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POLYNOMIAL ODES – EXAMPLES, SOLUTIONS, PROPERTIES

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ABSTRACT. Let P be a polynomial from $\mathbb{R}^n \to \mathbb{R}^n$ and $D \in \mathbb{R}^n$. I will consider the properties of the class of ODEs $Y' = P(y)$; $Y(0) = D$ and their solutions. The solution space to these ODEs form a proper subspace of the analytic functions with ODEs. They have many interesting algebraic and topological properties. I will present efficient methods for generating the power series solutions to these polynomial initial value ODEs. These methods give *a priori* error estimates that are optimal in the class of polynomial ODEs. Examples showing the power of these theorems are presented, including Newton's N-Body problem.

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1. INTRODUCTION

The properties and solutions of ordinary differential equations (ODEs) have been studied and analyzed since Isaac Newton. However, in 1964, Erwin Fehlberg (best known for the Runge-Kutta-Fehlberg method) discovered an amazing 'trick' for solving some ODEs in an easy and efficient manner using what he called 'auxiliary variables'. Apparently, Fehlberg did not implement these ideas and the NASA report (Fehlberg, 1964) where he presented these 'auxiliary functions' did not receive much attention. In 1989 Parker and Sochacki (Parker & Sochacki, 1996) rediscovered and extended Fehlberg's idea (unaware of Fehlberg's work, until recently) and defined projectively polynomial functions as those functions that were a solution to a polynomial ODE.

First, we look at some examples of polynomial ODEs and interesting results. We then show what an auxiliary variable is and how it can be used to determine the power series of functions. We then give four equivalent definitions of projectively polynomial and discuss some of the more important and interesting properties. We then conclude with some more examples that demonstrate the power of auxiliary variables.

Example 1. Consider the quadratic ODE $x' = 1 + x^2$ with initial condition $x(0) = 0$. It is well known that the solution is $x = \tan t$. If we substitute $x = \sum a_i t^i$ into this

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ODE we find that

$$x' = \sum_{i=0}^{\infty}(i+1)a_{i+1}t^i = 1 + \left(\sum_{i=0}^{\infty} a_i t^i\right)^2 = 1 + a_0^2 + \sum_{i=1}^{\infty}\left[\sum_{j=0}^{i} a_j a_{i-j}\right] t^i.$$

The last sum on the right is given by Cauchy's formula for the product of power series (Knuth, 1998). If we equate like terms in the power series on the far left with the one on the far right, we see that

$$a_0 = x(0); \quad a_1 = 1 + a_0^2; \quad a_{i+1} = \frac{1}{i+1}\sum_{j=0}^{i} a_i a_{j-i}; \quad i = 1, 2, 3, \ldots$$

This is a simple formula for generating the power series for $\tan t$.

NOTATION. Let the function f have a power series; then we write

$$f(t) = \sum_{i=0}^{\infty} f_i t^i = [f_0, f_1, f_2, \ldots].$$

Knuth and others (Knuth, 1998; Graham, et al., 1994) have shown that if $h(t) = \frac{f(t)}{g(t)}$ then since $h(t)g(t) = f(t)$, Cauchy products for power series gives $h_0 = \frac{f_0}{g_0}$ and

$$h_i = \frac{1}{g_0}(f_i - \sum_{j=0}^{i-1} g_j h_{i-j}); \quad i = 1, 2, 3, \ldots \tag{K}$$

If we let $f(t) = \sin t$ and $g(t) = \cos t$ in Formula (K) then we have another recurrence relation for the coefficients of the power series for $\tan t$. However, it is fairly easy to see that the recurrence relation found above by substituting the power series into the ODE of Example 1 is much simpler.

