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A DETERMINISTIC PARTICLE METHOD FOR THE VLASOV-MAXWELL-FOKKER-PLANCK SYSTEM IN TWO DIMENSIONS

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ABSTRACT. A brief summary is given of a numerical procedure for solving the Vlasov-Maxwell-Fokker-Planck system in two spatial dimensions with periodic boundary conditions. A complete development of this work is given in [S. Wollman, Numerical approximation of the Vlasov-Maxwell-Fokker-Planck system in two dimensions, preprint]. The system of equations under consideration is a model for the time evolution of a collisional plasma in the presence of a self consistent electromagnetic field. The numerical method is a type of deterministic particle method and generalizes the numerical procedure of [S. Wollman, E. Ozizmir, Numerical approximation of the Vlasov-Poisson-Fokker-Planck system in two dimensions, J. Comput. Phys. 228 (2009) 6629–6669] to the case where a non constant, internally consistent, magnetic field is included. The problem of the long time asymptotic behavior of solutions is addressed.

Key words: collisional plasma, two dimensional Vlasov-Maxwell-Fokker-Planck system, deterministic particle method

AMS subject clasifications. 35Q83, 65M06, 65M25, 82D10

1. INTRODUCTION

A numerical method is developed for approximating the Vlasov-Maxwell-Fokker-Planck system in two spatial dimensions with periodic boundary conditions, [9]. A brief summary of this work is given here. Let $x = (x_1, x_2)$, $v = (v_1, v_2)$, $(x, v) \in R^4$ and t is time. For this two dimensional model the magnetic field acts in a direction perpendicular to the x_1, x_2 plane. The electromagnetic force, $E + v \times B$, can be understood in terms of its reduction from a three dimensional space. Let $\overline{v} = (v_1, v_2, v_3)$ and

$$E(x,t) = (E_1(x_1,x_2,t), E_2(x_1,x_2,t), 0), \quad B(x,t) = (0,0,b(x_1,x_2,t)).$$

Then

$$E + \overline{v} \times B = (E_1(x,t) + v_2 b(x,t), E_2(x,t) - v_1 b(x,t), 0).$$

Let

$$\nabla_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}), \quad \nabla_v = (\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}), \quad \nabla_x^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

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In computing the solution to Maxwell's equations it is convenient to introduce the scalar potential, $\phi(x,t)$, and the vector potential, $A(x,t) = (A_1(x,t), A_2(x,t))$. The electric field is then computed as

$$E(x,t) = (E_1, E_2) = -\nabla_x \phi - \frac{\partial A}{\partial t} = \left(-\frac{\partial \phi}{\partial x_1} - \frac{\partial A_1}{\partial t}, -\frac{\partial \phi}{\partial x_2} - \frac{\partial A_2}{\partial t} \right).$$

The component of the magnetic field perpendicular to the plane of x_1, x_2 is

$$b(x,t) = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}.$$

To define the problem with the periodic boundary conditions let the set $\mathcal{A} \in R^4$ be

$$\mathcal{A} = \{(x,v)/0 \le x_1, x_2 \le L,\ -\infty < v_1, v_2 < \infty\}.$$

Then for $(x,v) \in \mathcal{A}$ and $t \in [0,T]$ the system to be solved is

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + (E_1 + v_2 b, E_2 - v_1 b) \cdot \nabla_v f = \nabla_v \cdot (\beta v f + q \nabla_v f), \tag{1.1}$$

$$f(x,v,0) = f_0(x,v),$$

$$f(0,x_2,v,t) = f(L,x_2,v,t), \quad f(x_1,0,v,t) = f(x_1,L,v,t).$$

$$\lim_{|v| \to \infty} f(x,v,t) = 0.$$

For the solution $f(x,v,t)$ to (1.1) the charge density $\rho(x,t)$ is defined by

$$\rho(x,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f dv - h(x). \tag{1.2}$$

The function $h(x)$ represents a fixed neutralizing background charge density. The current density $J(x,t)$ is given by

$$J(x,t) = (J_1, J_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v f dv = \Big(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_1 f dv, \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_2 f dv \Big) \tag{1.3}$$

