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PRETRIGONOMETRIC AND PREHYPERBOLIC FUNCTIONS VIA LAPLACE TRANSFORMS

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ABSTRACT. Present paper computes the Laplace transforms of pretrigonometric, prehyperbolic and extended pretrigonometric and prehyperbolic functions. Then obtain the prefunctions by solving initial value problems for second order non-homogeneous differential equations using Laplace Transforms. In the later part considered the matrix differential equations whose solutions leads to extended matrix trigonometric functions.

Key words: Pretrigonometric and prehyperbolic functions, extended pretrigonometric and prehyperbolic functions

1. INTRODUCTION

In [2] we have defined preexponential, pretrigonometric and prehyperbolic functions and studied some properties. The aim of present paper is to obtain Laplace transforms of these functions. It is known that the method of Laplace transforms helps us to solve homogeneous and non-homogeneous ordinary differential equations. We now extend this method to get the solutions of differential equations of the form $X^n(t,\alpha) + X(t,\alpha) = f(t,\alpha)$ with initial conditions and obtain prefunctions as solutions. At the end we use Laplace transform method to solve matrix differential equations with suitable initial conditions and obtain matrix functions as solution.

2. DEFINITIONS OF PREFUNCTIONS

1. The preexponential function is denoted by $\operatorname{pexp}(t,\alpha)$ and is defined as

$$\operatorname{pexp}(t,\alpha) = 1 + \sum_{n=1}^{\infty} \frac{t^{n+\alpha}}{\Gamma(n+1+\alpha)}. \tag{2.1}$$

for any real number t and for any $\alpha \geq 0$, α being a parameter. The preexponential function $\operatorname{pexp}(-t,\alpha)$ is defined as

$$\operatorname{pexp}(-t,\alpha) = 1 + (-1)^{\alpha} \sum_{n=1}^{\infty} (-1)^n \frac{t^{n+\alpha}}{\Gamma(n+1+\alpha)}, \tag{2.2}$$

for $0 \leq t < \infty$ and for any $\alpha \geq 0$

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2. The prefunction $\mathrm{pcos}(t,\alpha)$ of trigonometric function $\cos t$ is defined as

$$\mathrm{pcos}(t,\alpha) = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{t^{2n+\alpha}}{\Gamma(2n+1+\alpha)} \quad \text{for } t \in R \text{ and } \alpha \geq 0. \tag{2.3}$$

and prefunction $\mathrm{psin}(t,\alpha)$ of trigonometric function *sint* is defined as

$$\mathrm{psin}(t,\alpha) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1+\alpha}}{\Gamma(2n+2+\alpha)} \quad \text{for } t \in R \text{ and } \alpha \geq 0. \tag{2.4}$$

3. The prefunction $\mathrm{pcosh}(t,\alpha)$ of hyperbolic function $\cosh t$ is defined as

$$\mathrm{pcosh}(t,\alpha) = 1 + \sum_{n=1}^{\infty} \frac{t^{2n+\alpha}}{\Gamma(2n+1+\alpha)} \quad \text{for } t \in R \text{ and } \alpha \geq 0. \tag{2.5}$$

and prefunction $\mathrm{psinh}(t,\alpha)$ of hyperbolic function *sinht* is defined as

$$\mathrm{psinh}(t,\alpha) = \sum_{n=0}^{\infty} \frac{t^{2n+1+\alpha}}{\Gamma(2n+2+\alpha)} \quad \text{for } t \in R \text{ and } \alpha \geq 0. \tag{2.6}$$

Series (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6) are absolutely convergent.

