Neural, Parallel, and Scientific Computations 19 (2011) 111-128

SOLUTIONS PROPERTIES AND THE EXTENSION OF CLASSICAL RANK CONTROLLABILITY CRITERION FOR HIGHER ORDER LINEAR DESCRIPTOR MATRIX DIFFERENTIAL SYSTEMS

Athanasios A. Pantelous[1], Athanasios D. Karageorgos[2], Vasilis N. Tsoulkas[3, 4], and Grigoris I. Kalogeropoulos[2]

[1]Department of Mathematical Sciences, University of Liverpool, U.K.

[2]Department of Mathematics, University of Athens, Greece

[3]General Secretariat for Research and Technology, Athens, GR

[4]National Delegate to the Joint Communications Board of the European Space Agency, Athens, Greece

E-mails A.Pantelous@liverpool.ac.uk, {athkar, gkaloger}@math.uoa.gr

and btsu@gsrt.gr

Abstract. The aim of this paper is two fold. First, we want to discuss the solution properties of higher order linear descriptor matrix differential systems. Secondly, we want to extend the classical matrix rank criterion for the controllability of first order to higher order linear descriptor matrix differential systems. This criterion has been chosen and studied, since it is a typical property and requirement for many modern control systems while at the same time it emerges in a plethora of different control science applications.

AMS (classification): 93B05, 93C15

Key Words: Linear Descriptor Matrix Systems; Controllability; Rank Criterion

1. INTRODUCTION

In the last two decades, higher order linear descriptor systems are considered for the control of constrained mechanical systems, see Müller et al [11], Hou [6], Rabier and Rheinboldt [14] etc; the control of electrical systems, e.g. Bai et al [3] and more recently; the control of more complex structures which are constructed by a mixture of different systems, see Otter et al [12]. Moreover, it is pointed out that higher order linear descriptor differential systems might result from linearization procedures of general nonlinear higher order descriptor systems of the form

$$F\left(t,\underline{\dot{x}},\underline{\ddot{x}},\ldots,\underline{x}^{(k)}\right)=0,$$

around reference solutions, see Mehrmann and Shi [10], Pantelous et al [13] and references therein.

Generally speaking, we consider the algebra $\mathrm{M}\left(n;\mathrm{F}\right)$, where F is a field.

A higher order descriptor matrix differential system with arbitrary constant coefficients, appears to be significant -above all, if one is interested in the behaviour of the original state variables, see Apostol [1], Ben Taher and Rachidi [2], see eq. (1.1)

Received November 25, 2010

$$EX^{(k)}(t) = A_k X^{(k-1)}(t) + A_{k-1} X^{(k-2)}(t) + \ldots + A_1 \dot{X}(t) + A_o X(t) + BU(t), \quad (1.1)$$

where $E, A_i \in \mathrm{M}(n \times n; \mathrm{F})$, $i = 0,1,\ldots,k-1$ with E singular, i.e. $rank(E) < n$, t is the independent time variable, $U(t) \in C^{\infty}(\mathrm{F}; \mathrm{M}(m \times l, \mathrm{F}))$ is the piecewise continuous input (control) vector and $B \in \mathrm{M}(n \times m; \mathrm{F})$ is the input matrix.

If $X^{(k-1)}(t_o^-)$, $X^{(k-2)}(t_o^-), \ldots, \dot{X}(t_o^-)$, $X(t_o^-)$ stand for the value of the state trajectory $X(t)$ immediately before starting the dynamical process described by eq. (1.1), then every point X_o is *consistent*, i.e. if for every point X_o there exists a sufficiently smooth input function $U(t)$ and an associated state trajectory $X(t)$ of eq. (1.1) such that

$$X(t_o) = X(t_o^+) = \lim_{t \downarrow t_o} X(t).$$

Usually, in the classical theory of ordinary differential equations and *classical state space systems*, (i.e. descriptor systems where the leading coefficient is the identity matrix), higher order differential systems are turned into first order systems by introducing new variables for the $k-1$ derivative, see Kalogeropoulos et al [8]. This transformation gives rise to linear first order descriptor (or generalized state-space) systems of the form (1.2) which appears next.

Thereafter, if we consider the transformation $\underline{\mathrm{X}}(\mathrm{t}) = \begin{bmatrix} X(t) \\ \dot{X}(t) \\ \vdots \\ X^{(k-1)}(t) \end{bmatrix} \in \mathrm{M}(kn \times l; \mathrm{F})$, the

eq. (1.1) can be rewritten as a first order descriptor differential system, see eq. (1.2),

$$\mathbf{E}\dot{\underline{\mathrm{X}}}(\mathrm{t}) = \mathbf{A}\underline{\mathrm{X}}(\mathrm{t}) + \mathbf{B}\mathrm{U}(\mathrm{t}) \quad (1.2)$$

where

$$\mathbf{E} = \begin{bmatrix} I_n & & & & \\ & I_n & & \mathbf{O} & \\ & & \ddots & & \\ & \mathbf{O} & & I_n & \\ & & & & E \end{bmatrix} \in \mathrm{M}(kn \times kn; \mathrm{F}), \quad \mathbf{A} = \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \\ A_o & A_1 & A_2 & \cdots & A_{k-1} \end{bmatrix} \in \mathrm{M}(kn \times kn; \mathrm{F})$$

and

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ B \end{bmatrix} \in \mathrm{M}(kn \times m; \mathrm{F}).$$