Since it is clear that we can determine the power series for a solution to a polynomial ODE using Cauchy products for power series, we develop some notation for this process. Note that $t[g_0, g_1, g_2, \ldots] = [0, g_0, g_1, g_2, \ldots]$, $t^2[g_0, g_1, g_2, \ldots] = [0, 0, g_0, g_1, g_2, \ldots]$, etc. That is, t^k times a power series g is a power series for g with the first k coefficients being 0 and the remaining coefficients the same as that of g. Therefore,

$$\begin{aligned} g(t)h(t) &= [g_0, g_1, g_2, \ldots][h_0, h_1, h_2, \ldots] \\ &= g_0[h_0, h_1, h_2, \ldots] + g_1[0, h_0, h_1, h_2, \ldots] + g_2[0, 0, h_0, h_1, h_2, \ldots] + \cdots. \end{aligned}$$

Therefore, the coefficients of the power series $g(t)h(t)$ is given by the product of the circulant matrix

$$\begin{pmatrix} h_0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ h_1 & h_0 & 0 & \cdot & \cdot & \cdot & 0 \\ h_2 & h_1 & h_0 & 0 & \cdot & \cdot & 0 \\ h_3 & h_2 & h_1 & h_0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \text{and the vector} \quad \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix},$$

and we can find the first k terms of the product of two power series by multiplying the $k \times k$ circulant matrix above by the $k \times 1$ vector $[g_0, g_1, g_2, \dots, g_k]^T$. This notation allows us to find the Cauchy products needed in polynomial ODEs rather quickly.

Since $h(t) = \tan t = \frac{\sin t}{\cos t}$, we can multiply the power series for $\frac{1}{\cos t}$ by the power series for $\sin t$ using Cauchy products to get the power series for $\tan t$. We generalize this process.

Suppose we know the power series for g and we want to determine the power series for $p = 1/g$. We have that $pg = 1$ and that $(pg)' = p'g + pg' = 0$. If we use Formula (K), we find that

$$p_0 = \frac{1}{g_0}; \quad p_1 = \frac{1}{g_0}(1 - g_1 p_0); \quad p_i = -\frac{1}{g_0} \sum_{j=0}^{i-1} g_j p_{i-j}, \qquad i = 2, 3, 4, \dots$$

We can now determine the power series for $h = \frac{f}{g}$ by multiplying the power series for f by the power series for $p = 1/g$ using Cauchy products. If we use the equation $p'g + pg' = 0$ instead, we find that we also obtain this recurrence relation for p. However, if we use the auxiliary variable $p = 1/g$ and differentiate we obtain

$$p' = -\frac{1}{g^2} g' = -p^2 g'.$$

If we now introduce the auxiliary variable $w = p^2$, we obtain the system of ODEs

$$p' = -wg'; \quad w' = 2pp'.$$

Substituting in the power series for g, g', p and w we obtain the recurrence relations

$$(i+1)p_{i+1} = \sum_{j=0}^{i} (j+1) g_{j+1} w_{i-j}; \quad (i+1) w_{i+1} = 2 \sum_{j=0}^{i} (j+1) p_{j+1} p_{i-j}.$$

Since we solve for p_{i+1} in the first equation, we can substitute it into the second equation to determine w_{i+1}. Although this is not as efficient as Formula (K) to determine $1/g$, it demonstrates a process for using auxiliary variables to get new recurrence relations. We note that we have shown that if the power series for g is a known recurrence relation then the one for p should be similarly known. We will come back to this point after we define projectively polynomial. Before we make this definition, we consider two more examples.

Example 2. Consider Euler's function $x(t) = \frac{t}{e^t-1}$. It is not easy to obtain the power series for this function, but it is easy to find the power series for $y(t) = 1/x(t) = \frac{e^t-1}{t}$. In fact, $y(t) = [1, \frac{1}{2}, \frac{1}{3!}, \frac{1}{4!}, \ldots]$, which is the power series for e^t shifted right. We note that, in general,

$$\frac{f(t) - f_0}{t} = [f_1, f_2, f_3, \ldots].$$

That is, to get this power series we just do a right shift on the power series for f. The power series for Euler's function using our algorithm for $1/y$ is

$$\begin{aligned} x(t) &= \frac{t}{e^t - 1} \\ &= \Bigg[1, -1/2, 1/12, 0, -\frac{1}{720}, 0, \frac{1}{30240}, 0, -\frac{1}{1209600}, 0, \frac{1}{47900160}, \\ &\qquad 0, -\frac{691}{1307674368000}, 0, \frac{1}{74724249600}, 0, -\frac{3617}{10670622842880000}, \\ &\qquad 0, \frac{43867}{5109094217170944000}, 0, -\frac{174611}{802857662698291200000}, 0, \ldots\Bigg]. \end{aligned}$$

We see that this series is very different from the series for x. In that series it was easy to see a pattern for the coefficients. We will bring this point up again.