3. DEFINITIONS OF EXTENDED PREFUNCTIONS

In this section we define extended pretrigonometric and prehyperbolic functions. Trisecting the series (2.2) we form three infinite series absolutely convergent for $t \in R$ and $\alpha \geq 0$. The extended pretrigonometric functions are defined by

$$pM_{3,0}(t,\alpha) = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{t^{3n+\alpha}}{\Gamma(3n+1+\alpha)}, \tag{3.1}$$

$$pM_{3,1}(t,\alpha) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{3n+1+\alpha}}{\Gamma(3n+2+\alpha)}, \tag{3.2}$$

$$pM_{3,2}(t,\alpha) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{3n+2+\alpha}}{\Gamma(3n+3+\alpha)}. \tag{3.3}$$

Trisecting the series (2.1) we form three infinite series absolutely convergent for $t \in R$ and $\alpha \geq 0$. The extended prehyperbolic functions are defined by

$$pN_{3,0}(t,\alpha) = 1 + \sum_{n=1}^{\infty} \frac{t^{3n+\alpha}}{\Gamma(3n+1+\alpha)}, \tag{3.4}$$

$$pN_{3,1}(t,\alpha) = \sum_{n=0}^{\infty} \frac{t^{3n+1+\alpha}}{\Gamma(3n+2+\alpha)}, \tag{3.5}$$

$$pN_{3,2}(t,\alpha) = \sum_{n=0}^{\infty} \frac{t^{3n+2+\alpha}}{\Gamma(3n+3+\alpha)}. \tag{3.6}$$

Series (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6) are absolutely convergent.

4. LAPLACE TRANSFORMS OF PREFUNCTIONS

In view of the relations for $\mathrm{pcos}(t,n)$, $\mathrm{psin}(t,n)$, $\mathrm{pcosh}(t,n)$, $\mathrm{psinh}(t,n)$ for an integer $n > 0$, which have been discussed in [2], it is clearly seen that the Laplace transform of all such prefunctions can be obtained and as applications can be employed in solving differential equation involving preexponential, pretrigonometric and prehyperbolic functions. We obtain Laplace transform of these functions by following the definition.

We find Laplace transform of $\mathrm{pexp}(at,\alpha)$ $a > 0$.

$$\mathrm{pexp}(at,\alpha) = 1 + \sum_{1}^{\infty} \frac{(at)^{n+\alpha}}{\Gamma(n+1+\alpha)}$$

Clearly,

$$\begin{aligned} L\{\mathrm{pexp}(at,\alpha)\} &= \int_0^\infty e^{-st}\,\mathrm{pexp}(at,\alpha)dt \\ &= \int_0^\infty e^{-st}\left\{1 + \sum_{n=1}^{\infty} \frac{(at)^{n+\alpha}}{\Gamma(n+1+\alpha}\right\}dt \\ &= \int_0^\infty e^{-st} 1\,dt + \sum_{n=1}^{\infty} \frac{1}{\Gamma(n+1+\alpha)} \int_0^\infty e^{-st}(at)^{n+\alpha}dt \\ &= \frac{1}{s} + \sum_{n=1}^{\infty} \frac{1}{\Gamma(n+1+\alpha)} \frac{a^{n+\alpha}\Gamma(n+1+\alpha)}{s^{n+1+\alpha}} \\ &= \frac{1}{s} + \sum_{n=1}^{\infty} \frac{a^{n+\alpha}}{s^{n+1+\alpha}} \\ &= \frac{1}{s} + \frac{a^{1+\alpha}}{(s-a)s^{1+\alpha}} \end{aligned}$$

When $a = 1$,

$$L\{\mathrm{pexp}(t,\alpha)\} = \frac{1}{s} + \frac{1}{(s-1)s^{1+\alpha}}$$

Further when $\alpha = 0$, $L\{\mathrm{pexp}(t,0)\} = L\{\exp t\} = \frac{1}{s-1}$. Clearly, $L^{-1}\left\{\frac{1}{s} + \frac{a^{1+\alpha}}{(s-a)s^{1+\alpha}}\right\} = \mathrm{pexp}(at,\alpha)$

Laplace transforms of remaining prefunctions can be computed similarly, we omit the details. We list below the Laplace transforms of all the prefunctions.