The scalar potential ϕ is the solution to

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla_x^2 \phi = \rho, \tag{1.4}$$

$$\phi(x,0) = \phi_0(x), \;\; \phi_t(x,0) = 0,$$

The function $\phi_0(x)$ is the solution to

$$\nabla_x^2 \phi_0(x) = -\rho(x,0), \quad \rho(x,0) = \int_v f_0(x,v) dv - h(x).$$

The vector potential $A = (A_1, A_2)$ is the solution to

$$\frac{\partial^2 A}{\partial t^2} - \nabla_x^2 A = J, \tag{1.5}$$

$$A(x,0) = 0, \; A_t(x,0) = 0.$$

The equations (1.4) and (1.5) have periodic boundary conditions in x_1 and x_2. The electric field $E = (E_1, E_2)$ and magnetic field b in (1.1) are computed as

$$E(x,t) = -\nabla_x \phi - \frac{\partial A}{\partial t}, \qquad b(x,t) = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}. \tag{1.6}$$

With the given initial conditions for equations (1.4), (1.5) it can be proved that the Lorentz condition for ϕ and A holds. That is

$$\frac{\partial \phi}{\partial t} + \nabla_x \cdot A = 0.$$

It then follows that the electric and magnetic field variables (1.6) are solutions to Maxwell's equations in 2-D given as

$$\frac{\partial E_1}{\partial t} = \frac{\partial b}{\partial x_2} - J_1(x,t), \tag{1.7}$$

$$\frac{\partial E_2}{\partial t} = -\frac{\partial b}{\partial x_1} - J_2(x,t), \tag{1.8}$$

$$\frac{\partial b}{\partial t} = -\frac{\partial E_2}{\partial x_1} + \frac{\partial E_1}{\partial x_2}, \tag{1.9}$$

$$\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} = \rho(x,t). \tag{1.10}$$

such that

$$E(x,0) = -\nabla_x \phi_0(x), \quad b(x,0) = 0.$$

and with periodic boundary conditions.

The system comprised of the Vlasov-Fokker-Planck equation (1.1) and Maxwell's equations (1.7)–(1.10) with periodic boundary conditions will be referred to as VMFP. This system represents the time evolution of a plasma in the presence of a self consistent electromagnetic field and for which collisonal effects are included. The function $f(x,v,t)$ in (1.1) is the phase space distribution function for electrons. The function $h(x)$ in (1.2) is the density for a fixed background of heavy ions. The Fokker-Planck equation of type (1.1) was initially derived by Chandrasekhar in connection with the theory of Brownian motion, [3]. To briefly mention some other papers related to the present work the problem of existence of solutions is considered in [4], [7]. The numerical approximation of solutions is carried out in [5], [6]. An analysis of high field asymptotics is done in [1].

2. DESCRIPTION OF THE NUMERICAL METHOD

The numerical method is based on putting the Vlasov-Fokker-Planck equation (1.1) into a form so that finite difference methods for parabolic type PDE's can be applied. This procedure is described in detail in [8] as relates to the Vlasov-Poisson-Fokker-Planck system. The characteristic system associated with the first order transport part of (1.1) is

$$\begin{aligned} &\frac{dx_1}{dt} = v_1,\ \frac{dx_2}{dt} = v_2,\ x_1(0) = \xi_1,\ x_2(0) = \xi_2 \\ &\frac{dv_1}{dt} = E_1(x(t),t) + b(x(t),t)v_2 - \beta v_1,\ v_1(0) = \eta_1 \\ &\frac{dv_2}{dt} = E_2(x(t),t) - b(x(t),t)v_1 - \beta v_2,\ v_2(0) = \eta_2 \end{aligned} \tag{2.1}$$

