/80$ and different values of ε (a) $\varepsilon = 2^{-4}$, and (b) $\varepsilon = 2^{-8}$.

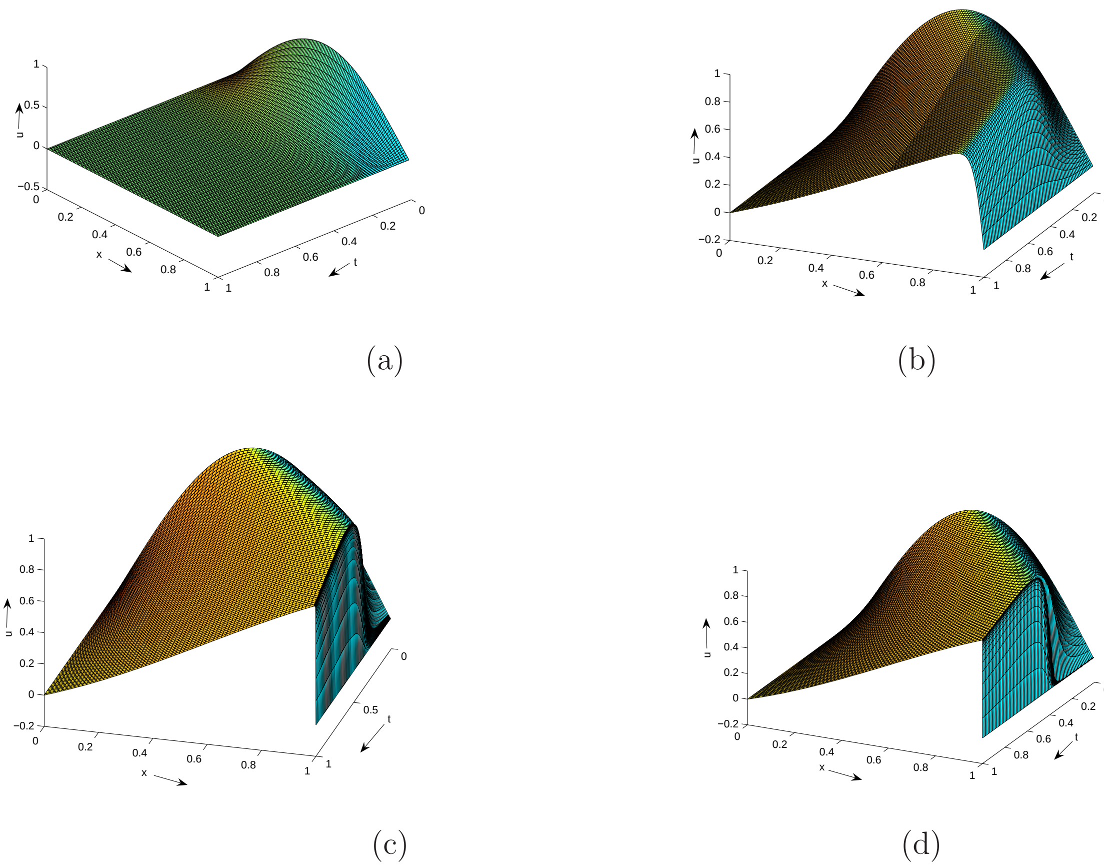

FIGURE 2. Numerical solution profiles of Example-2 with $N = 128$ and $\Delta t = 1/80$ and different values of ε (a) $\varepsilon = 2^0$, (b) $\varepsilon = 2^{-6}$, (c) $\varepsilon = 2^{-12}$ and (d) $\varepsilon = 2^{-24}$.

that the quasilinearization process converges quadratically to the solution of the original non-linear problem. A brief analysis has been carried out to prove the uniform convergence of the proposed scheme and show the parameter free linear convergence in the temporal direction and first order uniform convergence in the region Ω_1 outside from the boundary layer region and almost quadratic uniform convergence in the boundary layer region Ω_2 for the spatial variable. ε-uniform error estimate for simple upwind scheme is bounded by $N^{-1}(\log N)^2$(see [6]), whereas for the proposed hybrid monotone difference operator, error estimate is bounded by N^{-1} in spatial domain $\bar{\Omega}_x$ with Shishkin mesh. Thus hybrid finite difference method in spatial direction has superior convergence properties than simple upwinding [6], but is of same computational cost.

The numerical accuracy of the present scheme is tested at low viscosity coefficient ε for both the problems and the results are presented in Table 1 and Table 2. Numerical results show that for a fixed value of ε, pointwise errors and maximal nodal errors decrease as the number of mesh points increases. We observe that the computational order of local convergence are in good agreement with the theoretical estimates. Numerical results for Burgers-Huxley equation are also compared with the scheme proposed by Kaushik and Sharma [6]. From the numerical solution profiles given in Figures 1, 2, we observe that the propagation front is steeper in the neighborhood of Γ_r, the right part of the lateral surface for the small values of the parameter ε, which validates the physical behavior of the solution.

Thus the present method works nicely for both the singularly perturbed modified Burgers' and Burgers-Huxley equauions and the numerical results support the theoretical predictions and exhibit good physical behavior. The performance of the proposed scheme is investigated by comparing the results and observed that the accuracy in the numerical results is comparable and better to those by existing methods. The technique presented in this paper may also be applicable to the construction and study of ε-uniform direct numerical methods for more complicated non-linear problems.

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