The $kn \times kn$ matrix $\mathbf{A}$ is known as the (block) companion matrix of (1.3)

$$L(X) = A_k X^{(k-1)}(t) + A_{k-1} X^{(k-2)}(t) + \ldots + A_1 \dot{X}(t) + A_o X(t), \quad (1.3)$$

and the vector $\underline{X}(t)$ is called the state vector of system (1.2), see also Gantmacher [5]. Note also that $rank\ \mathbf{E} < nk$.

Consequently, the aim of the paper is twofold. Firstly, in the 2nd section, to calculate the solution of the (homogeneous) linear matrix descriptor differential equation (1.4) with consistent and non-consistent initial conditions,

$$EX^{(k)}(t) = A_k X^{(k-1)}(t) + A_{k-1} X^{(k-2)}(t) + \ldots + A_1 \dot{X}(t) + A_o X(t). \tag{1.4}$$

Secondly, since the matrix criteria in the verification of controllability are of special importance and with many applications, in the 3rd section, to provide the controllability property for higher order linear descriptor differential systems, see eq. (1.1).

2. SOLUTION PROPERTIES FOR A CLASS OF HIGHER ORDER LINEAR DESCRIPTOR MATRIX DIFFERENTIAL SYSTEMS

As a short introduction of this chapter, some preliminary concepts and definitions from Matrix Pencil theory are introduced. In the literature of control and system theory, descriptor systems are closely related to matrix pencil theory, see Dai [4], Gantmacher [5].

Definition 2.1 Given $F, G \in \mathrm{M}(n \times m, \mathrm{F})$ and an indeterminate $s \in \mathrm{F}$, the matrix pencil $sF - G$ is called *regular* when $m = n$ and $\det(sF - G) \neq 0$. In any other case, the pencil will be called *singular*.

Definition 2.2 The pencil $sF - G$ is said to be *strictly equivalent* to the pencil $s\tilde{F} - \tilde{G}$ if and only if there exist nonsingular matrices $P \in \mathrm{M}(n \times n, \mathrm{F})$ and $Q \in \mathrm{M}(m \times m, \mathrm{F})$ such as

$$P(sF - G)Q = s\tilde{F} - \tilde{G}.$$

In this paper, we consider the case that pencil is *regular*. In this case, i.e. where the pencil $sF - G$ is a regular, we have elementary divisors (e.d.) of the following type:

- e.d. of the type s^p are called *zero finite elementary divisors* (**z. f.e.d.**)
- e.d. of the type $(s - a)^{\pi}$, $a \neq 0$ are called *nonzero finite elementary divisors* (**nz. f.e.d.**)

- e.d. of the type $\hat{s}^q$ are called *infinite elementary divisors* (**i.e.d**.).

Let $B_1, B_2, ..., B_n$ be elements of M_n. The direct sum of them denoted by $B_1 \oplus B_2 \oplus ... \oplus B_n$ is the *block diag* $\{B_1, B_2, ..., B_n\}$.

Then, the complex Weierstrass form $sF_w - Q_w$ of the regular pencil $sF - G$ is defined by $sF_w - Q_w \cong sI_p - J_p \oplus sH_q - I_q$, where the first normal Jordan type element is uniquely defined by the set of f.e.d.

$$(s-a_1)^{p_1}, \ldots, (s-a_\nu)^{p_\nu}, \ \sum_{j=1}^{\nu} p_j = p$$

of $sF - G$ and has the form

$$sI_p - J_p \cong sI_{p_1} - J_{p_1}(a_1) \oplus \ldots \oplus sI_{p_\nu} - J_{p_\nu}(a_\nu).$$

And also the q blocks of the second uniquely defined block $sH_q - I_q$ correspond to the i.e.d.

$$\hat{s}^{q_1}, \ldots, \hat{s}^{q_\sigma}, \ \sum_{j=1}^{\sigma} q_j = q$$

of $sF - G$ and has the form

$$sH_q - I_q \cong sH_{q_1} - I_{q_1} \oplus \ldots \oplus sH_{q_\sigma} - I_{q_\sigma}.$$