Letting $x = (x_1, x_2),\ \xi = (\xi_1, \xi_2),\ \eta = (\eta_1, \eta_2)$ the solution to (2.1) is written

$$x(t) = x(\xi, \eta, t),\ v(t) = v(\xi, \eta, t). \tag{2.2}$$

Taking into account the periodicity the functions (2.2) define a transformation of $\mathcal{A}$ onto $\mathcal{A}$ given by

$$(\xi, \eta) \to (x(\xi, \eta, t), v(\xi, \eta, t)).$$

The Jacobian of this transformation is $\partial(x, v)/\partial(\xi, \eta) = e^{-2\beta t} \neq 0$. Therefore the transformation is invertible with the inverse transformation given by

$$\xi = \xi(x, v, t),\ \eta = \eta(x, v, t). \tag{2.3}$$

A change of variables is made from (x, v) to (ξ, η), and equation (1.1) is then given an expression in terms of $f(\xi, \eta, t)$ that is the same as that preceding (2.4) in [8]. A change of dependent variable $f(\xi, \eta, t) = e^{2\beta t} g(\xi, \eta, t)$ then leads to the equation for $g(\xi, \eta, t)$ given by

$$\begin{aligned} \frac{\partial g}{\partial t} = q\Big[&c_1\frac{\partial^2 g}{\partial \xi_1^2} + c_2\frac{\partial^2 g}{\partial \xi_2^2} + c_3\frac{\partial^2 g}{\partial \eta_1^2} + c_4\frac{\partial^2 g}{\partial \eta_2^2} + 2\Big(c_5\frac{\partial^2 g}{\partial \xi_1 \partial \xi_2} + \ldots + c_{10}\frac{\partial^2 g}{\partial \eta_1 \partial \eta_2}\Big) \\ &+ c_{11}\frac{\partial g}{\partial \xi_1} + c_{12}\frac{\partial g}{\partial \xi_2} + c_{13}\frac{\partial g}{\partial \eta_1} + c_{14}\frac{\partial g}{\partial \eta_2}\Big],\quad g(\xi, \eta, 0) = f_0(\xi, \eta). \end{aligned} \tag{2.4}$$

The coefficients $c_i = c_i(\xi, \eta, t),\ i = 1, \ldots, 14$, are expressed in terms of first and second partial derivatives with respect to ξ, η of the solution to (2.1).

To reduce the velocity space computation to a finite domain a further transformation of independent variables is made. Let $\eta(u) = (\eta_1, \eta_2)$ where

$$\eta_1 = \eta_1(u_1) = \frac{cu_1}{\sqrt{1 - u_1^2}},\quad \eta_2 = \eta_2(u_2) = \frac{cu_2}{\sqrt{1 - u_2^2}},\quad -1 < u_1, u_2 < 1 \tag{2.5}$$

with c is a positive constant. Let

$$s_1(u_1) = \frac{1}{c}(1 - u_1^2)^{3/2},\ s_2(u_2) = \frac{1}{c}(1 - u_2^2)^{3/2}.$$

In terms of variables $\xi_1, \xi_2, u_1, u_2, t$ the equation (2.4) is written as an initial, boundary value problem for $g(\xi, u, t)$ given as

$$\begin{aligned}
\frac{\partial g}{\partial t} &= q\Big[c_1\frac{\partial^2 g}{\partial \xi_1^2} + c_2\frac{\partial^2 g}{\partial \xi_2^2} + c_3 s_1(u_1)\frac{\partial}{\partial u_1}\Big(s_1(u_1)\frac{\partial g}{\partial u_1}\Big) + c_4 s_2(u_2)\frac{\partial}{\partial u_2}\Big(s_2(u_2)\frac{\partial g}{\partial u_2}\Big) \\
&+2\Big(c_5\frac{\partial^2 g}{\partial \xi_1 \partial \xi_2} + c_6 s_1(u_1)\frac{\partial^2 g}{\partial \xi_1 \partial u_1} + \ldots + c_{10} s_1(u_1) s_2(u_2)\frac{\partial^2 g}{\partial u_1 \partial u_2}\Big) \\
&+c_{11}\frac{\partial g}{\partial \xi_1} + c_{12}\frac{\partial g}{\partial \xi_2} + c_{13} s_1(u_1)\frac{\partial g}{\partial u_1} + c_{14} s_2(u_2)\frac{\partial g}{\partial u_2}\Big], \qquad (2.6)
\end{aligned}$$