1. $$L\{\mathrm{pexp}(-at,\alpha)\} = \frac{1}{s} - \frac{(-1)^\alpha a^{1+\alpha}}{(s+a)s^{1+\alpha}}$$

2. $$L\{\mathrm{pcos}(at,\alpha)\} = \frac{1}{s} - \frac{a^{\alpha+2}}{(s^2+a^2)s^{1+\alpha}}$$

3. $$L\{\mathrm{psin}(at,\alpha)\} = \frac{a^{1+\alpha}}{(s^2+a^2)s^{\alpha}}$$

4. $L\{\text{pcosh}(at,\alpha)\} = \frac{1}{s} + \frac{a^{\alpha+2}}{(s^2-a^2)s^{1+\alpha}}$

5. $L\{\text{psinh}(at,\alpha)\} = \frac{a^{1+\alpha}}{(s^2-a^2)s^{\alpha}}$

6. $L\{pM_{30}(at,\alpha)\} = \frac{1}{s} - \frac{a^{3+\alpha}}{(s^3+a^3)s^{1+\alpha}}$

7. $L\{pM_{31}(at,\alpha)\} = \frac{a^{1+\alpha}}{(s^3+a^3)s^{-1+\alpha}}$

8. $L\{pM_{32}(at,\alpha)\} = \frac{a^{2+\alpha}}{(s^3+a^3)s^{\alpha}}$

9. $L\{pN_{30}(at,\alpha)\} = \frac{1}{s} + \frac{a^{3+\alpha}}{(s^3-a^3)s^{1+\alpha}}$

10. $L\{pN_{31}(at,\alpha)\} = \frac{a^{1+\alpha}}{(s^3-a^3)s^{\alpha-1}}$

11. $L\{pN_{32}(at,\alpha)\} = \frac{a^{2+\alpha}}{(s^3-a^3)s^{\alpha}}$

Note that by substituting $a = 1$ and $\alpha = 0$ in the above results we get Laplace transforms of exponential, trigonometric and hyperbolic functions.

5. PREFUNCTIONS AS SOLUTION OF INITIAL VALUE PROBLEMS (IVP)

The infinite series expansion of preexponential, pretrigonometric and prehyperbolic functions is given in sections 2 and 3. We have obtained Laplace transforms of these functions by following definition. Below we consider a sequence of differential equations of the type $X^n(t,\alpha) + X(t,\alpha) = f(t,\alpha)$, $n = 1, 2, \ldots$ with suitable initial conditions. It is interesting to note that the solutions of these IVP's are prefunctions.

1. Consider the following IVP for first order nonhomogeneous differential equation

$$x'(t,\alpha) - x(t,\alpha) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} - 1 \tag{5.1}$$

together with the initial conditions

$$x(0,\alpha) = 1, x'(0,\alpha) = 0 \tag{5.2}$$

We take the Laplace transform of both sides of the equation (5.1) to get

$$L\{x'(t,\alpha)\} - L\{x(t,\alpha)\} = L\left\{\frac{t^{\alpha}}{\Gamma(1+\alpha)} - 1)\right\}.$$

i.e.

$$sL\{x(t,\alpha)\} - x(0,\alpha) - L\{x(t,\alpha)\} = \frac{1}{s^{1+\alpha}} - \frac{1}{s}$$

i.e.

$$(s-1)L\{x(t,\alpha)\} = \left\{\frac{1}{s^{1+\alpha}} - \frac{1}{s} + 1\right\}$$

Simplifying we get,

$$L\{x(t,\alpha)\} = \frac{1}{s} + \frac{1}{(s-1)s^{1+\alpha}}$$

$$x(t,\alpha) = L^{-1}\left\{\frac{1}{s} + \frac{1}{(s-1)s^{1+\alpha}}\right\} = \text{pexp}(t,\alpha)$$

Preexponential function $\text{pexp}(t,\alpha)$ is the solution of IVP (13) - (14).