Thus, H_q is a nilpotent matrix with elements into the algebra $\mathrm{M}(n \times n, \mathrm{F})$. Moreover, the index of nilpotency of H_q is $\tilde{q} = \max\{q_j : j = 1, 2, \ldots, \sigma\}$, where

$$H_{\tilde{q}} = \mathrm{O}, \tag{2.1}$$

and $I_{p_j}, J_{p_j}(a_j), H_{q_j}$ are defined as

$$I_{p_j} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{M}(p_j \times p_j, \mathrm{F}), \quad J_{p_j}(a_j) = \begin{bmatrix} a_j & 1 & 0 & \cdots & 0 \\ 0 & a_j & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & a_j & 1 \\ 0 & 0 & 0 & 0 & a_j \end{bmatrix} \in \mathrm{M}(p_j \times p_j, \mathrm{F})$$

and $$H_{q_j} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathrm{M}\left(q_j \times q_j, \mathrm{F}\right).$$

(2.2)

Consequently, from the regularity of matrix $sF - G$, there exist nonsingular matrices P and Q into the algebra $\mathrm{M}\left(n \times n, \mathrm{F}\right)$ such that.

$$PFQ = F_w = I_p \oplus H_q, \tag{2.3}$$

$$PGQ = G_w = J_p \oplus I_q, \tag{2.4}$$

where I_p, J_p, H_q and I_q are given by (2.2) where

$$I_p = I_{p_1} \oplus \ldots \oplus I_{p_\nu},$$

$$J_p = J_{p_1}\left(a_1\right) \oplus \ldots \oplus J_{p_\nu}\left(a_\nu\right),$$

$$H_q = H_{q_1} \oplus \ldots \oplus H_{q_\sigma},$$

and $$I_q = I_{q_1} \oplus \ldots \oplus I_{q_\sigma}.$$

Note that $\sum_{j=1}^{\nu} p_j = p$ and $\sum_{j=1}^{\sigma} q_j = q$, where $p + q = n$.

Thereafter, if we consider the transformation

$$\underline{\mathrm{X}}\left(\mathrm{t}\right) = \begin{bmatrix} X\left(t\right) \\ \dot{X}\left(t\right) \\ \vdots \\ X^{\left(k-1\right)}\left(t\right) \end{bmatrix} \in \mathrm{M}\left(kn \times l; \mathrm{F}\right),$$

eq. (1.4) can be rewritten as a first order generalized differential system,

$$\mathbf{E}\dot{\underline{\mathrm{X}}}\left(\mathrm{t}\right) = \mathbf{A}\underline{\mathrm{X}}\left(\mathrm{t}\right), \tag{2.5}$$

where $\mathbf{E} = I_n \oplus I_n \oplus \cdots \oplus I_n \oplus E \in \mathrm{M}\left(kn \times kn; \mathrm{F}\right)$, and

$$\mathbf{A}=\begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \\ A_o & A_1 & A_2 & \cdots & A_{k-1} \end{bmatrix} \in \mathrm{M}\left(kn\times kn;\mathrm{F}\right).$$

Now, considering the transformation

$$\underline{X}\left(t\right)=Q\underline{Y}\left(t\right), \tag{2.6}$$

the following well-known lemma is derived.

Lemma 2.1 [Dai, 4] System (2.5) can be divided into two sub-systems:

The so-called *slow* sub-system

$$\underline{Y}_p'\left(t\right)=J_p\underline{Y}_p\left(t\right), \tag{2.7}$$

with initial conditions, $\underline{Y}_p\left(0\right)=\underline{Y}_{p,o}=Q_{p,kn}^{-1}\underline{X}_o$.

and the relative *fast* sub-system

$$H_q\underline{Y}_q'\left(t\right)=\underline{Y}_q\left(t\right). \tag{2.8}$$

with *consistent* initial conditions, $\underline{Y}_q\left(0\right)=\underline{Y}_{q,o}=Q_{q,kn}^{-1}\underline{X}_o$.

Remark 2.1 The initial conditions of system (1.4) are give by

$$X^{(k-1)}\left(0\right),\ X^{(k-2)}\left(0\right),\ldots,\ \dot{X}\left(0\right),\ X\left(0\right).$$

So, $\underline{X}_o=\begin{bmatrix} X\left(0\right) \\ \dot{X}\left(0\right) \\ \vdots \\ X^{(k\text{-}1)}\left(0\right) \end{bmatrix}\in \mathrm{M}\left(kn\times l;\mathrm{F}\right)$, and $\underline{Y}\left(0\right)=Q^{-1}\underline{X}_o$.

Actually, the system (2.7) is an ordinary linear differential system and has a unique solution for any initial condition $\underline{Y}_q\left(0\right)=\underline{Y}_{q,o}=Q_{q,kn}^{-1}\underline{X}_o$.

It is well known that the solution of (2.7) is given by (2.9)

$$\underline{Y}_p\left(t\right)=e^{J_pt}Q_{p,kn}^{-1}\underline{X}_o. \tag{2.9}$$

Proposition 2.1 [Dai, 4] The fast system (2.8) has only the zero solution, i.e. $\underline{Y}_q\left(t\right)=\mathrm{O}$, when we have (*consistent*) initial conditions, i.e. $\underline{Y}_q\left(0\right)=\underline{Y}_{q,o}=Q_{q,kn}^{-1}\underline{X}_o$.