$$g(\xi, u, 0) = f_0(\xi, \eta(u)),$$

$$g(0, \xi_2, u, t) = g(L, \xi_2, u, t), \quad g(\xi_1, 0, u, t) = g(\xi_1, L, u, t),$$

$$g(\xi, -1, u_2, t) = g(\xi, 1, u_2, t) = g(\xi, u_1, -1, t) = g(\xi, u_1, 1, t) = 0.$$

The coefficients in the PDE (2.6) are of the form $c_i = c_i(\xi, \eta(u), t),\ i = 1, \ldots, 14$, and are expressed in terms of the first and second partial derivatives with respect to ξ, η of the functions $x(\xi, \eta(u), t),\ v(\xi, \eta(u), t)$. These functions are solution to

$$\begin{aligned}
&\frac{dx_1}{dt} = v_1, \quad \frac{dx_2}{dt} = v_2, \quad x_1(0) = \xi_1, \quad x_2(0) = \xi_2, \\
&\frac{dv_1}{dt} = E_1(x(\xi, \eta(u), t), t) + b(x(\xi, \eta(u), t), t)v_2 - \beta v_1,\ v_1(0) = \eta_1(u_1) \qquad (2.7) \\
&\frac{dv_2}{dt} = E_2(x(\xi, \eta(u), t), t) - b(x(\xi, \eta(u), t), t)v_1 - \beta v_2,\ v_2(0) = \eta_2(u_2).
\end{aligned}$$

In terms of the solution to (2.6) the solution to (1.1) is

$$f(x, v, t) = e^{2\beta t} g(\xi(x, v, t), u(\eta(x, v, t)), t). \qquad (2.8)$$

where $(\xi(x, v, t), \eta(x, v, t))$ is the inverse transformation (2.3) and $u(\eta) = (u_1(\eta_1), u_2(\eta_2))$ is the inverse of (2.5). With the function $f(x, v, t)$ given by (2.8) the charge and current densities are obtained from (1.2), (1.3). The equations (1.4), (1.5) are then solved for the scalar and vector potentials ϕ and A. The self consistent electromagnetic field is computed from (1.6).

The numerical method is a type of deterministic particle method in which the Vlasov-Fokker-Planck equation (1.1) and wave equations (1.4), (1.5) are solved in terms of a sequence of solutions to the PDE (2.6) and (1.4), (1.5). Given the time interval $[0, T]$ let T_1 be such that $T/T_1 = M$ an integer. The interval $[0, T]$ is divided into subintervals $[mT_1, (m+1)T_1]$ for $m = 0, 1, \ldots, M-1$. Let $f(x, v, \bar{t})$ be the solution to (1.1), (1.4), (1.5) for $0 \le \bar{t} \le T$. On the time interval $mT_1 \le \bar{t} \le (m+1)T_1$ let $t = \bar{t} - mT_1$. Then

$$f(x, v, \bar{t}) = e^{2\beta t} g(\xi(x, v, t), u(\eta(x, v, t)), t), \quad t \in [0, T_1] \qquad (2.9)$$

and such that $g(\xi, u, t)$ is the solution to (2.6), (1.4), (1.5) with $g(\xi, u, 0) = f(\xi, \eta(u), mT_1)$. If $m = 0$ then $f(x, v, mT_1) = f_0(x, v)$. If $m > 0$ then $f(x, v, mT_1) =$

$e^{2\beta T_1} g(\xi(x,v,T_1), u(\eta(x,v,T_1)), T_1)$ such that $g(\xi,u,t)$ is the solution to (2.6), (1.4), (1.5) for $t \in [0,T_1]$ with $g(\xi,u,0) = f(\xi,\eta(u),(m-1)T_1)$. The numerical approximation is a discretization of this procedure.