2. Consider the following IVP for second order nonhomogeneous differential equation

$$x''(t,\alpha) + x(t,\alpha) = 1 - \frac{t^\alpha}{\Gamma(1+\alpha)} \tag{5.3}$$

with the initial conditions

$$x(0,\alpha) = 1, \qquad x'(0,\alpha) = 0 \tag{5.4}$$

Taking the Laplace transform of both sides of the equation (5.3) to get

$$L\{x''(t,\alpha)\} + L\{x(t,\alpha)\} = L\left\{1 - \frac{t^\alpha}{\Gamma(1+\alpha)})\right\}.$$

$$s^2L\{x(t,\alpha)\} - sx(0,\alpha) - x'(0,\alpha) + L\{x(t,\alpha)\} = L\left\{1 - \frac{t^\alpha}{\Gamma(1+\alpha)}\right\}$$

Simplifying we get,

$$L\{x(t,\alpha)\} = \frac{1}{s} - \frac{1}{(s^2+1)s^{\alpha+1}}$$

$$x(t,\alpha) = L^{-1}\left\{\frac{1}{s} - \frac{1}{s^{\alpha+1}(s^2+1)}\right\} = \text{pcos}(t,\alpha)$$

Preexponential function $\text{pcos}(t,\alpha)$ is the solution of IVP (15)–(16).

Note that the solution of IVP

$$x''(t,\alpha) + x(t,\alpha) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}; \quad x(0,\alpha) = 0, \quad x'(0,\alpha) = 0$$

is $\text{psin}(t,\alpha)$.

3. Next we consider the IVP for third order nonhomogeneous differential equation

$$x'''(t,\alpha) + x(t,\alpha) = 1 - \frac{t^\alpha}{\Gamma(1+\alpha)} \tag{5.5}$$

with the initial conditions

$$x(0,\alpha) = 1, \quad x'(0,\alpha) = 0, \quad x''(0,\alpha) = 0 \tag{5.6}$$

Taking the Laplace transform of both sides of the equation (5.5) to get

$$L\{x'''(t,\alpha)\} + L\{x(t,\alpha)\} = L\left\{1 - \frac{t^\alpha}{\Gamma(1+\alpha)}\right\}.$$

i.e. $s^3L\{x(t,\alpha)\} - s^2x(0,\alpha) - sx'(0,\alpha) - x''(0,\alpha) + L\{x(t,\alpha) = \frac{1}{s} - \frac{1}{s^{1+\alpha}}$

Simplifying we get,

$$L\{x(t,\alpha)\} = \frac{1}{s} - \frac{1}{(s^3+1)s^{1+\alpha}}$$

$$x(t,\alpha) = L^{-1}\left\{\frac{1}{s} - \frac{1}{(s^3+1)s^{1+\alpha}}\right\} = M_{30}(t,\alpha)$$

Extended Pretrigonometric function $M_{30}(t,\alpha)$ is the solution of IVP (17)–(18).

We state below Initial Value Problems having solutions prefunctions and these can be solved on similar lines.

A. $$x'''(t,\alpha) + x(t,\alpha) = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)};\quad x(0,\alpha) = x'(0,\alpha) = x''(0,\alpha) = 0$$

$x(t,\alpha) = pM_{31}(t,\alpha)$ is the solution.

B. $$x'''(t,\alpha) + x(t,\alpha) = \frac{t^{\alpha-1}}{\Gamma(\alpha)};\qquad x(0,\alpha) = x'(0,\alpha) = x''(0,\alpha) = 0$$

$x(t,\alpha) = pM_{32}(t,\alpha)$ is the solution.

C. $$x''(t,\alpha) - x(t,\alpha) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} - 1;\qquad x(0,\alpha) = 1, x'(0,\alpha) = 0$$

$x(t,\alpha) = \text{pcosh}(t,\alpha)$ is the solution.

D. $$x''(t,\alpha) - x(t,\alpha) = \frac{t^{\alpha-1}}{\Gamma(\alpha)};\qquad x(0,\alpha) = 0, x'(0,\alpha) = 0$$

$x(t,\alpha) = \text{psinh}(t,\alpha)$ is the solution.

E. $$x'''(t,\alpha) - x(t,\alpha) = \frac{t^{\alpha}}{\Gamma(1+\alpha)};\qquad x(0,\alpha) = 1, x'(0,\alpha) = x''(0,\alpha) = 0$$

$x(t,\alpha) = pN_{30}(t,\alpha)$ is the solution.