Hence, the set of consistent initial conditions for the system $\mathbf{E}_w\underline{Y}'(t)=\mathbf{A}_w\underline{Y}(t)$ is given by the form $\left\{\underline{Y}(0)=\begin{bmatrix}\underline{Y}_p(0)\\ \mathrm{O}_q\end{bmatrix}, p+q=kn\right\}$.

Proposition 2.2 The solution of the higher order linear descriptor regular system (1.4) with *consistent* initial conditions is given by

$$X(t)=LQ_{nk,p}e^{J_pt}Q^{-1}_{p,kn}\underline{X}_o, \tag{2.10}$$

where $L=[I_n \quad \mathrm{O} \quad \cdots \quad \mathrm{O}]\in \mathrm{M}(n\times kn,\mathrm{F})$, $\underline{X}_o=\begin{bmatrix}X(0)\\ \dot{X}(0)\\ \vdots\\ X^{(k\text{-}1)}(0)\end{bmatrix}\in \mathrm{M}(kn\times l;\mathrm{F})$, and

$Q_{nk,p}$, $Q^{-1}_{p,kn}$ are elements of the matrices Q and Q^{-1}, respectively.

Proof When we have *consistent* initial conditions,

$$\underline{Y}(t)=\begin{bmatrix}\underline{Y}_p(t)\\ \mathrm{O}_q\end{bmatrix}=\begin{bmatrix}e^{J_pt}Q^{-1}_{p,kn}\underline{X}_o\\ \mathrm{O}_q\end{bmatrix}, p+q=kn.$$

Here, since we have used the transformation (2.6), i.e. $\underline{X}(t)=Q\underline{Y}(t)$, we take

$$\underline{X}(t)=Q\underline{Y}(t)=\begin{bmatrix}Q_{nk,p} & Q_{nk,q}\end{bmatrix}\begin{bmatrix}\underline{Y}_p(t)\\ \underline{Y}_q(t)\end{bmatrix}=Q_{nk,p}\underline{Y}_p(t)=Q_{nk,p}e^{J_pt}Q^{-1}_{p,kn}\underline{X}_o.$$

Finally, we have used that $X(t)=Z_1(t)=L\underline{X}(t)$, where

$$L=[I_n \quad \mathrm{O} \quad \cdots \quad \mathrm{O}]\in \mathrm{M}(n\times kn,\mathrm{F}).$$

So, the expression (2.10) is derived. □

In this brief section, we describe the impulse behavior of the system (2.5), at time $t_o=0$. In that case, we have to reformulate only the Proposition 2.1, so the impulse solution is finally obtained.

Proposition 2.3 [Dai, 4; Kalogeropoulos et al, 8] The nilpotent system (2.8) has the following solution

$$\underline{Y}_q(t)=-\sum_{j=0}^{q^*-2}\delta^{(j)}(t)H_q^{j+1}Q^{-1}_{q,kn}\underline{X}_o, \tag{2.11}$$

where $\delta^{(j)}$ for $j=0,1,2,...,q^*-2$ is the Dirac function and its derivates.

Proposition 2.4 The solution of the higher order linear descriptor regular system (1.4) with *non-consistent* initial conditions is given by

$$X(t)=L\left\{Q_{nk,p}e^{J_p t}Q_{p,kn}^{-1}-Q_{nk,q}\sum_{j=0}^{q^*-2}\delta^{(j)}(t)H_q^{j+1}Q_{q,kn}^{-1}\right\}\underline{X}_o, \tag{2.12}$$

where $\delta^{(j)}$ for $j=0,1,2,...,q^*-2$ is the Dirac function and its derivates. Moreover,

$$L=\begin{bmatrix}I_n & \mathrm{O} & \cdots & \mathrm{O}\end{bmatrix}\in \mathrm{M}(n\times kn,\mathrm{F}),\quad \underline{X}_o=\begin{bmatrix}X(0)\\ \dot{X}(0)\\ \vdots\\ X^{(k\text{-}1)}(0)\end{bmatrix}\in \mathrm{M}(kn\times l;\mathrm{F}),\quad \text{and}\quad Q_{nk,p},$$

$Q_{p,kn}^{-1}$ are elements of the matrices Q and Q^{-1}, respectively.

Proof When we have *non-consistent* initial conditions,

$$\underline{Y}(t)=\begin{bmatrix}\underline{Y}_p(t)\\ \underline{Y}_q(t)\end{bmatrix}=\begin{bmatrix}e^{J_p t}Q_{p,kn}^{-1}\underline{X}_o\\ -\sum_{j=0}^{q^*-2}\delta^{(j)}(t)H_q^{j+1}Q_{q,kn}^{-1}\underline{X}_o\end{bmatrix}, p+q=kn.$$