Example 3. We now consider the advection-diffusion equation $u_t = u_x + \mu^2 u_{xx}$ and consider t as time and x as space. As shown in Parker-Sochacki (Parker & Sochacki, 2000), we suppose $u(x,t) = \sum_{i=0} a_i(x) t^i$ and solve for the $a_i' s$. Substituting this form for u into the advection-diffusion equation, we find that

$$a_1(x) = Da_0(x) + \mu^2 D^2 a_0(x)$$

$$a_2(x) = \frac{D^2 a_0(x) + 2\,\mu^2 D^3 a_0(x) + \mu^4 D^4 a_0(x)}{2}$$

$$a_3(x) = \frac{D^3 a_0(x) + 3\,\mu^2 D^4 a_0(x) + 3\,\mu^4 D^5 a_0(x) + \mu^6 D^6 a_0(x)}{3!}, \quad \text{and}$$

$$a_4(x) = \frac{D^4 a_0(x) + 4\,\mu^2 D^5 a_0(x) + 6\,\mu^4 D^6 a_0(x) + 4\,\mu^6 D^7 a_0(x) + \mu^8 D^8 a_0(x)}{4!},$$

where D is differentiation with respect to x. One can see that we are generating a Pascal triangle with $a_i(x) = \frac{[(D+\mu^2 D^2)^i] a_0(x)}{i!}$. Therefore, substituting a power series for time into a *polynomial* PDE will give a recurrence relation for the coefficients in terms of the space variables. In some cases, as in this example, this recurrence relation can be easily computed. Introducing auxiliary variables will project a PDE into polynomial form (Parker & Sochacki, 2000).

Examples 1–3 and the derivations presented show that polynomial ODEs are amenable to power series solutions using Cauchy products and properties of Cauchy products. We now define projectively polynomial.

2. PROJECTIVELY POLYNOMIAL

DEFINITION. The function $f : \mathbb{R} \to \mathbb{R}$ is *projectively polynomial* if there are natural numbers n, k, a polynomial $P : \mathbb{R}^n \to \mathbb{R}^n$ of degree k and a vector $A \in \mathbb{R}^n$ so that if

$$y' = P(y) \ ; \ y(0) = A$$

then $f = y_1$. Note that this implies f is analytic on a neighborhood of 0 (i.e. $f \in \mathbb{A}$) and $f(0) = A_1$. We write $f \in \mathbb{P}_{n,k}$. We let $\mathbb{P}_n = \cup_k \mathbb{P}_{n,k}$ and $\mathbb{P} = \cup_n \mathbb{P}_n$. Parker and Sochacki have shown that many functions are projectively polynomial and that many ODEs can be made polynomial through auxiliary variables (Parker & Sochacki, 1996).

The following have been shown to be equivalent statements for f being projectively polynomial (Carothers et al., 2005).

(1) There exists a polynomial $Q : \mathbb{R}^{n+1} \to \mathbb{R}$ so that $Q(f, f', \ldots, f^{(n)}) = 0$.

(2) There exists a natural number N and real numbers $a_1, \ldots, a_N$; and $b_{1,1}, \ldots, b_{1,N}, \ldots, b_{N,1}, \ldots, b_{N,N}$; and $c_{1,1,1}, \ldots, c_{1,1,N}, c_{1,2,2}, \ldots, c_{1,2,N}, \ldots, c_{1,N,N}, \ldots, c_{2,1,1}, \ldots, c_{2,1,N}, c_{2,2,2}, \ldots, c_{2,2,N}, \ldots, c_{2,N,N}, \ldots, \ \ldots c_{N,1,1}, \ldots, c_{n,1,N}, c_{N,2,2}, \ldots, c_{N,2,N}, \ldots, c_{N,N,N}$; and $B_1, \ldots, B_N$; together with functions $x_1, \ldots, x_N$; so that if for $j = 1, \ldots, N$

$$x_j' = a_j + \sum_{i=1}^{N} b_{j,i} x_i + \sum_{i=1}^{N} \sum_{k=i}^{N} c_{j,i,k} x_i x_k \ ; \ x_j(0) = B_j$$

then $f = x_1 = \sum_{k=0}^{\infty} \alpha_{1,k} t^k$.