F. $$x'''(t,\alpha) - x(t,\alpha) = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)};\qquad x(0,\alpha) = x'(0,\alpha) = x''(0,\alpha) = 0$$

$x(t,\alpha) = pN_{31}(t,\alpha)$ is the solution.

G. $$x'''(t,\alpha) - x(t,\alpha) = \frac{t^{\alpha-1}}{\Gamma(\alpha)};\qquad x(0,\alpha) = x'(0,\alpha) = x''(0,\alpha) = 0$$

$x(t,\alpha) = pN_{32}(t,\alpha)$ is the solution.

6. MATRIX LAPLACE TRANSFORM TECHNIQUES

In this section we extend the definition of Laplace transform to matrix-valued functions and apply this extension to solve matrix differential equations. We consider matrix differential equations whose solutions leads to matrix trigonometric or extended matrix trigonometric functions. Consider the matrix function $X(t) = [x_{ij}(t)]$ of order $m \times n$. Note that m may be equal to n. All elements of the matrix $X(t)$ are integrable functions on R. The Laplace transform of $X(t)$ is

$$L\left[X(t)\right] = \int_0^\infty X(t)e^{-st}dt \tag{6.1}$$

Observe that Laplace transform of each element exists. In short, the Laplace transform of matrix $X(t)$ is defined as the Laplace transform of each element of the matrix $X(t)$. If Laplace transform of each element exists then we say $X(t)$ is Laplace transformable. Next, we have the following result which gives us the linear property of the Laplace transform.

Linear Property: Let $A = [a_{ij}]$ be a constant matrix of order n and $B = [b_{ij}(t)]$ be matrix function of order n then $L\{A[B(t)]\} = AL\{B(t)\}$

$$L\left[A.B(t)\right] = \left[L\left\{\sum_{k=1}^{n} a_{ik}b_{kj}(t)\right\}\right] = \left[\sum_{k=1}^{n} a_{ik}L\{b_{kj}(t)\}\right]$$

It follows that

$$L\{A \cdot Bt\} = A \cdot L[B(t)]. \tag{6.2}$$

Applications of the Laplace Transform enables us to replace the operations of calculus by simple algebraic operations. We have the following result.

Theorem 6.1. *Suppose that $X(t)$ defined by (6.1) is continuous for $t \geq 0$ and each element satisfy the condition that there exists a positive constant M, a number $\boldsymbol{a}$ and a finite number t_o such that $|X_{i,j}(t)| < Me^{at}$, for $t > t_o$, for all i, j and suppose that the derivatives of elements are continuous for all $t \geq 0$ then the following relation is true [3].*

$$L[X^{(n)}(t)] = s^n L[X(t)] - s^{n-1}X(0) - \cdots - X^{n-1}(0) \tag{6.3}$$

7. MATRIX DIFFERENTIAL EQUATIONS

In this section we use the Laplace transform method to solve the following matrix differential equation with suitable initial conditions.

1. Consider

$$X''(t) = -A^2X(t); \quad X(0) = I, \quad X'(0) = O \tag{7.1}$$

A is a constant matrix of order n.

Taking Laplace transform of matrix equation (7.1),

$$s^2L\{X(t)\} - sX(0) - X'(0) = -A^2L\{X(t)\}$$
$$(s^2I + A^2)L\{X(t)\} = s$$
$$L\{X(t)\} = \frac{s}{s^2I + A^2}$$

$s \in R$ except for the eigenvalues of A and $\det(s^2I + A^2) \neq 0$.

Using the series expansion of the matrix , we get

$$L\{X(t)\} = \frac{s}{s^2I + A^2} = \frac{1}{s}\left(I + \frac{A^2}{s^2}\right)^{-1} = \frac{I}{s} - \frac{A^2}{s^3} + \frac{A^4}{s^5} - \cdots$$

$$X(t) = L^{-1}\left\{\frac{I}{s} - \frac{A^2}{s^3} + \frac{A^4}{s^5} - \cdots\right\} = I - \frac{A^2t^2}{2!} + \frac{A^4t^4}{4!} - \frac{A^6t^6}{6!}\cdots = \cos(At)$$

$X(t) = \cos(At)$ is a solution of matrix differential equation (7.1)

We state below IVP having $X(t) = \sin(At)$ as it solution.