Using again the transformation (2.6), i.e. $\underline{X}(t)=Q\underline{Y}(t)$, we take

$$\begin{aligned}\underline{X}(t)=Q\underline{Y}(t)&=\begin{bmatrix}Q_{nk,p} & Q_{nk,q}\end{bmatrix}\begin{bmatrix}\underline{Y}_p(t)\\ \underline{Y}_q(t)\end{bmatrix}=Q_{nk,p}\underline{Y}_p(t)+Q_{nk,q}\underline{Y}_q(t)\\ &=Q_{nk,p}e^{J_p t}Q_{p,kn}^{-1}\underline{X}_o-Q_{nk,q}\sum_{j=0}^{q^*-2}\delta^{(j)}(t)H_q^{j+1}Q_{q,kn}^{-1}\underline{X}_o\end{aligned}.$$

Finally, since also $X(t)=Z_1(t)=L\underline{X}(t)$, where

$$L=\begin{bmatrix}I_n & \mathrm{O} & \cdots & \mathrm{O}\end{bmatrix}\in \mathrm{M}(n\times kn,\mathrm{F}).$$

So, the expression (2.12) is derived.□

Remark 2.2 For $t>0$, it is obvious that (2.10) is satisfied. Thus, we can point out that the system (1.4) has the above impulse behaviour at instant time, where a *non-consistent* initial value is assumed, while it appears to have a smooth behaviour at any other instant subsequent time.

3. EXTENDING THE CLASSICAL CONTROLLABILITY RANK CRITERION

In the literature, the notion of the controllability property of dynamical systems has attracted considerable attention for some years. It refers to the ability of systems to transfer the state vectors from one specified vector value to another in finite time by suitable inputs.

In particular, systems of the form (1.2) (or equivalently systems (1.1)) are called controllable if, for any $t_1 > t_o$, $\underline{\mathrm{X}}(\mathrm{t_o}) \in \mathrm{M}\,(kn;\mathrm{F})$ and $\underline{\mathrm{W}} \in \mathrm{M}\,(kn;\mathrm{F})$, there exists a control input $U(t) \in C^{\infty}[t_o,\infty)$ such that $\underline{\mathrm{X}}(\mathrm{t_1}) = \underline{\mathrm{W}}$.

Generally speaking, in this section, we extend a very classical controllability criterion for a class of higher order linear descriptor matrix differential systems.

Let us briefly recall a result for first order linear descriptor matrix differential systems. In this part of the section, we want to remind the well-known Proposition 3.1, see [Dai, 4].

Proposition 3.1 [Dai, 4] The system $E\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$, where $E, A \in \mathrm{M}\,(s\times s;\mathrm{F})$ and $B \in \mathrm{M}\,(s\times r;\mathrm{F})$ is controllable if and only if the following matrix

$$D_{s-1} = \begin{bmatrix} -A & & & & B & & & \\ E & -A & & & & B & & \\ & E & -A & & & & B & \\ & & \ddots & -A & & & & \ddots \\ & & & E & & & & B \end{bmatrix} \in \mathrm{M}\,\left(s^2 \times s(s+r-1);\mathrm{F}\right)$$

is of full row rank, i.e. $rank\ D_{s-1} = s^2$.

Theorem 3.1 Higher order linear descriptor matrix differential systems of type (1.1) are controllable *if and only if* the following matrix

$$C_{kn-1} \triangleq \begin{bmatrix} -A_{k-1} & -A_{k-2} & -A_{k-3} & \cdots & 0 & B & 0 & 0 & \cdots & 0 \\ E & -A_{k-1} & -A_{k-2} & \cdots & 0 & 0 & B & 0 & \cdots & 0 \\ 0 & E & -A_{k-1} & \cdots & 0 & 0 & 0 & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -A_o & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -A_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -A_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -A_3 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & E & 0 & 0 & 0 & \cdots & B \end{bmatrix}$$

$$\in \mathrm{M}\left(kn^2 \times kn\left(n+m-1\right); \mathrm{F}\right) \tag{3.4}$$

is full row rank, i.e.

$$rank\ C_{kn-1} = kn^2 \tag{3.5}$$

Proof According to the Proposition 3.1, system (1.2) is controllable if and only if the following matrix D_{kn-1}, where

$$D_{kn-1} \triangleq \begin{bmatrix} \mathbf{-A} & & & & \mathbf{B} & & & \\ \mathbf{E} & \mathbf{-A} & & & & \mathbf{B} & & \\ & & \ddots & & & & \ddots & \\ & & & \mathbf{-A} & & & & \mathbf{B} & \\ & & & \mathbf{E} & & & & & \mathbf{B} \end{bmatrix} \in \mathrm{M}\left(\left(kn\right)^2 \times kn\left(kn+m-1\right); \mathrm{F}\right)$$

is full row rank, i.e. $rank\ D_{kn-1} = \left(kn\right)^2$, or equivalently,

$$rank\ D_{kn-1} = rank \begin{bmatrix} \mathbf{-A} & & & & \mathbf{B} & & & \\ \mathbf{E} & \mathbf{-A} & & & & \mathbf{B} & & \\ & & \ddots & & & & \ddots & \\ & & & \mathbf{-A} & & & & \mathbf{B} & \\ & & & \mathbf{E} & & & & & \mathbf{B} \end{bmatrix} = \left(kn\right)^2.$$