(3) There exists a natural number N and real numbers $a_1, \ldots, a_N$; and $b_{1,1}, \ldots, b_{1,N}, \ldots, b_{N,1}, \ldots, b_{N,N}$; and $c_{1,1,1}, \ldots, c_{1,1,N}, c_{1,2,2}, \ldots, c_{1,2,N}, \ldots, c_{1,N,N}, \ldots, c_{2,1,1}, \ldots, c_{2,1,N}, c_{2,2,2}, \ldots, c_{2,2,N}, \ldots, c_{2,N,N}, \ldots, \ \ldots c_{N,1,1}, \ldots, c_{n,1,N}, c_{N,2,2}, \ldots, c_{N,2,N}, \ldots, c_{N,N,N}$; together with sequences $\alpha_1, \ldots, \alpha_N$; so that $\alpha_{1,0} = f(0) = A_1$ and

$$\alpha_{j,1} = a_j + \sum_{i=1}^{N} b_{j,i} \alpha_{i,0} + \sum_{m=1}^{N} \left(\sum_{i=m}^{N} c_{j,m,i} \alpha_{m,0} \alpha_{i,0} \right)$$

$$\alpha_{j,k} = \frac{1}{k} \left(\sum_{i=1}^{N} b_{j,i} \alpha_{i,k-1} + \sum_{m=1}^{N} \left(\sum_{i=m}^{N} (c_{j,m,i} \sum_{l=0}^{k-1} \alpha_{m,j} \alpha_{i,k-l-1}) \right) \right)$$

for $j = 1, \ldots, N$ and $f = \sum_{k=0}^{\infty} \alpha_{1,k} t^k$.

We point out that Statement (3) is determined by substituting Maclaurin polynomials into the Initial Value (IV) ODE in Statement (2). That is, we substitute $x_j = \sum_{k=0}^{\infty} \alpha_{j,k} t^k$, for $j = 1, \ldots, N$ into the IV ODE in (2) and use Cauchy products to determine $\alpha_{j,k}$. Uniqueness of solutions to IV ODEs of the form in Statement (2) guarantees these have to be the Maclaurin polynomials. Statement (1) shows that if one has a polynomial system of N ODEs then one can decouple these N ODEs into N non-linear N^{th} order ODEs for $x_1, \ldots, x_N$.

We note that a function $f : \mathbb{R} \to \mathbb{R}$ may be analytic at 0, but not projectively polynomial at 0, that is, $\mathbb{P} \subset \mathbb{A}$ (Carothers et al., 2005). Statement (3) gives a specific form for the coefficients $[\alpha_{1,0}, \alpha_{2,0}, \alpha_{3,0}, \ldots]$ of a function that is projectively polynomial. Example 2 indicates that Euler's function and it's reciprocal are not projectively polynomial. Since if one is, we can generate the other using a polynomial ODE. It is known that $\mathbb{P}$ is closed under addition, multiplication, composition and *division* (Parker & Sochacki, 1996).

We note that Statement (3) also gives both a symbolic and numerical environment for solving general polynomial ODEs. We now look at some examples where we make use of these equivalent definitions for projectively polynomial and auxiliary variables. In these examples one sees that the solutions to the ODEs are projectively polynomial functions.

3. EXAMPLES OF PROJECTIVELY POLYNOMIAL

Example 4. Consider the IV ODE $x' = \sqrt{x}$; $x(0) = 0$. This ill-posed IV ODE has two solutions, namely $x = 0$ and $x = t^2/4$ for $t \geq 0$. We now let $y = \sqrt{x}$ and obtain the equivalent system of polynomial IV ODEs

$$x' = y; \quad x(0) = 0$$

$$y' = \frac{1}{2}x^{-\frac{1}{2}}x' = \frac{1}{2}; \quad y(0) = 0.$$

The only solution to this system of ODEs is $y = t/2$ and $x = \int_0^t y dt = t^2/4$. Introducing an auxiliary variable has converted the non-linear ill-posed ODE into a linear well-posed ODE.