The numerical approximation proceeds in the following way:

1) On the time interval $[mT_1,(m+1)T_1]$ the path of a particle in phase space is determined for $t \in [0,T_1]$ from the discrete approximation of the trajectory equation (2.7).

2) The charge along the approximate trajectory is determined at each time step from the solution to the PDE (2.6).

3) The equation (2.6) is approximated by a finite difference equation which is solved on a fixed grid by either an iterative SOR algorithm or by a direct Douglas-Rachford method.

4) The coefficients in (2.6) are obtained by approximating the first and second partial derivatives with respect to ξ,η of the solution to the trajectory equations (2.7) and following the procedure described in [8, Sections 3.2.4,3.2.5].

5) To approximate the field variables the charge and current densities are computed on a fixed grid from the charge along approximate trajectories by a particle-in-cell method. The solutions to the wave equations (1.4) and (1.5) are then approximated at each time step by applying a discrete Fourier transform in the spatial variables, integrating the resulting transformed equations exactly in time, applying a discrete inverse Fourier transform.

6) Given the resulting approximations to the scalar and vector potentials the discrete versions of the field expressions (1.6) are computed on the fixed grid for the electric and magnetic fields. The field at particle positions is obtained by applying the particle-to-grid assignment function.

7) At time $\bar{t} = (m+1)T_1,\ t = T_1$ the solution along trajectories is interpolated onto the fixed grid as initial data for the PDE (2.6). The solution to (2.6) for g is restarted, and the particle computation is repeated for the time interval $[(m+1)T_1,(m+2)T_1]$. This regriding of the solution greatly improves the long term stability and accuracy of the numerical method.

3. THE STEADY STATE SOLUTION

The Vlasov-Poisson-Fokker-Planck system in two dimensions with periodic boundary conditions is given for $(x,v) \in \mathcal{A}$ and $t \in [0,T]$ as

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + E(x,t) \cdot \nabla_v f = \nabla_v \cdot (\beta v f + q \nabla_v f), \tag{3.1}$$

$$f(x,v,0) = f_0(x,v),$$

$$f(0,x_2,v,t) = f(L,x_2,v,t), \quad f(x_1,0,v,t) = f(x_1,L,v,t).$$

combined with

$$E(x,t) = -\nabla_x \phi$$

where

$$\nabla_x^2 \phi = -\rho(x,t) \tag{3.2}$$

$$\phi(0,x_2,t) = \phi(L,x_2,t), \quad \phi(x_1,0,t) = \phi(x_1,L,t)$$

and

$$\rho(x,t) = \int f(x,v,t)dv - h(x).$$

This system, which will be referred to as VPFP, is a model for a collisional electrostatic plasma. In [2] it is proved that the solution to the Vlasov-Poisson-Fokker-Planck system converges to a time independent steady state solution as $t \to \infty$. For the above 2-D problem this steady state solution is given as

$$f_s(x,v) = \frac{K}{2\pi q/\beta} \frac{\exp(-\frac{|v|^2/2+\phi(x)}{q/\beta})}{\left(\int_0^L \int_0^L \exp(-\frac{\phi(x)}{q/\beta})dx\right)} \tag{3.3}$$

where $K = \int_{\mathcal{A}} f_0(x,v)dv\,dx$. The function $\phi(x)$ is the solution to

$$\nabla_x^2 \phi = -\Big(K \frac{\exp(-\frac{\phi(x)}{q/\beta})}{\int_0^L \int_0^L \exp(-\frac{\phi(x)}{q/\beta})dx} - h(x)\Big) \tag{3.4}$$

with periodic boundary conditions in x_1, x_2. It is assumed that $h(x)$ is such that $\int_0^L \int_0^L h(x)dx = K$. A determination is made that the solution to VMFP converges to to this same steady state solution (3.3), (3.4) as $t \to \infty$.