Then, by substituting $\mathbf{E}$, $\mathbf{A}$ and $\mathbf{B}$ we obtain

$$
rank\begin{bmatrix}
0 & -I_n & 0 & \cdots & 0 & 0 & & & & & & & & 0 & \\
0 & 0 & -I_n & \cdots & 0 & 0 & & & & & & & & 0 & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & & & & & & & \vdots & \\
0 & 0 & 0 & \cdots & 0 & -I_n & & & & & & & & 0 & \\
-A_o & -A_1 & -A_2 & \cdots & -A_{\kappa-2} & -A_{\kappa-1} & & & & & & & & \mathrm{B} & \\
I_n & 0 & 0 & \cdots & 0 & 0 & 0 & -I_n & 0 & \cdots & 0 & 0 & \cdots & & 0 \\
0 & I_n & 0 & \cdots & 0 & 0 & 0 & 0 & -I_n & \cdots & 0 & 0 & \cdots & & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & & \vdots \\
0 & 0 & 0 & \cdots & I_n & 0 & 0 & 0 & 0 & \cdots & 0 & -I_n & \cdots & & 0 \\
0 & 0 & 0 & \cdots & 0 & \mathrm{E} & -A_o & -A_1 & -A_2 & \cdots & -A_{\kappa-2} & -A_{\kappa-1} & \cdots & & \mathrm{B} \\
 & & & & & & I_n & 0 & 0 & \cdots & 0 & 0 & & & \\
 & & & & & & 0 & I_n & 0 & \cdots & 0 & 0 & & & \\
 & & & & & & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & & \\
 & & & & & & 0 & 0 & 0 & \cdots & I_n & 0 & & & \\
 & & & & & & 0 & 0 & 0 & \vdots & 0 & \mathrm{E} & \ddots & & \ddots
\end{bmatrix}
$$

Now, by taking each matrix block as an element, we make the following rearrangements:

- the k-row is transferred to $k-1$ places below, and simultaneously
- the $k+1$-row is rearranged with the first row, the $k+2$- row is rearranged with the second one and so on, as the following expression shows.

$$
rank\begin{bmatrix}
I_n & 0 & 0 & \cdots & 0 & 0 & 0 & -I_n & 0 & \cdots & 0 & 0 & & 0 & \\
0 & I_n & 0 & \cdots & 0 & 0 & 0 & 0 & -I_n & \cdots & 0 & 0 & & 0 & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots & \\
0 & 0 & 0 & \cdots & I_n & 0 & 0 & 0 & 0 & \cdots & 0 & -I_n & & 0 & \\
0 & -I_n & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & \\
0 & 0 & -I_n & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -I_n & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \vdots & \vdots \\
-A_o & -A_1 & -A_2 & \cdots & -A_{\kappa-2} & -A_{\kappa-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mathrm{B} & 0 \\
0 & 0 & 0 & \cdots & 0 & \mathrm{E} & -A_o & -A_1 & -A_2 & \cdots & -A_{\kappa-2} & -A_{\kappa-1} & \cdots & & \mathrm{B} \\
 & & & & & & 0 & 0 & 0 & \cdots & 0 & \mathrm{E} & & & \\
 & & & & & & I_n & 0 & 0 & \cdots & 0 & 0 & & & \\
 & & & & & & 0 & I_n & 0 & \cdots & 0 & 0 & & & \\
 & & & & & & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & & \\
 & & & & & & 0 & 0 & 0 & \vdots & I_n & 0 & \ddots & & \ddots
\end{bmatrix}
$$

Afterwards, we add to the $(k+2)$-column the first column, to the $(k+3)$- column the second one and so on, as it is demonstrated next:

$$
rank\begin{bmatrix}
I_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & \\
0 & I_n & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots & \\
0 & 0 & 0 & \cdots & I_n & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & \\
0 & -I_n & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & \\
0 & 0 & -I_n & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -I_n & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \vdots & \vdots \\
-A_o & -A_1 & -A_2 & \cdots & -A_{\kappa-2} & -A_{\kappa-1} & 0 & -A_o & -A_1 & \cdots & -A_{\kappa-3} & -A_{\kappa-2} & \cdots & B & 0 \\
0 & 0 & 0 & \cdots & 0 & E & -A_o & -A_1 & -A_2 & \cdots & -A_{\kappa-2} & -A_{\kappa-1} & \cdots & & B \\
 & & & & & & 0 & 0 & 0 & \cdots & 0 & E & & & \\
 & & & & & & I_n & 0 & 0 & \cdots & 0 & 0 & & & \\
 & & & & & & 0 & I_n & 0 & \cdots & 0 & 0 & & & \\
 & & & & & & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & & \\
 & & & & & & 0 & 0 & 0 & \vdots & I_n & 0 & \ddots & & \ddots
\end{bmatrix}
$$

Additionally, we add to k-row the second row, to $(k+1)$-row the third one and so on. Afterwards, we add to $(2k-1)$-row the first row multiplied by A_o, the second row multiplied by A_1 and so on, until the $(k-1)$-row.