Example 5. Consider the following generalization of Example 4; $x' = x^r$ for r any real number ($r = 1/2$ in Example 4). We introduce the auxiliary variable $y = x^{r-1}$ and obtain the polynomial ODE

$$x' = xy; \quad y' = (r-1)x^{r-2}x' = (r-1)x^{2r-2} = (r-1)y^2.$$

Using Statement (3) of the equivalent definitions of projectively polynomial gives us recurrence relations to determine the Maclaurin polynomials for x and y. Using Cauchy products on xy gives us the power series for x^r. In fact, we can generalize this example to obtain g^r for any power series g, by letting $p = g^r$. We now differentiate p and proceed analogously as we did for $p = 1/g$ above.

We now show how one can find the sin of any power series x.

Example 6. Let $x = [x_0, x_1, x_2, \dots]$. We will determine $y = \sin x$. We let $z = \cos x$. We now have

$$y' = zx'; \quad z' = -yx'.$$

We substitute in $x' = [x_1, 2x_2, 3x_3, 4x_4, \dots]$, $y = [y_0, y_1, y_2, \dots]$ and $z = [z_0, z_1, z_2, \dots]$ and use Cauchy products as before and obtain power series for both $\sin x$ and $\cos x$.

One now sees that we can generate arbitrary functions of power series using auxiliary variables (McGuire & Childs, 1991). We now present auxiliary variables for one of the most famous ODEs, Newton's N body problem.

Example 7. Newton's N body ODEs for the positions (x_i, y_i, z_i) of $i = 1, \dots N$ bodies is

$$x_i''(t) = \sum_{j \neq i} \frac{m_j(x_j - x_i)}{r_{i,j}^{\frac{3}{2}}}; \quad y_i''(t) = \sum_{j \neq i} \frac{m_j(y_j - y_i)}{r_{i,j}^{\frac{3}{2}}}; \quad z_i''(t) = \sum_{j \neq i} \frac{m_j(z_j - z_i)}{r_{i,j}^{\frac{3}{2}}}$$

where $r_{i,j} = [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]$, $j = 1, \dots, N$. If we let $s_{i,j} = r_{i,j}^{-\frac{1}{2}}$ this system for the N-body problem can be posed as

$$x_i' = u_i; \quad y_i' = v_i; \quad z_i' = w_i$$

$$u_i' = \sum_{j \neq i} m_j(x_j - x_i)s_{i,j}^3; \quad v_i' = \sum_{j \neq i} m_j(y_j - y_i)s_{i,j}^3; \quad w_i' = \sum_{j \neq i} m_j(z_j - z_i)s_{i,j}^3$$

$$s_{i,j}' = -\frac{1}{2}s_{i,j}^3[2(x_i - x_j)(u_i - u_j) + 2(y_i - y_j)(v_i - v_j) + 2(z_i - z_j)(w_i - w_j)], \quad i, j = 1, \dots N.$$

Pruett, Rudmin and Lacy (Pruett, et al. 2003) used Statement (3) of the equivalent definitions of projectively polynomial to build a numerical environment for solving this system of ODEs. Pruett, Ingham and Herman (Pruett et al., 2008) then parallelized this algorithm and obtained linear speed up for large N. Using the *a priori* error estimate presented in (Carothers et al. 2006) Pruett, Ingham and Howard obtained machine precision accuracy for the orbits of the bodies and the energy of the system over long time periods. An important question to be asked is whether there is a more efficient projection of the N-body problem in terms of ease of computation in determining Cauchy products for the power series and the *a priori* error estimate as demonstrated in Example 5.

Newton also developed a method for determining roots of functions, including polynomials.

Example 8. Consider Newton's Method $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k}$ for determining the zeroes of f. This algorithm can be obtained from the Forward Euler approximation to the ODE $x' = -\frac{f(x)}{f'(x)}$ (Neuberger, 1999). If we use the auxiliary variable $u = [f'(x)]^{-1}$ in this ODE we obtain the system

$$x' = -uf(x); \quad u' = -u^2 f''(x)x' = -u^3 f''(x)f(x).$$

If f is a polynomial then this is a polynomial system of ODEs. If f is not a polynomial we can usually use auxiliary variables to form a polynomial system. However, it is not easy to extend this concept to higher dimensions because then $f'(x) = Df(x)$ is a matrix. However, if we consider

$$x' = -[Df(x)]^T f(x),$$

no matrix inversion is needed and we can make this system polynomial. Many search methods can be posed as approximate solutions to ODEs. By working with these we can make the search methods polynomial.

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