In studying the time asymptotic behavior of the solutions to VMFP and VPFP one makes use of the following integral quantities that are functions of time, t. The electrostatic energy is

$$ese(t) = \frac{1}{2}\int_0^L \int_0^L |E(x,t)|^2 dx = \frac{1}{2}\int_0^L \int_0^L [(E_1(x,t))^2 + (E_2(x,t))^2]dx_1 dx_2.$$

The magnetic energy is

$$emg(t) = \frac{1}{2}\int_0^L \int_0^L (b(x,t))^2 dx.$$

For the given distribution function $f(x,v,t)$ the kinetic energy is

$$ke(t) = \frac{1}{2}\int_{\mathcal{A}} |v|^2 f(x,v,t)dv\,dx = \frac{1}{2}\int_{\mathcal{A}} (v_1^2 + v_2^2) f(x,v,t)dv\,dx.$$

The entropy of the system is

$$ent(t) = -\int_{\mathcal{A}} f(x,v,t) ln(f(x,v,t))dv\,dx.$$

The total energy of the system is $U(t) = ke(t) + ese(t) + emg(t)$, and the free energy is defined as

$$FE(t) = U(t) - q/\beta ent(t). \tag{3.5}$$

For VPFP the free energy has the expression given by (3.5) in which $emg(t) = 0$. In [2] it is proved in the context of a 3-D initial value problem that the free energy for the Vlasov-Poisson-Fokker-Planck system is a non increasing function of time, t, that is bounded from below. This result in the context of the 2-D periodic boundary value problem, VPFP, is given in [8, (4.3)] as

$$\frac{d\big(FE(t)\big)}{dt} = -\beta \int_{\mathcal{A}} \left| v\sqrt{f} + 2q/\beta \nabla_v \sqrt{f} \right|^2 (t) dv\, dx. \tag{3.6}$$

Since $d(FE)/dt \le 0$ then $FE(t)$ is non increasing. The proof that $FE(t)$ is bounded from below is proved for VPFP similarly as in [2]. It therefore follows that for VPFP $FE(t)$ converges to a limit as $t \to \infty$, and

$$\lim_{t\to\infty} \int_{\mathcal{A}} \left| v\sqrt{f} + \frac{2q}{\beta} \nabla_v \sqrt{f} \right|^2 (t) dv\, dx = 0. \tag{3.7}$$

Given (3.7) a next step in the analysis is to determine that

$$\lim_{t\to\infty} |v\sqrt{f} + \frac{2q}{\beta} \nabla_v \sqrt{f}| = 0. \tag{3.8}$$

A version of this result is proved in [2] for the 3-D initial value problem and leads to the conclusion that the solution to the Vlasov-Poisson-Fokker-Planck system in three dimensions converges to the steady state solution [2, (2.14), (2.15)]. In [8] the assumption is made that the analysis in [2] can be readily adapted to the 2-D periodic problem and that a result of the type (3.8) holds for the solution to VPFP. Proceeding with a development as in [2] it is determined that the solution to VPFP converges to the steady state solution (3.3), (3.4).

For VMFP with $b(x,t) \neq 0$ then in (3.5) $emg(t) \neq 0$. It is proved in [9] that for VMFP with $emg(t) \neq 0$ the quantity $d(FE)/dt$ is given precisely by the expression (3.6). Hence $FE(t)$ is non increasing. That $FE(t)$ is bounded from below is similarly proved as in [2]. Thus for VMFP the quantity $FE(t)$ converges to a limit, and the expression (3.7) holds. However, because the $v \times B$ force in (1.1) is an unbounded function of v the analysis in [2] cannot be directly adapted to the VMFP solution to obtain a result equivalent to (3.8). If it is assumed that (3.8) holds then one can continue the development similarly as in [2] and reach the conclusion that the solution to VMFP converges to the steady state solution (3.3), (3.4) as $t \to \infty$. Thus, the analysis in [9] gives a good indication that the steady state solution for VMFP is the same as that for VPFP but does not provide a complete proof. However, numerical approximation supports the conclusion suggested here on the long time asymptotics of VMFP. In [9] computations are done that apply the numerical method of Section 2 and which demonstrate that the solution to VMFP converges to a steady state given by (3.3), (3.4) as t gets large.

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