Furthermore, in the same row we add the $(2k+3)$-row multiplied by A_o, the $(2k+4)$-row multiplied by A_1, and so on until the $(3k-1)$- row. Finally, we add to $2k$-row the $(2k+2)$-row multiplied by A_o, the $(2k+3)$-row multiplied by A_1 and so on. Thus, we obtain the following rank expression

$$
rank\begin{bmatrix}
I_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & \\
0 & I_n & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots & \\
0 & 0 & 0 & \cdots & I_n & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & 0 & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -I_n & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -A_{\kappa-1} & 0 & 0 & 0 & \cdots & 0 & -A_{\kappa-2} & \cdots & \mathrm{B} & 0 \\
0 & 0 & 0 & \cdots & 0 & \mathrm{E} & 0 & 0 & 0 & \cdots & 0 & -A_{\kappa-1} & \cdots & & \mathrm{B} \\
 & & & & & & 0 & 0 & 0 & \cdots & 0 & \mathrm{E} & & & \\
 & & & & & & I_n & 0 & 0 & \cdots & 0 & 0 & & & \\
 & & & & & & 0 & I_n & 0 & \cdots & 0 & 0 & & & \\
 & & & & & & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & & \\
 & & & & & & 0 & 0 & 0 & \vdots & I_n & 0 & \ddots & & \ddots
\end{bmatrix}.
$$

Now, the k-column is transferred $k-1$ places on the right and the $(2k-2)$-row is multiplied by -1. Additionally, the $(2k+2)$, $(2k+3),\ldots,3k$ -rows are transferred $k-1$ places above. Consequently the following expression appears next:

$$
rank\begin{bmatrix}
I_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & & & \\
0 & I_n & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & & & & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & & & \\
0 & 0 & 0 & \cdots & I_n & 0 & 0 & 0 & \cdots & 0 & 0 & & & & \\
0 & 0 & 0 & \cdots & 0 & I_n & 0 & 0 & \cdots & 0 & 0 & & & & \\
0 & 0 & 0 & \cdots & 0 & 0 & I_n & 0 & \cdots & 0 & 0 & & & & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & & & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & I_n & 0 & & & & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -A_{\kappa-1} & -A_{\kappa-2} & \cdots & B & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & E & -A_{\kappa-1} & \ldots & 0 & B & 0 \\
 & & & & & 0 & 0 & 0 & \cdots & 0 & E & \cdots & 0 & 0 & B \\
 & & & & & 0 & 0 & 0 & \cdots & 0 & 0 & & & & \\
 & & & & & 0 & 0 & 0 & \cdots & 0 & 0 & & & & \\
 & & & & & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & & & \\
 & & & & & 0 & 0 & 0 & \cdots & 0 & 0 & & \ddots & & \ddots
\end{bmatrix}.
$$

Finally, we obtain the following expression

$$
rank\begin{bmatrix}
\mathbf{I_n} & & & & & & & & & & & & & \\
 & \mathbf{I_n} & & & & & & & & & & & & \\
 & & \ddots & & & & & & & & & & & \\
 & & & \mathbf{I_n} & & & & & & & & & & \\
 & & & & -A_{k-1} & -A_{k-2} & -A_{k-3} & \cdots & 0 & B & 0 & 0 & \cdots & 0 \\
 & & & & E & -A_{k-1} & -A_{k-2} & \cdots & 0 & 0 & B & 0 & \cdots & 0 \\
 & & & & 0 & E & -A_{k-1} & \cdots & 0 & 0 & 0 & B & \cdots & 0 \\
 & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 & & & & 0 & 0 & 0 & \cdots & -A_o & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & \cdots & -A_1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & \cdots & -A_2 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & 0 & 0 & 0 & \cdots & -A_{k-2} & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & \cdots & -A_{k-1} & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & \cdots & E & 0 & 0 & 0 & 0 & B
\end{bmatrix},
$$

where $\mathbf{I_n} \triangleq I_n \oplus I_n \oplus \ldots \oplus I_n \in \mathbb{R}^{(k-1)n\times(k-1)n}$.

Thus

$$
kn(k-1)n + rank\begin{bmatrix}
-A_{k-1} & -A_{k-2} & -A_{k-3} & \cdots & 0 & B & 0 & 0 & \cdots & 0 \\
E & -A_{k-1} & -A_{k-2} & \cdots & 0 & 0 & B & 0 & \cdots & 0 \\
0 & E & -A_{k-1} & \cdots & 0 & 0 & 0 & B & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -A_o & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & -A_1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & -A_2 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & -A_3 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & E & 0 & 0 & 0 & \cdots & B
\end{bmatrix} = (kn)^2
$$

if and only if

$$rank\begin{bmatrix} -A_{k-1} & -A_{k-2} & -A_{k-3} & \cdots & 0 & B & 0 & 0 & \cdots & 0 \\ E & -A_{k-1} & -A_{k-2} & \cdots & 0 & 0 & B & 0 & \cdots & 0 \\ 0 & E & -A_{k-1} & \cdots & 0 & 0 & 0 & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -A_o & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -A_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -A_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -A_3 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & E & 0 & 0 & 0 & \cdots & B \end{bmatrix} = kn^2 .$$