(7.2) $$X''(t) = -A^2X(t); \quad X(0) = I, \quad X'(0) = A$$

A is a constant matrix of order n.

2. Next consider the following IVP for third order matrix differential equation.

(7.3) $$X'''(t) = -A^3X(t); \quad X(0) = I, \quad X'(0) = O, X''(0) = O$$

A is a constant matrix of order n. Taking Laplace transform of matrix equation (7.3),

$$s^3L\{X(t)\} - s^2X(0) - sX'(0) - X''(0) = -A^3L\{X(t)\}$$
$$(s^3I + A^3)L\{X(t)\} = s^2$$
$$L\{X(t)\} = \frac{s^2}{s^3I + A^3}$$

$s \in R$ except for the eigenvalues of A and s is such that $\det(s^3I + A^3) \neq 0$. Using the series expansion of matrix, we get

$$L\{X(t)\} = \frac{s^2}{s^3I + A^3} = \frac{I}{s} - \frac{A^3}{s^4} + \frac{A^6}{s^7} - \cdots$$

$$X(t) = L^{-1}\left\{\frac{I}{s} - \frac{A^3}{s^4} + \frac{A^6}{s^7} - \cdots\right\} = I - \frac{A^3t^3}{3!} + \frac{A^6t^6}{6!} - \frac{A^9t^9}{9!}\cdots = M_{30}(At)$$

Extended matrix trigonometic function $X(t) = M_{30}(At))$ is a solution of matrix differential equation (7.3)

$X(t) = M_{31}(At))$ is a solution of IVP below.

(7.4) $$X'''(t) = -A^3X(t); \quad X0) = O, X'(0) = A, \quad X''(0) = O$$

A is a constant matrix of order n.

$X(t) = M_{32}(At))$ is a solution of IVP below.

$$X'''(t) = -A^3X(t); \quad X(0) = O, \quad X'(0) = O, \quad X''(0) = A^2 \tag{7.5}$$

A is a constant matrix of order n. We omit the details.

The following example illustrates and verifies IVP (7.3)

Example 7.1. To solve: $X'''(t) = -A^3X(t)$, $X(0) = I$, $X'(0) = O$, $X''(0) = O$ where $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

Using IVP 7.2 we get

$$X(t) = L^{-1}\left\{\frac{I}{s} - \frac{A^3}{s^4} + \frac{A^6}{s^7} - \dots\right\}$$
$$= \begin{bmatrix} M_{30}(t) & M_{30}(t) - M_{30}(2t) \\ \\ 0 & M_{30}(2t) \end{bmatrix} = M_{30}(At)$$

Other initial value problems having solutions extended matrix functions such as $N_{30}(At), N_{31}(At), N_{32}(At)$ can be solved on similar lines.

Conclusion: We observe the nature of the differential equations $X^n(t,\alpha) \pm X(t,\alpha) = f(t,\alpha), n = 1, 2, \dots$ and their solutions. The solution of $X'(t,\alpha) \pm X(t,\alpha) = f(t,\alpha)$ leads to $\text{pexp}(\pm t, \alpha)$. The solution of $X''(t,\alpha) \pm X(t,\alpha) = f(t,\alpha)$ is clearly a partition of the series of $\text{pexp}(t,\alpha)$ or $\text{pexp}(-t,\alpha)$. The solution of subsequent differential equations reveal similar partitioning characters. We conclude that $X^n(t,\alpha) \pm X(t,\alpha) = f(t,\alpha)n = 1, 2, \dots$ forms a family of differential equations whose solution partition the series of $\text{pexp}(t,\alpha)$ or $\text{pexp}(-t,\alpha), t \in R$.

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