□

In the end of this section, two simple application-examples are provided for second order linear descriptor differential systems (for instance, it may be simple examples of mathematical pendulums)

$$E\ddot{X}(t) = A_1\dot{X}(t) + A_0X(t) + BU(t).$$

Example 3.1 Denote the following system with

$$E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ \det E = 0,\ A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$rank\begin{bmatrix} -A_1 & -A_0 & B & 0 & 0 \\ E & -A_1 & 0 & B & 0 \\ 0 & E & 0 & 0 & B \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} = 6$$

Example 3.2 Denote the following system with

$$E = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix},\ \det E = 0,\ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A_0 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$rank\begin{bmatrix} -A_1 & -A_0 & B & 0 & 0 \\ E & -A_1 & 0 & B & 0 \\ 0 & E & 0 & 0 & B \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = 5 < 6$$

For example 3.2, it is not difficult to verify that the derived system is un-controllable, see Dai [4].

4. CONCLUSIONS – FURTHER RESEARCH

In this paper, using well-known results for first order linear descriptor matrix differential systems, we calculate the solution properties with consistent and with non-consistent initial conditions. Furthermore, a new controllability property for higher order linear descriptor matrix differential systems is provided.

Finally, two application examples of second order linear descriptor differential systems are solved numerically confirming the validity of the central theorem providing the matrix criterion.

Further current research efforts include the investigation of similar properties regarding higher order descriptor discrete systems, as well as stochastic type descriptor differential systems, see for instance Kalogeropoulos and Pantelous [7], and Mahmudov [9].

REFERENCES

[1] T.M. Apostol (1975), Explicit formulas for solutions of the second order matrix differential equation $Y'' = AY$, *American Mathematical Monthl*y, Vol. 82, pp. 159-162.

[2] R. Ben Taher and M. Rachidi (2008), Linear matrix differential equations of higher-order and applications, *Electronic Journal of Differential Equations*, Vol. 2008, No. 95, pp. 1-12.

[3] Z. Bai, P. De Wilde, and R. W. Freund (2005), *Reduced order modeling*. In W. Schilders and E. J. W. ter Maten, editors, Handbook of Numerical Analysis. Vol. XIII, Numerical Methods in Electromagnetics, p. 825-895.

[4] L. Dai (1989), *Singular Control Systems*, Springer-Verlag, Berlin, Germany.

[5] F.R. Gantmacher (1959), *The Theory of Matrices* Vol. I and II. Chelsea Publishing Company, New York, USA.

[6] M. Hou (1999), *Descriptor Systems: Observer and Fault Diagnosis*. Fortschr. – Ber. VDI Reihe 8, Nr. 482. VDI Verlag, Düsseldorf, Germany.

[7] G.I. Kalogeropoulos and A.A. Pantelous (2008), On Generalized regular stochastic differential-algebraic delay systems with time invariant coefficients, *Stochastic Analysis and Applications*, Vol. 26, No. 5, pp. 1076-1094.

[8] G.I Kalogeropoulos, A.D. Karageorgos, A.A. Pantelous (2009), Higher order linear matrix descriptor differential equations of Apostol-Kolodner type, *Electronic Journal of Differential Equations*, Vol. 2009, No. 25, pp. 1–13.

[9] N.I. Mahmudov (2001), Controllability of linear stochastic systems, *IEEE Transactions on Automatic Control*, Vol. 46, No. 5, pp. 724-731.

[10] V. Mehrmann and C. Shi (2006), Transformation of high order linear differential -algebraic systems to first order, *Numerical Algorithms* Vol. 42, pp. 281–307.

[11] P.C. Müller, P. Rentrop, W. Kortüm, and W. Führer (1993), *Constrained mechanical systems in descriptor form: identification, simulation and control*. In. Schiehlen W. (ed.): Advanced multi body system dynamics, p. 451-456. Kluwer Academic Publichers, Stuttgart, Germany.

[12] M. Otter, H. Elmqvist, and S.E. Mattson (2006), *Multi-domain Modeling with Modelica*. In Paul Fishwick, editor, CRC Handbook of Dynamic System Modeling. CRC Press.

[13] A.A. Pantelous, A.D. Karageorgos, G.I. Kalogeropoulos and K.G. Arvanitis (2010), Solution properties of linear descriptor (singular) matrix differential systems of higher-order with (non-)consistent initial conditions, *Abstract and Applied Analysis* Vol. 2010, Article ID 897301, 24 pages.

[14] P.J. Rabier and W.C. Rheinboldt (2000), *Nonholonomic Motion of Rigid Mechanical Systems from a DAE Viewpoint*. SIAM Publications, Philadelphia, PA.